CHAPTER 1

Introduction to Ordinary Differential Equations (Online)

1.7 Coupled Equations

When two or more dependent variables depend on each other, their equations are "coupled." The rules we have presented so far can be generalized to such situations.

1.7.1 Discovery Exercise: Coupled Equations

Consider a population of foxes and rabbits. They each reproduce, but their populations are both limited by the fact that foxes eat rabbits. For this exercise we'll adopt a simplified model of this relationship.

1. Write a differential equation for the rabbit population $R(t)$ that expresses the sentence: "Each year every rabbit produces 5 babies on average, but 10 rabbits are killed by each fox." Use $F(t)$ for the number of foxes.
   See Check Yourself #7 in Appendix L
2. Explain why you cannot solve this equation for $R(t)$ with the information given. What more information would you need? (*Hint*: The answer is not the initial number $R(0)$. Not knowing that just means there would be an arbitrary constant in your answer.)
3. In Question 1 we gave you a verbal description of the rabbit population and asked you for the differential equation. Here we will do the opposite for the foxes. The fox population is described by the differential equation $dF/dt = R/2 - F$. Give the verbal description that explains where this equation comes from.
4. If $dR/dt = dF/dt = 0$ then both populations remain constant. What would have to be true about the values of R and F for this condition to hold? (They would not both have to be zero, although that is one way to get this result.)
5. Which of the following pairs of functions solve the equations for R and F? (More than one answer may be correct. Indicate all of the correct solutions.)
   (a) $R(t) = 2000$, $F(t) = 1000$.
   (b) $R(t) = 2\cos t$, $F(t) = 2\cos t$
   (c) $R(t) = 1000e^{4t} + 200$, $F(t) = 100e^{4t} + 100$
   (d) $R(t) = e^{4t}$, $F(t) = 2e^{4t}$

1.7.2 Explanation: Coupled Equations

A differential equation in the form dy/dt = "some function of y and t" says that the change in y depends on the current values of y (the dependent variable) and t (the independent variable). In many physical situations, however, multiple dependent variables depend on *each other*. Two reactants decrease in response to each other's concentrations. A planet and a moon change positions in response to each other's positions. In cases like these you need a differential equation for each quantity that depends on itself and on the other variables.

To see what it means to solve coupled differential equations, recall simultaneous algebraic equations.

$$4x^2 - 3y = -5$$
$$2x + y^2 = 53$$

You cannot say that $x = 2$ is a solution, or that it is *not* a solution, because either statement depends on what y is. You can say that $x = 2$, $y = 7$ is a solution because you can plug that in and it works. Similarly, if you have a pair of differential equations for two functions $x(t)$ and $y(t)$ a solution is a *pair* of functions that, taken together, make both equations work.

As an example, consider the famous love of Romeo and Juliet.[3] Romeo's love for Juliet, $R(t)$, grows when it is returned. The more Juliet loves Romeo, $J(t)$, the more his love for her grows. When she hates him, this makes his love diminish. Juliet, on the other hand, is coy. The more Romeo loves her the more bored she becomes with him, but when he despises her this inflames her love. Using positive numbers for love and negative for hatred, we can express all this in a pair of coupled differential equations.[4]

$$\frac{dR}{dt} = J \tag{1.7.1}$$
$$\frac{dJ}{dt} = -R \tag{1.7.2}$$

Try to think of a solution before reading further. We're looking for two functions with the property that the derivative of the first one equals the second one, and the derivative of the second one equals negative the first. One obvious answer is $R(t) = J(t) = 0$, which might correspond to "they haven't met," but see if you can find something more interesting than that.

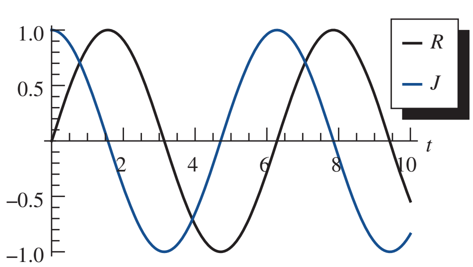


FIGURE 1.3 Romeo and Juliet's feelings for each other oscillate out of phase.

You may have landed on $R(t) = \sin t$, $J(t) = \cos t$. In this particular solution Romeo starts out indifferent to Juliet while Juliet starts out in love $(J > 0)$. Juliet's love causes Romeo's love to grow, which in turn causes Juliet's to fade. They remain fond of each other until $t = \pi/2$, when Juliet begins to dislike Romeo. This causes his love to fade as well until starting at time $t = \pi$ they both dislike each other, which causes Juliet's disdain for Romeo to diminish, etc.

This solution only matches one initial condition, however, so it can't be the general solution. Experimenting with a few constants, we find that $R(t) = 3 \sin t, J(t) = 3 \cos t$ works but $R(t) = 2 \sin t, J(t) = 3 \cos t$ does not. You can multiply R and J by an arbitrary constant, but you must multiply them by the *same* constant. So now we have $R(t) = A \sin t$, $J(t) = A \cos t$. We give below the mathematical criterion for identifying

[3]This example is adapted with permission from *Nonlinear Dynamics and Chaos* by Steven Strogatz. It is not particularly adapted from Shakespeare, although we like to think he would be amused.
[4]We're writing this with incorrect units for simplicity. In Problem 1.158 you'll put in constants of proportionality to fix the units and re-solve the problem.

the general solution to a set of coupled differential equations, but even without the official rules you can see that we're not there yet. You could start with any combination of Romeo's love and Juliet's love, so you need two arbitrary constants to be able to match any possible pair of initial conditions.

We thus need to find another independent solution. Once again, this can be done by squinting at the equations for a while: $R(t) = \cos t, J(t) = -\sin t$. As before, you can multiply this solution by an arbitrary constant and it still works, so adding our two solutions gives us the general solution.

$$R(t) = A\sin t + B\cos t, \quad J(t) = A\cos t - B\sin t$$

This pattern occurs frequently in coupled equations: the same arbitrary constants appear in both solutions, but they appear in front of different functions.

The box below generalizes the definitions and rules for single differential equations to coupled differential equations. We present them all together without much explanation because they are similar to what we have seen before.

Linear Superposition and General Solutions for Coupled Equations

Linearity A set of coupled differential equations is said to be linear if every term in all of the equations is linear in one of the dependent variables. Equations 1.7.1–1.7.2 for Romeo and Juliet are linear because each term is linear in either R or J. Note that this means no term can include more than one dependent variable. If the equations for R and J included a term with R^2, RJ, or $R(dJ/dt)$ they would be non-linear.

Order of a set of equations A set of coupled differential equations is nth order in a given variable if the nth derivative is the highest derivative of that variable that appears anywhere in the set of equations. For example, Equations 1.7.1–1.7.2 are first order in both R and J.

General solution If the differential equations in a set are all linear, then the general solution to that set will have as many independent arbitrary constants as the sum of the orders of all the dependent variables.

Homogeneity A set of linear coupled differential equations is homogeneous if every term includes one of the dependent variables or one of their derivatives.

Linear superposition If a set of coupled differential equations is linear and homogeneous then any linear combination of solutions to the set of equations is also a solution. If a set of coupled differential equations is linear and inhomogeneous then you can define the complementary set of equations by replacing all inhomogeneous terms with zero. In this case the sum of any particular solution to the inhomogeneous equations with any linear combination of solutions to the complementary set of equations is a solution to the inhomogeneous equations. (That's a mouthful, but it's the same thing we said earlier about single differential equations.)

The rule about having as many arbitrary constants as the sum of the orders makes sense if you think about it. If your equations are second order in $y(x)$ you need to specify $y(0)$ and $y'(0)$ as initial conditions, and if they are first order in $z(x)$ you also need to specify $z(0)$, so a set of equations that's first order in one variable and second order in another needs three arbitrary constants.

EXAMPLE

A Molecule with Two States

Problem:

Suppose a certain type of molecule can exist in two possible states. Let $a(t)$ represent the number of molecules in state A and $b(t)$ the number of molecules in state B. Let p be the rate at which state A converts to state B, meaning each second pa molecules change from A to B. Molecules in state B convert to state A at twice that rate, so every second $2pb$ molecules switch from B to A. Putting all that together:

$$\frac{da}{dt} = -pa + 2pb, \quad \frac{db}{dt} = pa - 2pb$$

Which of the following are valid solutions to this pair of equations?

1. $a(t) = 2e^{-2pt}, \ b(t) = -e^{-2pt}$
2. $a(t) = 2 + e^{-3pt}, \ b(t) = 1 - e^{-3pt}$
3. $a(t) = 6 - 2e^{-3pt}, \ b(t) = 3 + 2e^{-3pt}$

What is the general solution?

Solution:

To check a solution you plug *both* functions in simultaneously.

1. First take derivatives of the two functions: $a'(t) = -4pe^{-2pt}$, $b'(t) = 2pe^{-2pt}$. Plugging these in to the first differential equations gives $-4pe^{-2pt} = -2pe^{-2pt} - 2pe^{-2pt}$, which works. Plugging these into the second differential equation gives $2pe^{-2pt} = 2pe^{-2pt} + 2pe^{-2pt}$. This equation is not satisfied, so this is not a solution.
2. The derivatives are $a'(t) = -3pe^{-3pt}$, $b'(t) = 3pe^{-3pt}$. Plugging these in gives $-3pe^{-3pt} = -2p - pe^{-3pt} + 2p - 2pe^{-3pt}$ and $3pe^{-3pt} = 2p + pe^{-3pt} - 2p + 2pe^{-3pt}$. Both equations are satisfied, so this is a solution.
3. We will leave the calculations to you, but you can similarly show that plugging in these solutions works, so this is also a solution.

Because these linear equations are first order in both a and b, the general solution must have two arbitrary constants. We can get it, as usual, by writing a linear combination of the two independent solutions we just found.

$$\begin{aligned} a(t) &= C\left(2 + e^{-3pt}\right) + D\left(6 - 2e^{-3pt}\right) \\ b(t) &= C\left(1 - e^{-3pt}\right) + D\left(3 + 2e^{-3pt}\right) \end{aligned}$$

(This solution can be simplified by combining terms and renaming arbitrary constants; we'll leave that to you to think about.)

Solving Coupled Equations

You may be assuming by this point that we are building up to solving all sorts of coupled differential equations. As with uncoupled equations, there is no one method that applies to all of them, and computers can do that step quite well as a rule. Our main focus is on writing a set of equations for a given scenario, and on interpreting the equations and their solutions. Nonetheless, we will say a few words here about solving coupled differential equations.

The Romeo and Juliet example above demonstrated one important technique for solving coupled equations, which is just thinking about them. "The derivative of R is J, and the derivative of J is $-R$ … what would do that? … ah yes, a sine and cosine." The more you practice this skill the better you will be at it, and the better you will understand such equations.

(Try your hand at these: what if $f'(x) = g(x)$ and $g'(x) = f(x)$ with no negative signs? Or what if $f'(x) = -g(x)$ and $g'(x) = -f(x)$ with negative signs on both? See Problem 1.140.)

In Chapter 6 we will use matrices to represent and solve coupled equations. In Chapter 10 we will discuss Laplace transforms, a powerful technique for solving both single and coupled equations. Chapter 10 will also introduce "phase portraits" which do for coupled equations roughly what slope fields do for single equations: they give you a way of graphing and visualizing the entire space of solutions even if you cannot solve the problem. Here we will present one simple but fairly general method for approaching coupled equations, which is to "decouple" them.

To illustrate the method, let's return again to Romeo and Juliet, but imagine you failed to just think of the solutions. To decouple the equations take either one of them and take the derivative of both sides. Here we start with Equation 1.7.1.

$$R' = J \qquad \rightarrow \qquad R'' = J'$$

That equation gives us an expression for J'. When we substitute that in for J' in Equation 1.7.2, we get a single decoupled equation for R.

$$J' = -R \qquad \rightarrow \qquad R'' = -R$$

This is the simple harmonic oscillator equation, with solution $R = A\sin t + B\cos t$. Finally, plugging this into $R' = J$ gives $J = A\cos t - B\sin t$, the same solution we found before. In general, the way to decouple two differential equations is to differentiate one of them and then use the other to eliminate one of the two variables.

1.7.3 Problems: Coupled Equations

For Problems 1.135–1.139 check whether the given functions solve the set of coupled differential equations.

1.135 $x'(t) = y$, $y'(t) = x$; $x(t) = Ae^t$, $y(t) = Ae^t$

1.136 $x'(t) = -x + y$, $y'(t) = x - y$; $x(t) = A(1 + e^{-2t}) + B(1 - e^{-2t})$, $y(t) = A(1 - e^{-2t}) + B(1 + e^{-2t})$

1.137 $x'(t) = 2y$, $y'(t) = x + y$; $x(t) = Ae^{2t} + 2Be^{-t}$, $y(t) = Ae^{2t} - Be^{-t}$

1.138 $x'(t) = 2xy$, $y'(t) = x^2 + y^2$; $x(t) = 1/(t^2 - 1)$, $y(t) = -t/(t^2 - 1)$

1.139 $x''(t) = y$, $y''(t) = -x$; $x(t) = Ae^t$, $y(t) = Ae^t$

1.140 Find the general solutions (two independent arbitrary constants) to the following sets of coupled equations. You should be able to do these mostly by inspection, but you can decouple them if you get stuck.

(a) $f'(x) = g(x)$, $g'(x) = f(x)$

(b) $f'(x) = -g(x)$, $g'(x) = -f(x)$

(c) $f'(x) = g(x)$, $g'(x) = 4f(x)$

1.141 The superstates of Oceania and Eastasia have a complicated ever-changing relationship. For each scenario below write a set of coupled differential equations that might represent their populations $O(t)$ and $E(t)$. Then describe—using equations or words—how you would expect their populations to evolve over time.

(a) Oceania and Eastasia are mutually supportive allies. The greater their combined (total) population, the more their individual populations grow.

(b) Oceania and Eastasia are at war. The greater the population of Oceania, the more Eastasians are killed each year, and vice versa.

(c) Oceania and Eastasia are at war, but a different kind of war. If Oceania has more people than Eastasia, then the population of Oceania will grow while the population of Eastasia shrinks. The reverse if Eastasia outnumbers Oceania. The greater the difference in population, the greater the changes.

(d) Oceania and Eastasia have a treaty of peace and mutual support, but Oceania cheats. The greater the population of Eastasia, the more Oceania thrives;

the greater the population of Oceania, the more Eastasia suffers.

1.142 A vat contains a molecules of substance A and b molecules of substance B. Each second, kab reactions occur, each of which turns one molecule of A and two molecules of B into a molecule of C.

(a) Write differential equations for the number of molecules of a, b, and c.

(b) Describe in words how you would expect these numbers to evolve over time.

(c) Now assume that in addition to the chemical reaction described above, 10^{23} molecules of A are being added to the vat each second. Write the new differential equations and discuss how this will change the results over time.

1.143 Tanks A and B with volumes V_A and V_B are both filled with a mixture of water and brine. Pure brine is being poured into tank A at a rate of r gallons per minute. Meanwhile the water/brine mix from tank A is being poured into tank B at the same rate, and the mix from tank B is being poured into the nearby river at the same rate. Since the rates are all the same the volume of liquid in each tank stays the same, and you can assume that the tanks are well mixed, so the fraction of brine leaving each tank equals the total fraction of brine in the tank at that moment.

(a) Write a pair of coupled differential equations for the number of gallons of brine in each tank, G_A and G_B.

(b) Physically, what would you expect to happen to G_A and G_B in the long run, and why?

(c) Mathematically, how can you look at the differential equations you wrote and confirm that the physical behavior you described is what G_A and G_B will do at late times? *Hint*: Consider what must be true of G_A and G_B in order for $G_A' = G_B' = 0$, and what will happen to G_A and G_B when that condition isn't met.

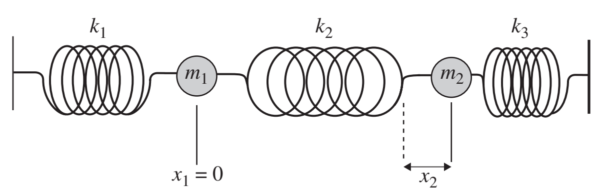
1.144 Tanks A and B, with volumes V_A and V_B, are both filled with a mixture of water and brine. Each minute r gallons of liquid flow from A to B and an equal amount flows from B to A. Since the rates are the same the volume of liquid in each tank stays the same, and you can assume that the tanks are well mixed, so the fraction of brine leaving each tank equals the total fraction of brine in the tank at that moment.

(a) Write a pair of coupled differential equations for the number of gallons of brine in each tank, G_A and G_B.

(b) Physically, what would you expect to happen to G_A and G_B in the long run, and why?

(c) Mathematically, how can you look at the differential equations you wrote and confirm that the physical behavior you described is what G_A and G_B will do at late times? *Hint*: Consider what must be true of G_A and G_B in order for $G_A' = G_B' = 0$, and what will happen to G_A and G_B when that condition isn't met.

1.145 The figure shows two balls connected to each other and to the walls by three springs. Ball 1 has mass m_1 and is displaced by an amount x_1 from its equilibrium position, and similarly for Ball 2.



(a) The position $x_1 = x_2 = 0$ represents the equilibrium position. Now imagine that Ball 1 is at this position precisely, but Ball 2 is slightly to the right of this position, as shown above. Which springs now exert force on Ball 1, and in which directions? Which springs now exert force on Ball 2, and in which directions?

(b) Now imagine that Ball 1 is displaced to the right by a distance x_1, and Ball 2 is displaced to the right by x_2.

i. The force of Spring 1 on Ball 1 depends only on the position x_1 (the other position is irrelevant). Write a formula for this force.

ii. How much is Spring 2 stretched? Your answer should be a function of x_1 and x_2. For example, if both balls are displaced the same amount to the right then Spring 2 isn't stretched at all. As a check on your answer, make sure it gives a positive answer when Spring 2 is longer than its equilibrium length, a negative answer when it is shorter, and 0 when Spring 2 is at its equilibrium length (neither stretched nor compressed.)

iii. Using your answer to Part (ii), write a formula for the force of Spring 2 on Ball 1.

iv. Putting the two forces together and using $F = ma$, write a differential equation for $x_1(t)$. Be sure you have the sign of each force correct.

(c) Repeat Part (b) for x_2.

You will solve this particular set of coupled differential equations twice: using matrices in Chapter 6 , and using Laplace transforms in Chapter 10.

1.146 *[This problem depends on Problem 1.145.]*

(a) Write the coupled differential equations for Problem 1.145 for the case where $k_1 = 1$, $k_2 = 4$, $k_3 = 28$, $m_1 = 1$, and $m_2 = 4$.

(b) The general solution to this problem would have four arbitrary constants. What are the four initial conditions you would need to find them?

(c) Verify that $x_1 = 4\cos(2t)$, $x_2 = \cos(2t)$ is one valid solution.

(d) Verify that $x_1 = \sin(3t)$, $x_2 = -\sin(3t)$ is one valid solution.

1.147 Define a coordinate system with the sun at the origin and the Earth's position given by (x, y). (The Earth orbits in a plane, so we can ignore the third direction.) We will for simplicity assume the sun doesn't move.

(a) The gravitational acceleration of a planet being pulled on by the sun has magnitude GM/r^2, where G is a constant, M is the sun's mass, and r is the distance between the two objects. Write down the magnitude of the gravitational acceleration of the Earth in terms of x and y.

(b) Using the fact that the gravitational acceleration points toward the origin (the sun), find the x- and y-components of the Earth's gravitational acceleration. *Hint*: You may find it helpful to draw a picture and label an angle θ, but your final answer should be in terms of x and y, not θ.

(c) Use the acceleration components you just wrote to write two coupled differential equations for $x(t)$ and $y(t)$.

(d) Verify that $x(t) = a\cos\left(\sqrt{GM/a^3}\ t\right)$, $y(t) = a\sin\left(\sqrt{GM/a^3}\ t\right)$ is a solution to the coupled equations you wrote for any value of a. What kind of motion does this solution represent?

1.148 **Walk-Through: Decoupling Equations.** In this problem you will solve the equations $f'(t) = f(t) + g(t)$, $g'(t) = 3f(t) - g(t)$. We will refer to these as the f' equation and the g' equation respectively.

(a) Differentiate both sides of the f' equation to get f'' in terms of f' and g'.

(b) Substitute g' from the g' equation into your answer to get f'' in terms of f, f', and g.

(c) Solve for g in the original f' equation and plug this into your answer to get a decoupled, second-order differential equation for f.

(d) Find the general solution for $f(t)$ by inspection.

(e) Plug this solution for f into the f' equation and solve (algebraically) for g.

(f) Plug your general solution for f and g into the original equations and verify that they work.

For Problems 1.149–1.152, find the general solutions to the differential equations by decoupling them. You should be able to solve the decoupled equations by inspection or guess-and-check. It may help to work through Problem 1.148 as an model.

1.149 $f' = 3g$, $g' = 6f$

1.150 $f' = f + g$, $g' = f$

1.151 $f' = af + bg$, $g' = cf + dg$

1.152 $f' = af^2/g$, $g' = 2af + bg$. *Hint*: You should end up with a simple linear equation for f''.

Newton's law of heating and cooling states that when you put two objects in contact the cold one will heat up at a rate proportional to the temperature difference between them. The hot object will cool down at a rate proportional to the same temperature difference; however, the constants of proportionality may be very different (the "heat capacity" of each object).[5]

In Problems 1.153–1.155 assume that any two objects in contact with each other obey Newton's law of cooling. Unless otherwise specified, assume that none of the objects gains or loses heat to any other part of the environment. The ODEs will include constants of proportionality, to which you should assign letters, but you should write your equations in

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[5] In practice this is only an approximation, but it generally works well unless the temperature differences are large.

such a way that those letters have positive values. In general the constants of proportionality will be different for the two objects in contact.

1.153 Bars A and B are in contact with each other.

- **(a)** Write a pair of coupled ODEs for T_A and T_B.
- **(b)** Looking at your equations, describe what will happen to T_A and T_B over time. If they start with $T_A > T_B$ what will happen to the two temperatures initially? How will it change over time? What will happen at very late times?
- **(c)** Based on your answers, draw a qualitative sketch showing T_A and T_B vs. time on the same plot. Your plot should show how the functions behave at early and late times. You do not need to include any numbers on your axes.
- **(d)** Solve the differential equations you wrote by decoupling them. Verify that the solutions match the behavior you described qualitatively above.

1.154 Bars A, B, and C are stacked so that A and B are touching and B and C are touching, but not A and C.

- **(a)** Write a set of coupled ODEs for T_A, T_B, and T_C.
- **(b)** What would have to be true of T_A, T_B, and T_C in order for none of them to change? You can use physical intuition to help guide you, but you must explain your answer in terms of the ODEs you wrote.
- **(c)** If the bars start out with $T_A > T_B = T_C$, describe what will happen to the temperatures initially. What will happen a short time later? What will happen a very long time later?

1.155 Bar A is touching bar B, which is in contact with a room at constant temperature T_R. (Bar A is insulated from the room, so it can only exchange heat with bar B. The room is large so it will affect bar B without being affected by it.)

- **(a)** Write a pair of coupled ODEs for T_A and T_B.
- **(b)** What do you expect to happen to T_A and T_B in the long term?
- **(c)** Solve the ODEs you wrote by decoupling them. Take the limit of your solution as $t \to \infty$ and verify that they match your predictions.

1.156 The Discovery Exercise (Section 1.7.1) presented a simplified model of a predator-prey relationship. A more commonly used model, although still too simple for some situations, is the "Lotka–Volterra equations," sometimes called the "predator-prey equations."

$$dR/dt = \alpha R - \beta RF$$
$$dF/dt = \gamma RF - \delta F$$

R and F represents the populations of rabbits and foxes, and the Greek letters represent positive constants.

- **(a)** What would happen to the rabbit population if there were no foxes? You may use common sense to check your answer, but you must explain how you can figure out your answer from the Lotka–Volterra equations.
- **(b)** Similarly, explain using these equations what would happen to the fox population if there were no rabbits.
- **(c)** What values would R and F have to have in order for their populations to be unchanging? Give two answers to this, one of which you can think of just by looking at the equations. The other one will require some calculation.
- **(d)** The product RF appears in both equations, adding to the number of foxes and subtracting from the rabbits. Why?
- **(e)** The simplest population growth for the foxes would be $dF/dt = \delta F$: "the more foxes you have, the more you get." But in the Lotka–Volterra equations, δF is subtracted from dF/dt: "the more foxes you have, the fewer you get." Why? (*Hint*: You cannot answer this question without thinking about the whole scenario.)

1.157 *[This problem depends on Problem 1.156.]* Decouple the Lotka–Volterra equations to get a differential equation for $R(t)$. Why is this technique not particularly useful for this case?

1.158 Consider the Romeo and Juliet problem with proper units. Here R and J are Romeo's and Juliet's love for each other and α and β are positive constants.

$$dR/dt = \alpha J$$
$$dJ/dt = -\beta R$$

(a) If R and J are each measured in love-units, what are the units of α and β?

(b) Solve for $R(t)$ and $J(t)$ by decoupling the equations. Your final answer should have two arbitrary constants in it.

(c) Using the units you found for α and β, what units must your arbitrary constants have in order for your answer to make sense? Make sure that all terms are added to things with equivalent units and that all arguments of trig functions are unitless.

(d) Suppose α is very large compared to β. What does that tell us about Romeo and Juliet? (Answer based on the original differential equations.) What effect will it have on their behavior? (Answer based on your solution.)

1.8 Differential Equations on a Computer

Some students consider computer use "cheating" in some mathematical sense, and believe that Real Scientists find all their solutions by hand. Other students rely too heavily on computers, believing they don't need to learn how to solve equations on their own.

Practical scientists use computers as a tool to solve problems efficiently. They know when and how to use them, and what to do with the answers they get. They also know when to rely on their own understanding of the math.

This section will introduce some of the ways computers can handle differential equations. It will *not* discuss specific commands or syntax, because we don't know what computer program(s) you happen to be using. We're introducing the concepts here, but you will need to learn how to use a particular program, either from your class or from other resources.

1.8.1 Discovery Exercise: Differential Equations on a Computer

1. Go to `http://www.wolframalpha.com`.
2. Type the following:

   ```
   solve dy/dx = e^x / sqrt(ky)
   ```

3. Note that the program shows you your question, written in standard mathematical notation, so you can make sure it interpreted your question correctly. The program also gives you solutions. Test one solution to confirm that it solves the differential equation.
4. Type the following:

   ```
   solve dy/dx = y/(1-x^3)
   ```

5. Once again, the program gives you a solution. Perhaps you'd better take its word for this one.
6. Choose three other differential equations, ranging from easy ones that you know how to solve to more complicated ones that you don't. Record your questions and the solutions. If you try one that the computer can't solve just try a different one instead. If you get an answer that includes a function you've never heard of, that's fine. We'll discuss in the section below what to do with answers like that.

1.8.2 Explanation: Differential Equations on a Computer

If you spend the rest of your life studying methods of solving ordinary differential equations, it's unlikely you will get better at it than computers are now. You ask the computer to solve $dy/dx + y = e^x y^2$ and it takes less than a second to identify "Riccati's equation" and offer the general solution $y = e^{-x}/(C - x)$. (You can check it; it works.) After you've had this experience a couple of times, you will never want to go back to a world in which people looked up differential equations in huge tables.

On the other hand, don't worry that computers are going to take your engineering job. Human beings are still needed to turn real-world problems into differential equations in the first place, and human beings are needed to interpret the results of those equations. The middle step, handling the differential equation, is where computers excel.

One way computers can help is by finding an "analytical solution": a function with the requisite number of arbitrary constants that makes the differential equation true. In the above example the computer solved Riccati's equation; you can now easily plug in an initial condition to find a specific function.

Unfortunately, some solutions can only be expressed as integrals or as infinite series, and may be quite difficult to work with. Worse still, you often can't find any analytical solution, even with a computer! That doesn't mean there's anything wrong with the differential equation. It presumably still has a solution for any given set of initial conditions;[6] you simply can't write that solution in terms of known functions. In such cases, you can often find practical answers by asking the computer for a "numerical solution."

You are already familiar with one numerical technique, which is solving an integral by doing a Riemann sum. (This is how many calculators find integrals, which is why they know that $\int_1^2 2x\,dx = 3$ but may not know that $\int 2x\,dx = x^2 + C$.) A Riemann sum allows you to evaluate any definite integral to any desired degree of accuracy, *without* clever tricks or symbolic manipulation, but *with* dozens (or thousands) of computations, so it is ideally suited to a computer.

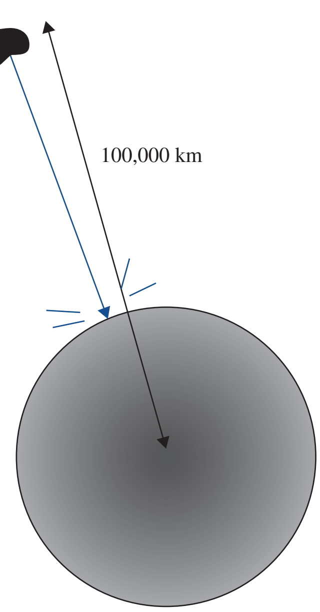
What does it mean to approach a differential equation numerically? Consider that a first-order differential equation tells you how much a function is changing at any given point. So once you have an initial condition, the differential equation tells you how the function will move up or down from that (known) point until you reach another (unknown) point. When you ask a computer to numerically solve a differential equation it generally uses some variation of this method, repeated many times over tiny intervals. We illustrate the use of numerical solutions below, and you will learn one such technique by hand in Problem 1.177.

Analytically or numerically, computers can be tremendously helpful in many different situations, once you learn to use them properly. The three examples below illustrate some of the situations you may run into, and some of the ways you may choose to handle them.

EXAMPLE **Mass Drivers**

Problem:

Centauri Prime won its war with rival planet Narn by hurling rocks from space. The rocks fell toward Narn under the influence of the planet's gravity, obeying Newton's law of universal gravitation $F = Gm_1 m_2 / r^2$. Assume the mass and radius of Narn are comparable to those of the Earth, but that Narn's atmosphere exerts negligible drag. If a rock starts at rest 100,000 km from the center of Narn, how long will it take to strike the planet's surface?



[6]This is not guaranteed mathematically, but for real systems it can often be justified on physical grounds.

Solution:
The first step, of course, is setting up the equation. Using $F = ma$ and plugging in appropriate constants leads you to the equation of motion:

$$\frac{d^2x}{dt^2} = -\frac{k}{x^2} \tag{1.8.1}$$

where $k = 4 \times 10^{14} \text{m}^3/\text{s}^2$. So now you plug that equation into a computer, sit back, and wait for it to solve all your problems! The bad news comes almost immediately: the computer can't find a function that happens to satisfy that particular differential equation. (This usually means there isn't one.) The analytical approach fails in this case.

This problem is ideal for a numerical solution, however. You need to look up the syntax for solving differential equations numerically on your computer system. Whatever system you are using you will need to tell the computer that:

- $d^2x/dt^2 = -4 \times 10^{14}/x^2$, and
- $x(0) = 10^8$, and
- $x'(0) = 0$

The computer generates a list of numbers representing x at different times t. You can make a plot of this function to see where it crosses the line $x = 6 \times 10^6$ (the radius of the planet) or you can use other computer functions to find that value for you. The answer you get is roughly $t = 55,000$. Since you put everything in SI units this value is in seconds, so as a last step you can divide it by 3600 to get the somewhat more useful answer $t \approx 15$ hours.

EXAMPLE An analytical solution that doesn't help

Problem:
A physical quantity is described by the equation:

$$\frac{ds}{dt} = s^3 - 2s^2 - s + 2$$

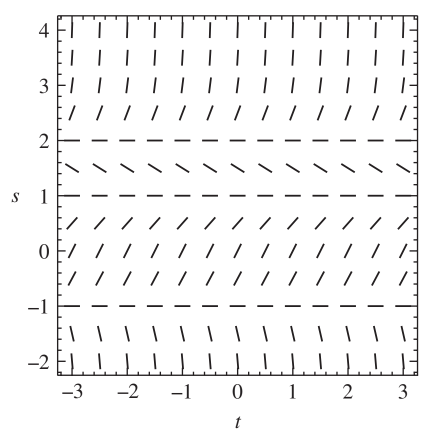
What are the possible long-term behaviors of the system?

Solution:
Unlike the previous example, this can be approached analytically, provided you happen to notice that the expression on the right can be factored. You separate variables, rewrite the left side using partial fractions, integrate, use the laws of logs, and simplify cleverly, and you arrive at $(s+1)(s-2)^2/(s-1)^3 = Ce^{6t}$. Now it's time to solve that for s … and at this point, let's say you start over and throw the problem at a

computer. It comes back almost immediately with an analytical answer:

$$\frac{\sqrt[3]{\sqrt{e^{6(C+t)}\left(e^{6(C+t)}-1\right)^3}-2e^{6(C+t)}+e^{12(C+t)}+1}}{e^{6(C+t)}-1}-\frac{1}{\sqrt[3]{\sqrt{e^{6(C+t)}\left(e^{6(C+t)}-1\right)^3}-2e^{6(C+t)}+e^{12(C+t)}+1}}+1$$

That solution, seeded with different values of C, represents every possible function that this quantity could follow, and therefore contains all the information about the long-term behaviors of this quantity.[7] It's easy to look at that solution and in mere minutes conclude that you no longer care about the long-term behaviors of this quantity and might prefer to go into dentistry. A better solution (not to imply anything wrong with dentistry) is to ask the computer to draw a slope field.



The solution is now clear. If s starts between -1 and 2 it will asymptotically approach $s = 1$. If it starts above 2 or below -1 it will rise or fall (respectively) without bound. The computer can draw the slope field that tells you this, but *you* have to figure out that a slope field will be more useful than an analytic solution for this situation. You might wonder how we knew that drawing the slope field in the range $-2 \le y \le 4$ would show us all the possible behaviors. You can try to figure this out with trial and error, but a more surefire method is to start by finding the equilibria. Recall that these are points where $ds/dt = 0$, so you can have a computer tell you that -1, 1, and 2 are the roots of $s^3 - 2s^2 - s + 2$. Then you know that any range that includes those three points will be sufficient.

[7]Actually, it's a bit worse than that because this is one of three solutions to this equation. If you include all possible values of C in all three solutions then you have all the possible behaviors of $s(t)$. Remember that for non-linear equations a solution with enough arbitrary constants isn't guaranteed to be the general solution.

EXAMPLE **An analytical solution with unknown functions**

Problem:
A function is known to obey the differential equation:

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - 9)y = 0 \; (x \geq 0)$$

where $y(0) = 0$. It is important to know for what *other* x-values this function equals zero. (This function represents an important class of equations that come up in mechanics, quantum mechanics, and probability. You'll explore some of its applications in the problems.)

Solution:
When you hand this differential equation to a computer, it comes back with something like this.

$$y(x) = C_1 J_3(x) + C_2 Y_3(x)$$

Huh?

We arrive now at one of the most important rules for using computer math systems: *when in doubt, look for help!* Look at the built-in help resources of your computer math program, or search online. It doesn't take long to find that $J_3(x)$ and $Y_3(x)$ are "Bessel functions." Don't be put off if you've never heard of one; look at a few graphs or properties and see if you can find the answers you need. In this case, $J_3(0) = 0$ and $Y_3(0)$ is undefined; since our problem stipulated that $y(0) = 0$, we must have $C_2 = 0$, leaving $y(x) = C_1 J_3(x)$.

Now, what about those zeros? Poking around a bit more, we find that J_3 has an infinite number of zeros. The first few are roughly 6.38, 9.76, and 13. Most computer math systems have a built-in function for generating as many of them as you need.

Analytical and Numerical Approaches

"Analytical" and "numerical" represent two very different approaches to problems involving differential equations. The computer can help with both approaches, but it *cannot* suggest which one you should use, so it's worth taking a moment to contrast them.

Disadvantages of analytical solutions include the following:

- Sometimes they are very messy.
- Sometimes they don't exist at all.
- Sometimes the answer you're looking for is just one simple number ("How long can the reactor run before all the fuel is exhausted?") and an analytical solution makes you work much harder to get it.

Disadvantages of numerical solutions include the following:

- To find a numerical solution you must specify initial and/or boundary conditions; there is no such thing as a *general* numerical solution to a differential equation. If you want to know the solution for different sets of initial conditions you have to find a separate numerical solution for each one.
- Similarly, all constants in your equations must be given numerical values. If you want to know how long it will take for a lunar lander to touch down on the surface of the moon you might find a numerical answer, but if you want to find an answer that applies

to landing on other moons or planets, or uses a different fuel burn rate or different size lander, you have to solve the equation again from scratch.

In cases where an analytical solution doesn't exist and you want to describe all the possible behaviors of your system, you have two main options. The first is to approach the problem graphically, using either slope fields (as described in this chapter) or phase portraits (described in Chapter 10). The second option is to approximate the differential equation with one that describes your system reasonably well, but which can be solved analytically. We discuss this method in Chapter 2.

Stepping Back

You can take entire courses or read volumes on computer approaches to differential equations, and this section is no substitute for all that time—or for the time you need to spend familiarizing yourself with the abilities and quirks of your particular software. Our main point here has been that a computer is an invaluable tool, but it is not a permission slip to turn off your brain; on the contrary, using the computer properly can require as much thinking and understanding as solving problems by hand.

But you should also be aware that knowing how to use the computer is quite different from knowing what it is doing "under the covers." Problem 1.177 walks you through an example of a numerical technique called Euler's method. Euler's method turns out to have a lot of drawbacks; modern software tends to use more complicated algorithms that are *based* on Euler's method but provide more accurate answers more quickly. But when you walk through even our brief introduction to that method, you get a beginning sense of what numerical solutions are about. If you find yourself having to program your own solutions (which is more common than you might suppose), you will need to learn the appropriate numerical recipes.[8] As obvious as this sounds, it's important to remember that the computer is not doing anything magic; it is doing the calculations that someone programmed it to do, and you can understand all those calculations even though you could never do so many so quickly.

1.8.3 Problems: Differential Equations on a Computer

Nearly all of the problems in this section require a computer. Whatever software you are using, remember that the first important skill is using the built-in help resources to identify the specific syntax required to solve these kinds of problems!

For any problem in this section where you find an answer numerically or by looking at a plot, you should give an answer that is accurate to at least two significant figures.

In Problems 1.159–1.163 you will be given a differential equation, a set of initial conditions, and a final time. Solve the equation numerically, plot the solution, and find the value of $y(t_f)$.

1.159 $dy/dt = t^3 - y^3$, $y(0) = 1/10$, $t_f = 2$

1.160 $d^2y/dt^2 = -y^3$, $y(0) = 1/10$, $y'(0) = 0$, $t_f = 2$

1.161 $d^2y/dt^2 = -\sin(y)$, $y(0) = 0$, $y'(0) = 1$, $t_f = 2\pi$

1.162 $d^2y/dt^2 + (dy/dt)^2 + y = 1$, $y(0) = 0$, $y'(0) = 0$, $t_f = 5$

1.163 $d^3y/dt^3 = -y^2$, $y(0) = 0$, $y'(0) = 0$, $y''(0) = .2$, $t_f = 2$

1.164 For the differential equation $f''(x) = -(1/x)f(x)$ with boundary condition $f(0) = 0$ how many times does the solution cross the x-axis in the range $0 < x < 100$? Notice that $x = 0$ is not included in this range, so you shouldn't count the initial condition at the origin as one of the axis crossings. (*Hint*: Because this is a second-order equation with only one boundary condition the solution will have an arbitrary constant in it. You should still be able to answer this question.)

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[8]For a wonderful introduction to many types of computer algorithms for numerical calculations see *Numerical Recipes: The Art of Scientific Computing* by Press, et. al.

1.165 For the differential equation $y''(x) + xy'(x) + y(x) = 0$ with initial conditions $y(0) = 0$, $y'(0) = 1$, the solution rises to a maximum, then falls, and then decreases toward zero. Find the value of x where this solution reaches its maximum value.

1.166 For the differential equation $y''(x) + y'(x) + xy(x) = 0$ with initial conditions $y(0) = 0$, $y'(0) = 1$, find the first positive value of x at which the solution $y(x)$ equals zero.

1.167 Important Quantity I follows the differential equation $dI/dt = I^4 - 6I^3 + 9I^2 - 4I$.

(a) Have a computer find the solutions to the equation $I^4 - 6I^3 + 9I^2 - 4I = 0$. Explain how you know that those solutions are the equilibrium values of I.

(b) Have a computer generate a slope field with a range of values for I that includes all the equilibrium values you found in Part (a).

(c) Using this slope field, predict the long-term behavior of I. Your answer will consist of several different statements of the form "If I starts in this range, then it will head toward…"

1.168 Find all the equilibrium values of the differential equation $dx/dt = 4x^4 - 4x^3 - 4x^2 + 4x$ and classify each one as stable, unstable, or neither.

1.169 Consider the differential equation $y'(t) = \sin y$.

(a) Have a computer solve this equation analytically.

(b) Based on your solution, what is $\lim_{t\to\infty} y(t)$ if $y(0) = \pi/2$?

(c) What are the equilibrium values for this equation? *Hint*: There are an infinite number of them.

(d) Draw a slope field for this equation. You can do this by hand or with a computer. Your graph should show at least three equilibrium values.

(e) Make sure your slope field confirms your answer to Part (b) and then use it to find $\lim_{t\to\infty} y(t)$ if $y(0) = -\pi/2$.

1.170 Consider $y'(t) = \sqrt{y - y^2}$

(a) Have a computer solve this analytically. Verify that the solution works.

(b) Draw a slope field for this equation for $0 \le y \le 1$. (Why did we have to restrict it to that range?)

(c) You may have found that the analytic solution and slope field seem to predict very different behavior. Explain. *Hint*: when you verified the analytic solution, what assumptions did you have to make?

(d) Describe the long-term behavior of $y(t)$ if $y(0) = 0$. This is a trick question because there is more than one possible answer. Give at least two. This is an example of the general fact that non-linear equations don't always have a unique solution for each initial condition.

(e) Describe the long-term behavior of $y(t)$ if $y(0) = 1/2$. There is only one answer for this condition.

1.171 An asteroid is detected heading straight toward Earth at 25 km/s. When it is first detected it is 500,000 km from the center of the Earth.

(a) How long will it take to reach the surface of the Earth? (*Hint*: You can find all the information you need in the "mass drivers" example on Page 11. Be careful to convert all units to SI before entering equations on a computer.)

(b) How long would it take an asteroid to reach the surface of Jupiter if were moving straight toward it at 25 km/s starting 500,000 km from the center of Jupiter? (*Hint*: The constant k in Equation 1.8.1 is the universal constant G times the mass of the planet, so you can calculate its value for Jupiter by looking up G and the mass of Jupiter. You will also need to look up the radius of Jupiter to solve the problem.)

1.172 *[This problem does not require a computer.]* In the example on Page 14 we solved the equation

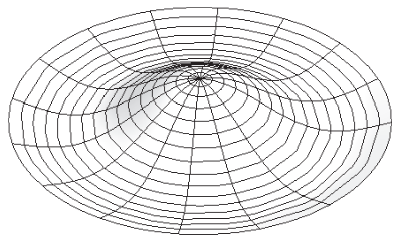
$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - 9)y = 0 \ (x \ge 0)$$

with the boundary condition $y(0) = 0$. Explain why we could have arrived at the same solution if all we had specified was that $y(0)$ must be finite. Using similar logic, explain why you *cannot* solve this equation with the boundary condition $y(0) = 1$.

1.173 A circular drumhead of radius R can have circular standing waves whose amplitude as a function of distance from the center $A(\rho)$ obeys the differential equation

$$A''(\rho) + \frac{1}{\rho}A'(\rho) + k^2A(\rho) = 0 \qquad (1.8.2)$$

where k is a constant related to the frequency of the wave. The boundary conditions for this equation are that the edges of the drum are clamped down, meaning $A(R) = 0$, and that $A(0)$ must be finite.



(a) Find the general solution to this differential equation. The result should be two special functions, each multiplied by an arbitrary constant.

(b) Using the condition that $A(0)$ must be finite explain why one of the two arbitrary constants in your solution must be zero. Write the resulting solution with one arbitrary constant.

(c) The condition that $A(R) = 0$ does *not* restrict your other arbitrary constant. Instead it restricts the possible values of k. By looking up the values at which the function you found equals zero, find the first three possible values of k for which the condition $A(R) = 0$ can be satisfied for a drum of radius $R = 0.1$ m.

(d) The frequency f is related to k by $f = kv/(2\pi)$, where v is the speed of sound on the drumhead (which depends on its tension). For a drumhead of radius 0.1 m with sound speed $v = 100$ m/s find the first three possible frequencies for circular waves. As a check on your work your answer should come out in units of 1/s, otherwise known as Hertz (Hz).

(Drums can also have waves with more complicated shapes. We'll consider the general solution for a vibrating drumhead in Chapter 11.)

1.174 *[This problem depends on Problem 1.173.]* A circular wave on a drumhead is described by the solution you found in Problem 1.173 multiplied by $\cos(2\pi ft)$, where f is the frequency you found at the end. Using the third value of k you found and the corresponding value of f, make a series of nine plots similar to the one at the beginning of Problem 1.173, each plot showing the drumhead at a different time. Your final plot should be at the time when it returns to its original shape. (If your program can make animations you can do a single animation instead of the sequence of nine plots.) Use 0.2 for your arbitrary constant (which gives the amplitude of the wave).

1.175 In quantum mechanics a particle is described by a "wavefunction" ψ that tells you the probabilities of finding the particle in different places. For a particle in a spherical region with no forces acting on it the wavefunction obeys the equation

$$\frac{d^2\psi}{dr^2} + \frac{2}{r}\frac{d\psi}{dr} - \left(\frac{l(l+1)}{r^2} - 1\right)\psi(r) = 0$$

where l is an integer related to the particle's angular momentum and the distance from the origin r is expressed in units that allow you to eliminate all other constants from the problem.

(a) Find the general solution for $\psi(r)$.

(b) Using the condition that $\psi(0)$ must be finite, set one of the arbitrary constants in your general solution to zero and write the remaining solution.

(c) The values of r where $\psi(r) = 0$ indicate radii where there is zero chance of finding the particle. Find the first such non-zero radius for the three cases $l = 0$, $l = 1$, and $l = 2$.

1.176 Superman's enemy Lex Luthor is holding a block of kryptonite, which is deadly to Superman. Superman is attempting to reach Luthor, but the closer he gets to the kryptonite the slower he moves. Assume his velocity is given by $v = -v_s x/(d + x)$ where v_s and d are constants and x is Superman's distance from the kryptonite.

(a) Sketch the function $v(x)$ and describe what happens to Superman's speed when he is very far from the kryptonite and when he is very close.

(b) Try to solve this differential equation by hand using separation of variables to find the function $x(t)$. Explain why this *doesn't* work.

(c) Assume Superman's normal speed when he is far away from kryptonite is 1000 m/s (faster than a speeding bullet) and that his speed drops to half that value when he is 20 m from the kryptonite. Find the values of v_s and d and solve for $x(t)$ numerically assuming he starts 100 m away. If he needs to get within 1 meter of the kryptonite before he can reach it and get rid of it, how long will that take him? (Be careful with signs!)

1.177 Exploration: Euler's Method By Hand *[This problem does not require a computer.]*
The simplest technique for numerically solving differential equations is "Euler's method." In this problem, you will use Euler's method to answer the question: If $dy/dx = y$ and $y(0) = 1$, what is $y(1)$?

This problem asks us to follow the function from $x = 0$ to $x = 1$. We choose to take this journey in three steps, each of length 1/3.

(a) You're going to draw a slope field for this equation. To start, draw the axes and draw a small segment of slope at the point $(0, 1)$. Explain how we know that the value of the slope at that point must be 1.

(b) Our next task is to find the value $y(1/3)$. Using the slope you found, explain how we can know that $y(1/3) \approx 4/3$.

(c) Calculate the slope of the curve at $(1/3, 4/3)$ and use the result to draw the next entry in your slope field at that point.

(d) Using the slope you just found, estimate the value of $y(2/3)$.

(e) Calculate the slope at the new point you just found and use it to draw another entry on your slope field.

(f) Use the last slope you calculated to estimate $y(1)$.

(g) We began with the question "What is $y(1)$?" and you answered that question in Part (f). Explain why your answer is not exactly correct. Your explanation should enable you to predict whether the actual $y(1)$ is higher or lower than your approximation.

(h) Solve the equation $dy/dx = y$ with initial condition $y(0) = 1$ analytically and find the exact, correct value for $y(1)$. If you didn't correctly predict whether your approximate answer would be too high or too low rethink your explanation and explain the result you did get.

1.178 Exploration: Euler's Method By Computer *[This problem depends on Problem 1.177.]*

In Problem 1.177 you used Euler's method to find an approximate value for $y(1)$ given the equation $dy/dx = y$ and the initial condition $y(0) = 1$. You did this in three steps and found a not-so-great approximation to the exact answer.

(a) Have a computer repeat the calculation, but this time using 10 steps. In other words, starting from the known value $y(0) = 1$ calculate the slope $y'(0)$ and use that to find an approximate value for $y(0.1)$. Then use that to find the slope $y'(0.1)$ and thus the value $y(0.2)$, and so on until you have found a value for $y(1)$. Record the resulting value for $y(1)$. *Hint*: We strongly suggest using a loop to do the ten calculations rather than writing all ten of them out one at a time. This will be faster and easier in this step, and essential for the next one.

(b) Repeat Part (a) with 20 steps instead of 10. You should find that as you increase the number of steps your answer gets closer to the exact answer. *Hint*: If you haven't done so already you should be able to write your calculation in such a way that you can change the number of steps simply by changing one number and rerunning.

(c) Keep doubling the number of steps until your answer for $y(1)$ is within 1% of the exact answer. How many steps do you need and how close is the resulting answer to the exact one?

You've now used Euler's method to get a fairly accurate answer to a problem that you could have answered more easily without it anyway. Of course, the real power of the method is in solving problems you couldn't easily solve analytically! So now consider the equation $dy/dx = \tan(x + y)$ with initial condition $y(0) = 1$.

(d) Ask your computer to *analytically* solve this differential equation. What result do you get?

(e) Use Euler's method to find $y(0.1)$ with four steps. In other words use the slope $y'(0)$ to estimate $y(.025)$ and so on.

(f) Try again with eight steps and keep doubling the number of steps until you get an answer that differs from its predecessor by less than 1%.

1.9 Additional Problems

1.179 Parts (a)–(e) below give five different scenarios for the mass of a spherical snowball $M(t)$. Match each one with the appropriate differential equation.

(a) Rolling down a hill, it gains 5 g/s.	I. $dM/dt = 2M$
(b) Every second, 5 g melt.	II. $dM/dt = M - 5$
(c) Rolling, the mass doubles every second.	III. $dM/dt = -5$
(d) Rolling, the mass triples every second.	IV. $dM/dt = 5$
(e) Every second the snowball picks up its own mass, but 5 g melt.	V. $dM/dt = M$

1.180 A thermostat is set to keep the house temperature at a constant 68°. If the temperature rises too high, the air conditioner brings it down; if the temperature falls too low, the heater brings it up. Using u for the room temperature, which of the following differential equations best represents this situation? Assume every k is a positive constant with the appropriate units.

(a) $du/dt = 68k$

(b) $du/dt = 68kt$

(c) $du/dt = 68ku$

(d) $du/dt = k(u - 68)$

(e) $du/dt = k(68 - u)$

(f) $du/dt = k(u + 68)$

(g) $du/dt = 68k_1(u + k_2 t)/(u - k_2 t)$

For Problems 1.181–1.186,

(a) Confirm that the given solution solves the Differential Equation (DE) for *any* value of the arbitrary constant(s).

(b) Find the specific solution that matches the given condition.

1.181 DE: $r'(\theta) + 3r(\theta) = \cos\theta$
Solution: $r = (\sin\theta + 3\cos\theta)/10 + Ce^{-3\theta}$
Condition: $r(0) = 1$

1.182 DE: $dx/dt = (tx - x^2)/t^2$
Solution: $x = t/(\ln t + C)$
Condition: $x(1) = e$

1.183 DE: $\dfrac{dy}{dx} = \dfrac{(x+y)\ln(x+y) - (x+2)}{x+2}$
Solution: $y = e^{C(x+2)} - x$
Condition: $y(0) = 10$

1.184 DE: $\dfrac{dy}{dx} = \dfrac{(\ln x - 1)(y^2+1)}{xy\ln x}$
Solution: $y = \sqrt{\dfrac{Cx^2}{(\ln x)^2} - 1}$
Condition: $y(e^2) = 3$

1.185 DE: $V''(x) - 4V(x) = 6x - 4x^3$
Solution: $V(x) = x^3 + C_1 e^{2x} + C_2 e^{-2x}$
Conditions: $V(0) = 0$, $V(\ln 2) = 1$

1.186 DE: $s''(t) - 9s + 8 = \sin t$
Solution: $s(t) = 8/9 - (\sin t)/10 + C_1 e^{3t} + C_2 e^{-3t}$
Conditions: $s(0) = 0$, $s'(0) = 0$

For Problems 1.187–1.192,
a) Find the general solution to the given differential equation. You may solve by simple inspection, by separation of variables, or by guess-and-check with an unknown constant.
b) Find the specific solution corresponding to the given condition(s).

1.187 $dy/dt = ky$, $y(0) = p$

1.188 $d^2y/dt^2 = -16y$, $y(0) = 0$, $y'(0) = 2$

1.189 $du/dx = u + 3$, $u(0) = 0$

1.190 $du/dx = (2x + 1)u$, $u(0) = 1$

1.191 $d^2f/dx^2 - 5df/dx + 6f = 0$, $f(0) = 0$, $f(1) = 1$

1.192 $d^2f/dx^2 - 3df/dx + 2f = 6e^{3x}$,
$f(0) = 0$, $f'(0) = 0$

1.193 With sufficient food and no predators, the population of tribbles multiplies by 121 every day. However, the population is kept in check by a ravenous horde of vermicious knids, who eat 12,000 tribbles a day.

(a) Write a differential equation for the number of tribbles as a function of time.

(b) What is the equilibrium solution to your equation? Is it a stable or unstable equilibrium? How can you tell?

(c) Solve your differential equation with the initial condition $P(0) = 125$.

1.194 Solve the differential equation $d^2y/dt^2 = 3y$ with initial conditions $y(0) = 0$, $y'(0) = 4\sqrt{3}$.

1.195 $dy/dx = e^{x+y}$.

(a) Solve.

(b) Find the solution that contains the point $y(0) = -2$.

(c) Demonstrate that your solution satisfies both the differential equation and the initial condition.

1.196 The function $y = f(x)$ is the solution to the differential equation $dy/dx = (y-2)^2/(x-2)$ that goes through the point $(3, 0)$.

(a) Find the slope of this curve at the point $(3, 0)$.

(b) Find a formula for the *concavity* of this curve as a function of x and y. (Remember that concavity is the derivative with respect to x of dy/dx. After you use the quotient rule, your formula for concavity will have x, y, and dy/dx in it. Then you can replace dy/dx with $(y-2)^2/(x-2)$ to get a function of x and y.)

(c) Find the concavity of this curve at the point $(3, 0)$.

(d) Based on your answers to parts (a) and (c), draw a quick sketch of the curve around the point $(3, 0)$.

(e) Find the function $y = f(x)$.

1.197 The function $V(t)$ follows the differential equation $dV/dt = 2V^2 + V^3 - V^4$. Predict the long-term behavior of V. Your answer will consist of several different statements of the form "If V starts in this range, then it will head toward…"

1.198 Consider the differential equation $dy/dt = y^2$.

(a) Begin by drawing a slope field at all integer points in $t \in [0, 5]$, $y \in [-2, 2]$.

(b) Based on looking at your slope field, what is $\lim_{t\to\infty} y(t)$ if $y(0) = 1$? If $y(0) = -1$?

(c) Now solve the differential equation using separation of variables. Your solution should have an arbitrary constant C.

(d) Find two specific solutions for the two initial conditions $y(0) = 1$ and $y(0) = -1$.

(e) Find $\lim_{t\to\infty} y(t)$ for your two specific functions.

(f) In one case, you should find that the limit you predicted based on your slope field does not match the limit you predicted based on your solution. Explain why one (or both) of your predictions was wrong.

1.199 Consider the differential equation $dy/dx = y - x^2$.

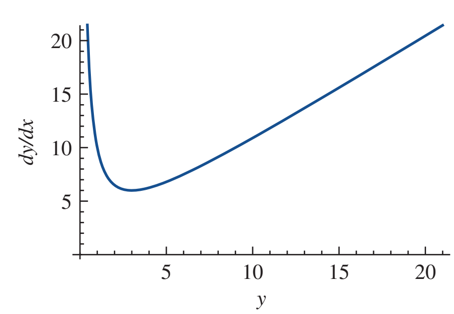
(a) Have a computer make a slope field for this equation. You may need to do some trial and error to find good ranges for x and y. You should have at least 80 points on your slope field.

(b) Have the computer plot the function $y = 2 + 2x + x^2$ on the same plot as the slope field you just found. You should be able to convince yourself that this function is following the slopes indicated on your slope field, and is thus a solution to the differential equation. (You can also check this analytically.)

(c) Predict how the function will behave if it has an initial condition that places it above the solution you just plotted. How will it behave if it has an initial condition below the one you plotted? Explain your predictions by referring to the behavior of the slope field above and below that solution.

(d) Have the computer numerically solve this differential equation with initial conditions $y(0) = 1$ and $y(0) = 3$. Make a final plot showing the slope field, the particular solution you plotted before, and these two numerical solutions. Does their behavior match your predictions? Explain.

1.200 The drawing below shows dy/dx as a function of y. $\lim_{y\to 0^+}(dy/dx) = \infty$ and $\lim_{y\to\infty}(dy/dx) = \infty$, and the absolute minimum occurs where $y = 3$ and $dy/dx = 6$. The function is undefined for $y \le 0$.



Sketch the function $y(x)$ that goes through the point $x = 0$, $y = 3$.

1.201 A dish contains an amount of bacteria $B(t)$. On average each bacteria cell produces one new offspring each day. At the same time an enzyme in the dish kills 1000 bacteria cells per day. Write a differential equation for $B(t)$ and solve it, assuming the dish starts with 1200 bacteria cells. (In writing your equation you should assume time is measured in days and B is measured in number of cells.) How many bacteria cells does the dish contain after ten days?

1.202 In 1930, psychologist Louis L. Thurstone proposed[9] that learning follows the differential equation $dp/dt = k[p(1-p)]^{3/2}$ where p represents mastery of a task or topic on a scale of 0 to 1 and k is a positive constant.

- **(a)** What are the equilibrium solutions, and what do they represent about learning?
- **(b)** At what p-value does a student learn the fastest?
- **(c)** Sketch the general shape of a learning curve according to Thurstone's model.

This equation can be solved analytically. We encourage you to have a computer solve it for you, which will give you a good appreciation for why sometimes an analytic solution to an ODE is not the best way to understand it.

1.203 We have worked several times with drag forces proportional to velocity. In some cases it is more accurate to model a drag force as proportional to velocity *squared*: $F = -kv^2$ where k is a positive constant.

- **(a)** Explain why this law, as written, only makes sense for positive v-values.
- **(b)** Using Newton's Second Law, write a first-order differential equation for the velocity of an object experiencing this drag force and no other forces.
- **(c)** Draw a slope field at all integer points in $t \in [0, 5]$, $v \in [0, 2]$.
- **(d)** Based on looking at your slope field, what is $\lim_{t\to\infty} v(t)$ if $v(0) = 0$? If $v(0) = 1$?
- **(e)** Solve the differential equation using the initial condition $v(0) = v_0$.
- **(f)** How long will it take the object to reach 1/10 of its original speed? Your answer will depend on k, m, and v_0.
- **(g)** Find the body's *position* function $x(t)$, assuming $x(0) = 0$.
- **(h)** Draw a quick sketch of $x(t)$ for $t \geq 0$.

1.204 Black Holes and Hawking Radiation
In 1974 Stephen Hawking predicted that, due to quantum effects, black holes should lose mass. This process, called "evaporation," follows the equation $dM/dt = -\hbar c^4/(15,360\pi G^2 M^2)$ where M is the mass of the black hole and all other quantities are constants.

- **(a)** We can make this equation more tractable by grouping the constants together. Let $k = \hbar c^4/(15,360\pi G^2)$ and rewrite the differential equation in a simpler form.
- **(b)** Looking at your differential equation, does it predict that black holes will grow or shrink? Does it predict that large black holes will change more slowly or more quickly, than small ones? Based on these facts, write a brief description of the life of a black hole.
- **(c)** Solve the differential equation with the initial condition $M = M_0$.
- **(d)** Look up values of $\hbar$ (Planck's constant divided by 2π), c (the speed of light), and G (the gravitational constant). Make sure they are all in standard SI units! Put them together to find the value of your constant k.
- **(e)** The most commonly observed type of black hole is a "stellar black hole," which has a mass comparable to that of the sun: 2×10^{31} kg. How long would such a black hole take to evaporate entirely?
- **(f)** The universe is roughly 1.4×10^{10} years old. Express the lifetime of a stellar black hole as a multiple of the current age of the universe.
- **(g)** It's possible that particle physics experiments will produce microscopic black holes with mass comparable to a proton: about 2×10^{-27} kg. As of this writing, there has been no evidence of such black holes being produced, but some people worry that if they were they would destroy the Earth. How long would such a black hole last? Is this something we should be worried about?[10]

[9] Thurstone, L.L. The Learning Function. *Journal of General Psychology,* 1930, 3, 469–493.
[10] Aside from the short lifetime of such black holes, there's a simpler argument for why they couldn't destroy Earth. Cosmic rays constantly strike the upper atmosphere with energies far greater than we can produce in a lab. If these experiments could produce anything that would destroy Earth it would have happened billions of years ago.

1.205 Rocket Science
A rocket engine produces thrust by burning and expelling fuel. Newton's Second Law $F = ma$ holds as always, but m is not a constant. Assume the rocket begins with a total mass of m_0 and burns off mass at a steady rate of k (measured in mass per unit time), producing a constant force of F.

(a) Write a function for the rocket's mass $m(t)$. (This is a high school algebra problem; don't try to make it complicated.)

(b) Write a second-order differential equation for the rocket's position $x(t)$.

(c) Because your equation gives d^2x/dt^2 as a function of t only (no x-dependence), you can integrate twice to find the general solution. Your final answer should have two independent arbitrary constants.

(d) Assuming the rocket begins its journey at rest at $x = 0$, solve for your arbitrary constants and find the function $x(t)$. Your answer will, of course, contain the constants F, m_0, and k.

(e) If the rocket begins with a mass of 5000 kg and burns mass at a rate of 60 kg/s, exerting a constant force of 5,000,000 N, how fast is it going after 50 s? How far has it traveled?

1.206 The Barometric Formula
A region is filled with fluid (gas or liquid). This fluid is everywhere subjected to the downward force of gravity, balanced by the "pressure gradient" (lower altitudes are at higher pressure). This equilibrium is represented by the equation $dP/dz = -\rho g$, where P is the pressure, z is height, ρ is density, and $g = 9.8$ m/s^2 is the acceleration due to gravity. (Deriving this equation is a physical problem unrelated to what we want to convey in this chapter, so we're just giving it to you.)

(a) For a liquid, density generally stays pretty constant. Treating ρ as a constant, find $P(z)$. Your answer should of course include an arbitrary constant.

(b) For a gas, the ideal gas law gives the density as $\rho = mP/(RT)$, where m is the molar mass and R is the ideal gas constant. Assuming constant temperature, find $P(z)$. This result is known as the "barometric formula."

1.207 *[This problem depends on Problem 1.206.]* In Problem 1.206(b) you derived the barometric formula for pressure variation with depth by assuming a constant temperature. For the lower portion of the Earth's atmosphere, known as the troposphere, a more realistic model is a linear variation of temperature with altitude: $T = T_0 - kz$. Calculate $P(z)$ under this assumption.

1.208 Newton's Law of Heating and Cooling
A small object (such as a hot cup of coffee or a cold soda) at temperature Q is placed inside a room with ambient temperature Q_r. According to "Newton's Law of Heating and Cooling," the rate of change of the object's temperature is proportional to the *difference* in temperature between the object and the room.

(a) Newton's Law can be expressed as $dQ/dt = k(Q - Q_r)$ or as $dQ/dt = k(Q_r - Q)$. Both are mathematically valid, but we want to choose the one that will lead to a positive k-value. (That way we know that e^{kt} is blowing up and e^{-kt} is approaching zero.) Which of these equations will give us a positive k-value? Will it work for both the hot coffee and the cold soda? How can you tell?

(b) Solve Newton's Law by separating variables. Note that your function $Q(t)$ will have three constants in it: Q_r (the temperature of the room), k (the constant of proportionality), and C (an arbitrary constant).

(c) What is $\lim\limits_{t\to\infty} Q(t)$?

(d) A 90°C cup of coffee placed in a 70°C room reaches 80°C in 10 min. How long will it take to reach 71°C?

1.209 Radioactive Decay
A radioactive sample consists of a large collection of unstable atoms. In any given day (or year or century), every one of those atoms has a certain chance of decaying, independent of all the other atoms. Therefore, the more atoms you have, the more atoms decay: $dM/dt = -kM$ where k is a positive constant.

(a) Draw a slope field for this differential equation on all integer points in $0 \le t \le 4$, $0 \le M \le 4$ using $k = 1/2$. Then draw two sample curves through your slope field.

(b) Based on your slope field, what is $\lim\limits_{t\to\infty} M(t)$?

(c) Solve the differential equation with the initial condition $M(0) = M_0$. (Do *not* set $k = 1/2$ as you did in Part (a); leave it as an unknown constant that varies from one substance to another.)

Does the resulting function match the curves you drew in Part (a)?

(d) The "half-life" $t_{1/2}$ of a radioactive substance is the amount of time elapsed before $M = M_0/2$. Write a formula for the half-life as a function of k.

(e) If you measure that 10% of a radioactive sample has decayed in one day, how long will it be until 90% decays?

1.210 *[This problem depends on Problem 1.209.]*

(a) Uranium-235 is used in atomic bombs. How long does it take a sample of ^{235}U to decay to a tenth of its original mass?

(b) Carbon-14 is used in dating ancient organic samples. How long does it take a sample of ^{14}C to decay to a tenth of its original mass?

1.211 Rate of a Chemical Reaction.
When iron and sulfur are heated together, they can combine to create iron sulfide. Each time an Fe molecule collides with an S molecule, there is a chance that they will form an FeS molecule: **Fe+S→FeS**.

Let f equal the number of g-moles of iron, s equal the number of g-moles of sulfur, and p equal the number of g-moles of iron sulfide produced. The system begins with $f = f_0$, $s = s_0$, and $p = 0$.

(a) Explain why, at any given time, $f = f_0 - p$.

(b) Suppose at a given moment $df/dt = -5$. What is ds/dt? What is dp/dt?

(c) Because the reaction depends upon random collisions between Fe and S molecules, the rate of reaction dp/dt is proportional to the *product fs*. Write a differential equation for $p(t)$. Note that your differential equation will include the variable p and the constants f_0 and s_0, but cannot include the variable quantities f and s.

(d) Solve your differential equation assuming that $f_0 = s_0$.

(e) Solve your differential equation assuming that $f_0 \neq s_0$. (*You can solve this on a computer or do it by hand using partial fractions.*)

(f) Find $\lim_{t\to\infty} p(t)$. The answer will depend on whether $s_0 > f_0$, $s_0 = f_0$, or $s_0 < f_0$.

1.212 Spread of a Disease
The *logistic equation* is used to predict, among other things, the spread of disease through a fixed population. For instance, a disease might follow the equation $dP/dt = 2P(1 - P)$ where P is the fraction of the population that is infected.

(a) For very low values of P, $1 - P$ is approximately 1, so the differential equation is approximately $dP/dt = 2P$. What kind of growth is predicted by this equation? What does it suggest about the mechanism of disease spread?

(b) For very high values of P, $1 - P$ approaches zero, so the growth slows down. Why does this make sense in terms of how the disease spreads?

(c) Draw a slope field for this equation for $P = 0, 1/8, 1/4, 3/8, 1/2, 5/8, 3/4, 7/8, 1$. Ignore negative t-values.

(d) Based on looking at your slope field, list the equilibrium solutions and classify them as stable or unstable.

(e) Trace a sample curve through your slope field, starting at $P(0) = 1/8$. Your curve should extend far enough to determine $\lim_{t\to\infty} P(t)$.

1.213 *[This problem depends on Problem 1.212.]*

(a) Solve the differential equation in Problem 1.212. (*You can solve this on a computer or do it by hand using partial fractions.*)

(b) Find the specific solution that corresponds to the initial condition $P(0) = 1/8$.

(c) Find $\lim_{t\to\infty} P(t)$. Make sure it matches your prediction in Problem 1.212.

(d) Graph your solution on a computer. Make sure the graph matches the graph you drew based on your slope field.

CHAPTER 2

Taylor Series and Series Convergence (Online)

2.8 Asymptotic Expansions

In introductory calculus classes the statement "this series diverges" is generally taken to mean "this series is useless." But with *asymptotic expansion* we can sometimes use divergent series to approximate functions.

2.8.1 Explanation: Asymptotic Expansions

On Page 70 we presented the following Taylor series.

$$\frac{1}{1-x} = -\frac{1}{2} + \frac{1}{4}(x-3) - \frac{1}{8}(x-3)^2 + \frac{1}{16}(x-3)^3 + \dots \qquad (1 < x < 5) \tag{2.8.1}$$

Equation 2.8.1 claims that the function on the left and the infinite series on the right are "equal"—that is, the partial sums $S_n(x)$ are roughly equal to the function values $f(x)$. This approximation works best if n is very large and x is very close to 3. We can therefore make this claim more specific in two ways.

1. Hold x constant and increase n. For instance if $x = 3.1$ then $1/(1-3.1) \approx -0.47619$.

$$\begin{aligned} S_1(3.1) &= -1/2 && = -0.5 \\ S_2(3.1) &= -1/2 + (0.1)/4 && = -0.475 \\ S_3(3.1) &= -1/2 + (0.1)/4 - (0.1)^2/8 && = -0.47625 \end{aligned}$$

First claim: $\mathbf{As\ n \to \infty, S_n(3.1) \to 1/(1-3.1)}$. This claim holds, not only for $x = 3.1$, but for any x-value between 1 and 5.

2. Hold n constant and let x approach 3. For instance, S_2 is the linear approximation $-1/2 + (x-3)/4$.

$$\begin{aligned} f(4) &= -0.333 & S_2(4) &= -0.25 \\ f(3.1) &= -0.47619 & S_2(3.1) &= -0.475 \\ f(3.01) &= -0.497512 & S_2(3.01) &= -0.4975 \end{aligned}$$

Second claim: **The closer x gets to 3, the closer $\mathbf{S_2(x)}$ comes to $\mathbf{1/(1-x)}$.** This claim holds, not only for $S_2(x)$, but for any partial sum in the series.

Take a moment to convince yourself that these two claims define our expectation for any Taylor series, and that both are true for Equation 2.8.1.

For a divergent series the first claim above cannot possibly be true. (In the limit as $n \to \infty$ such a series does not approach anything.) In some such cases, however, the second claim still holds true; each partial sum makes a better and better approximation for a given function as x gets closer to some designated value. Since most of our uses for power series involve using finite partial sums as approximations, that claim is enough to make a power series useful even if it will ultimately diverge.

The Asymptotic Expansion for the Complementary Error Function

As an example, consider the "complementary error function" erfc x.

$$\text{erfc } x = \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-t^2}\,dt$$

This function comes up in probability and statistics, and we discuss it in Chapter 11. For our purpose here you only need to know three things about the complementary error function: it is useful in many real-world situations, it is defined for all x-values, and it can be difficult to calculate. It is therefore desirable to approximate its values with a series.

$$\text{erfc } x \sim \frac{e^{-x^2}}{\sqrt{\pi}x} \sum_{n=0}^{\infty} (-1)^n \frac{(2n)!}{n!(2x)^{2n}} \tag{2.8.2}$$

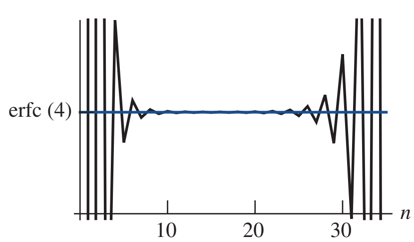


FIGURE 2.13 Partial sums of the asymptotic series for erfc x, evaluated at $x = 4$.

We will discuss below where that series comes from, but first let's see what it does. If you choose one particular x-value and start accumulating terms you will find that the partial sums approach the desired value for a while and then move away from it. For instance Figure 2.13 shows partial sums of this series evaluated at $x = 4$. By the 10th partial sum the series gives a very good approximation of erfc 4, but some time after the 20th it begins to move away.

You can see that this series does not ultimately converge to erfc 4. In fact Equation 2.8.2 diverges for any x-value you plug into it! (See Problem 2.183.) But you can also see that this series does approximate erfc 4 well if you add up the right number of terms. (The optimal result often comes from stopping after the smallest term—not a surprising result if you look at the graph.)

Asymptotic Expansions

The above example shows that a divergent series can still be useful for estimating a function value. But there are two other important points we need to make—about this example in particular, and asymptotic expansions in general.

Suppose you choose a particular partial sum—the best one in our example above is the 15th. Now instead of an infinite series you have a *finite* series that can be used to accurately approximate erfc 4. But that particular finite series makes an even better approximation for erfc 5, and it's spectacular for erfc 100. Every asymptotic expansion is built around a particular value or (as in this case) around infinity, and the approximation works better as you approach that value. Above we made two claims about Equation 2.8.1; here we are saying that the *second* of these two claims also holds for Equation 2.8.2.

If that were the end of the story we could just work with the finite series and forget the infinite divergent series that we started with. In practice we often do just that. But suppose you use more terms: say, the 40th partial sum. We saw above that for $x = 4$ that makes a lousy approximation. But for sufficiently high x-values it makes a very good approximation—better in fact than S_{15}. As x gets higher we can use more and more terms of the series.

In summary: when we claim that a series $\Sigma a_n(x)$ asymptotically approaches a function $f(x)$ around $x = x_0$ we are saying the following two things. (Note that x_0 could be a finite number or, as in the example above, infinity.)

1. Any given partial sum $S_n(x)$ can make an arbitrarily accurate approximation to $f(x)$ by allowing x to approach x_0. (We say below how we are defining accuracy.)
2. As x gets closer to x_0 you can use higher partial sums before they start diverging away from the correct value. This is a consequence of the first point, but we note it separately because it is useful for understanding asymptotic series.

The definition below does not just reexpress the points we made above in more formal language; it provides a specific requirement for how the partial sums must approach the function.

Definition: Asymptotic Expansion

Let S be the series $\sum_n a_n(x)$ and let S_n be the nth partial sum of S. We say S is an asymptotic expansion of a function $f(x)$ about the point $x = x_0$ if it obeys the following limit for any fixed, positive integer n.

$$\lim_{x \to x_0} \frac{f(x) - S_n}{a_n(x)} = 0$$

For a Taylor series that converges to a function we write $f(x) = S$. For a series that is divergent but asymptotically approaches $f(x)$ in the sense defined above, we write $f(x) \sim S$.

Deriving an Asymptotic Series

Equation 2.8.2 gives an asymptotic series for the complementary error function. Below we start the process of deriving that formula. In the problems you will continue this process and show why it gives us an asymptotic series.

Our strategy is use integration by parts to tackle the integral that defines the complementary error function.

$$\text{erfc } x = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2}\, dt$$

$$\begin{aligned} u &= \frac{1}{t} & dv &= te^{-t^2}\, dt \\ du &= -\frac{1}{t^2}\, dt & v &= -\frac{1}{2} e^{-t^2} \end{aligned}$$

$$\frac{2}{\sqrt{\pi}} \left[uv - \int v\, du \right] = \frac{2}{\sqrt{\pi}} \left[-\frac{1}{2t} e^{-t^2} - \frac{1}{2} \int \frac{1}{t^2} e^{-t^2}\, dt \right]_x^\infty$$

The uv term (before the integral) vanishes at $t = \infty$ so plugging in the limits of integration gets us here.

$$\text{erfc } x = \frac{e^{-x^2}}{\sqrt{\pi}\, x} - \frac{1}{\sqrt{\pi}} \int_x^\infty \frac{1}{t^2} e^{-t^2}\, dt \qquad (2.8.3)$$

Equation 2.8.3 is not an approximation; it is an exact rewriting of the complementary error function. The term before the integral represents the first term in the asymptotic series expansion for erfc x. The integral itself represents the remainder—that is, the difference between the actual function and the one-term series approximation.

In Problem 2.189 you will take this process to the next step, finding subsequent terms in the series and the integrals that represents their remainders. In Problem 2.190 you will use the remainder terms to explain the behavior of this asymptotic series.

2.8.2 Problems: Asymptotic Expansions

2.183 Prove that the asymptotic series for erfc x given in Equation 2.8.2 is divergent for any fixed value of x.

2.184 Equation 2.8.2 gives a series expansion for erfc x. Suppose you wish to use this series to approximate erfc 3.

(a) Evaluate terms of this series expansion at $x = 3$. You should see that the terms decrease in magnitude for a while and then increase. What term has the smallest magnitude?

(b) In Part (a) you found the n-value that minimizes the magnitude of the terms in this series. Make a graph of the *partial sums* of this series as n goes from 0 up to twice that value. Include on your graph a label showing the correct value of erfc 3. Describe the behavior you see.

(c) How many terms of this series do you need to obtain an approximation accurate to within 0.1%?

(d) Repeat Parts (a)–(c) for erfc 5. How is the behavior the same, and how is it different?

2.185 The "exponential integral" is defined as Ei $x = -\int_{-x}^{\infty}(e^{-t}/t)dt$. The following series converges to Ei x for all $x \neq 0$: Ei $x = \gamma + \ln|x| + \sum_{n=1}^{\infty} x^n/(nn!)$. Here γ is the "Euler–Mascheroni constant," roughly equal to 0.5772. You can also represent Ei x with the following asymptotic expansion, valid in the limit $x \to -\infty$.

$$\text{Ei } x \sim \frac{e^x}{x}\sum_{n=0}^{\infty}\frac{n!}{x^n}$$

You'll use both of these series to approximate Ei(−10).

(a) Prove that this asymptotic series diverges for any value of x.

(b) Calculate the value of Ei(−10) to at least 5 decimal places.

(c) Calculate the 40th partial sum of both series. Is either one a good approximation?

(d) Calculate the 30th partial sum of both series. Is either one a good approximation?

(e) Calculate the 3rd partial sum of both series. Is either one a good approximation?

(f) Plot the partial sums of both series up to $N = 50$ as a function of N (the maximum value of n used in the partial sum). Show on your plots the correct value of Ei(−10). Describe how each series behaves.

(g) Why is it useful to have a divergent, asymptotic expansion even though there is a convergent series that works for this function?

2.186 Suppose you use n terms of an infinite series to approximate a value X. The "remainder" is the difference between the actual value and your approximation: $R_n = |X - S_n|$.

(a) Draw a graph of R_n as a function of n for a convergent Taylor series. (Although the details vary from one convergent Taylor series to the next, the overall shape should be the same.)

(b) Draw a graph of R_n as a function of n for a divergent asymptotic series. (Same comment.)

2.187 Consider the function $f(x) = \int_x^{\infty} e^{-t^3}\,dt$.

(a) Use integration by parts to find the first two terms of an asymptotic series expansion for $f(x)$. Express the remainder as an integral.

(b) For what values of x does your two-term series best approximate $f(x)$? (Does it work best for values of x close to zero, values of x close to some other number, or values of x approaching ∞?) How can you tell?

(c) Use your two-term series to approximate $f(4)$, and compare it to the actual value of $f(4)$.

2.188 The "error function" is defined by the same integral as the complementary error function, but with different limits of integration.

$$\text{erf } x = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt$$

The behavior and use of the error function are discussed in Chapter 11.

Our purpose here is to approximate this function with two different series and compare their behavior. It can be shown that erf $x = 1-$ erfc x, so its asymptotic expansion is

$$\text{erf}\, x \sim 1 - \frac{e^{-x^2}}{\sqrt{\pi}x}\sum_{n=0}^{\infty}(-1)^n\frac{(2n)!}{n!(2x)^{2n}}$$

(a) Write the Maclaurin series for e^t. (You can derive it or look it up.) From there you should be able to easily generate the Maclaurin series for e^{-t^2} and from there the error function. You should not need a computer for this part, although you can use one to check your answer when you're done.

You now have two different series that can be used to estimate erf x. In the remainder of this problem you will compare these two series using a computer.

(b) Use the $n = 3$ partial sum of each series to estimate erf 1/2. Which estimate is more accurate?

(c) Use the $n = 3$ partial sum of each series to estimate erf 2. Which estimate is more accurate?

(d) On one plot, show erf x and the $n = 3$ partial sum of both series for $0 \le x \le 5$. Choose a vertical range that allows you to see when each series is and isn't a good approximation to the function, and estimate the ranges in which each one gives a good estimate.

(e) One one plot, show the partial sums of both series with $x = 2$ as a function of N, the highest n-value of the partial sums. Estimate the range of partial sums for which each series gives a good approximation to erf 2.

2.189 Equation 2.8.3 gives the first term in the series expansion of erfc x.

(a) Use integration by parts on that series to find the next term. Your final answer will be in the form erfc x = *<two terms plus an integral>*. *Hint*: You cannot integrate e^{-t^2} but you can integrate te^{-t^2}.

(b) After n integrations the series looks like $\sum_{n=0}^{\infty} a_n(x) + R_n$. Assume the remainder term is given by the following integral with some coefficient c_n.

$$R_n = c_n\int_x^{\infty}\frac{1}{t^{2n}}e^{-t^2}dt$$

Perform the next integration by parts to find the term a_{n+1} and the remainder R_{n+1}. You should verify that the pattern continues because R_{n+1} looks like R_n with n replaced by $n+1$ and a different coefficient.

(c) Do one more integration by parts to find a_{n+2}. (Don't worry about the remainder integral this time.) Simplify the ratio $|a_{n+2}/a_{n+1}|$.

(d) Given a fixed value of x, for which values of n will a_{n+2}/a_{n+1} be greater than 1, and for which will it be less than 1?

(e) Using your answer to Part (d), explain why Figure 2.13 looks like it does. In particular, explain how you can use that answer to predict the value on the horizontal axis at which the terms switch from converging toward a finite value to diverging away from it.

Your results in this problem demonstrate that for any fixed x the asymptotic series for erfc x should converge toward a finite value for a while, and then start diverging above a certain value of n. Problem 2.190 will continue this argument by arguing that the finite value the series converges to is in fact erfc x.

2.190 **Exploration: The Remainder of the Asymptotic Expansion for erfc** *[This problem depends on Problem 2.189.]*

At any particular value of n, the asymptotic series for erfc x looks like $\sum_{n=0}^{\infty} a_n(x) + R_n$, where R_n is the exact difference between the series expansion and the correct value of erfc x. In this problem you're going to show that as n increases this remainder term decreases for a while, indicating that the series is getting closer to erfc x, but that it starts to increase past a certain value of n. In fact you'll show that this is approximately the same value of n at which the terms a_n go from decreasing to increasing.

In Problem 2.189 you showed that if $R_n = c_n\int_x^{\infty} t^{-2n}e^{-t^2}dt$ then $R_{n+1} = -c_n[(2n+1)/2]\int_x^{\infty} t^{-(2n+2)}e^{-t^2}dt$. The coefficient in front of the R_{n+1} integral is bigger than the one in front of the R_n integral by a factor of $(2n+1)/2$. At the same time, however, the integrand decreased by a factor of t^2, and it's harder to say what effect that has on the entire integral. The key to figuring that out is to notice that the decaying exponential causes the integral to be dominated by values of t very close to x.

(a) Numerically calculate $\int_x^\infty t^{-2}e^{-t^2}dt$ and $\int_{x+.1}^\infty t^{-2}e^{-t^2}dt$ for values of x ranging from 2 to 20. Show that for sufficiently large x over 90% of the entire integral comes from $x < t < x + .1$. Estimate the lowest value of x for which is true.

(b) Numerically calculate the ratio of $\int_x^\infty t^{-2}e^{-t^2}dt$ to $\int_x^\infty t^{-4}e^{-t^2}dt$ for values of x ranging from 2 to 20. On the same plot, plot x^2.

(c) What does dividing the integrand by t^2 do to the value of the integral? Explain how your answer follows from the plots you made.

(d) Putting together your answers so far, write a simple approximation for R_{n+1}/R_n, valid for large x.

(e) Using your answer to Part (d), estimate the value of n at which the remainder term switches from decreasing to increasing as you increase n. Check your answer by verifying that it correctly predicts the appearance of Figure 2.13.

(f) Explain how the results you've derived in this problem lead to the two properties that we said define an asymptotic series. *Hint*: Remember that this integral represents the difference between the original function and the nth term of the asymptotic series!

2.9 Additional Problems

2.191 The first two terms in the Taylor series for a function $f(x)$ make up the linear approximation $f(x_0+\Delta x) \approx f(x_0) + f'(x_0)\Delta x$.

(a) Under what circumstances would this expression be exactly correct for all values of Δx?

(b) When the linear approximation is not exact the next order approximation is $f(x_0+\Delta x) \approx f(x_0) + f'(x_0)\Delta x + (1/2)f''(x_0)\Delta x^2$. Explain why this next term appears with a positive sign. In other words, why is the linear approximation generally too low when $f''(x_0) > 0$ and too high when $f''(x_0) < 0$?

For Problems 2.192–2.200 find the fourth-order Taylor series of the given function about the given point.

2.192 $1/(1+x)^3$ about $x=0$

2.193 $x^2 + e^x$ about $x=0$

2.194 $\tan\left(x^2\right)$ about $x=0$

2.195 $1/(\sin x + \cos x)$ about $x=0$

2.196 $e^{\ln x + 2}$ about $x=0$. (Think about this one for a moment before starting.)

2.197 $\ln\left(x^2\right)$ about $x=2$

2.198 e^{x+x^2} about $x=1$

2.199 $1/(1+\tan x)$ about $x=\pi/2$

2.200 $1/(1-x^2)$ about $x=-2$

For Problems 2.201–2.204 find the 15th-order Maclaurin series of the given function. *Hint*: There's an easy way and a hard way to do each of these. We recommend you find the easy way.

2.201 $\sin\left(x^3\right)$

2.202 $\sin\left(x^2\right)/x^2$

2.203 $\ln(5e^x)$

2.204 $x^3 e^{-x^2}$

For Problems 2.205–2.208 use a computer to calculate partial sums of the Taylor series for the function about the midpoint of the domain. On one plot, show the function in black and its partial sums in different colors. You should show enough partial sums to clearly see how they are changing as you add more terms, and your final partial sum should match the function well throughout the domain.

2.205 $\sin x,\ -5 \le x \le 5$

2.206 $e^{-x^2},\ -2 \le x \le 2$

2.207 $e^{-x^2},\ 0 \le x \le 2$

2.208 $\sin^3 x,\ -\pi/2 \le x \le \pi$

In Problems 2.209–2.213 you are given a non-linear differential equation that, like most non-linear differential equations, has no simple solution. For each one replace the right-hand side of the equation with a linear function that approximates it well under the specified assumptions, and solve the resulting approximate differential equation.

2.209 $d^2f/dx^2 = 1 - e^f$. Assume you know that $f(x)$ is going to stay close to 0.

2.210 $d^2x/dt^2 = -e^x$. Assume $x(t)$ stays close to $x=1$.

2.211 $d^2x/dt^2 = -\ln x$. Assume $x(t)$ stays close to $x=1$.

2.212 $dz/dt = 1 + \ln z$. Assume $z(t)$ is close to $z=2$ for the period of time you are interested in. Explain why this approximation can *not* be used out to arbitrarily late times.

2.213 $d^2x/dt^2 = -k\sinh x$. Assume x is the displacement from equilibrium of a mass on a non-ideal spring and that it is oscillating with a small amplitude. (If you don't know what sinh is see Appendix J.)

For Problems 2.214–2.224 show whether the given series converges or diverges.

2.214 $\sum_{n=1}^{\infty} n^{-5}$

2.215 $\sum_{n=1}^{\infty} n^5$

2.216 $\sum_{n=1}^{\infty} 1/(1+n^5)$

2.217 $\sum_{n=1}^{\infty} n/(1+n^5)$

2.218 $\sum_{n=1}^{\infty} 1 + n^{-5}$

2.219 $\sum_{n=1}^{\infty} (\tanh n)/n$ (see Appendix J for tanh)

2.220 $\sum_{n=1}^{\infty} n/(1+n^3)$

2.221 $\sum_{n=1}^{\infty} (\sin n)/(n!)$

2.222 $\sum_{n=2}^{\infty} 2!(n-2)!/(n!)$

2.223 $\sum_{n=1}^{\infty} e^{1/n}/n^2$

2.224 $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\cos(\pi n)}$

For Problems 2.225–2.232 determine the interval of convergence of the given power series. (In other words, for which x-values does this series converge?)

2.225 $\sum_{n=0}^{\infty} x^{2n}$

2.226 $\sum_{n=0}^{\infty} x^{2n}/(n!)$

2.227 $\sum_{n=0}^{\infty}(x+1)^n/(n!)$

2.228 $\sum_{n=0}^{\infty}(-1)^n(x+1)^n/(n!)$

2.229 $\sum_{n=0}^{\infty}\sin(\pi n/2)x^n/(n!)$

2.230 $\sum_{n=0}^{\infty}(x-1)^n/n^2$

2.231 $\sum_{n=0}^{\infty}(-1)^n(x-1)^{2n}/n$

2.232 $\sum_{n=0}^{\infty}e^{-n}(x-2)^n$

Problems 2.233–2.237 deal with finding bounds on the errors in series approximations. All of these problems use the formulas for those errors given in Appendix B.

2.233 Let $f(x)=e^x+e^{x^2}$. Use a first-order Maclaurin series to estimate $f(0.1)$. Use the Lagrange remainder to place an upper bound on the error of this approximation, and verify that the upper bound is correct.

2.234 Let $f(x)=e^x+e^{x^2}$. Use a 10th-order Maclaurin series to estimate $f(.8)$. Use the Lagrange remainder to place an upper bound on the error of this approximation, and verify that the upper bound is correct.

2.235 **(a)** Use a third-order Maclaurin expansion of $\sin x$ to estimate $\sin(1.5)$.

(b) Use a third-order Taylor series for $\sin x$ around $x=\pi/2$ to estimate $\sin(1.5)$.

(c) Which answer would you expect to be more accurate? Why?

(d) Use the Lagrange remainder to show that the error in your second approximation must be less than 1.1×10^{-6}.

2.236 Let $f(x)=e^{-x}$.

(a) Find the third-order Maclaurin series for $f(x)$ and use it to estimate $e^{-0.2}$.

(b) Use the rule for errors in alternating series to put an upper bound on the error in this estimate. Verify that your error is less than the upper bound.

(c) Use your series for e^{-x} to estimate $e^{0.2}$. Explain why you can't use the same technique to put an upper bound on this value. Instead, find an upper bound using the Lagrange remainder. Express your answer as a ratio of the possible error in your estimate to the correct value for $e^{0.2}$.

2.237 Let $f(x)=1/(1+x)$. Use a second-order Maclaurin series to estimate $f(0.1)$ and find the bounds on that estimate using each of the three methods described in Appendix B. Verify that the actual error is lower than all three bounds. Which technique gives you the strictest bound? Which one is easiest to find?

2.238 Does the series $\sum_{n=1}^{\infty}(\pi/3)^n\sin(n\pi/2)/n!$ converge or diverge? If it does converge, what (exactly!) does it converge to?

2.239 The differential equation $dx/dt=x+\sin x+\cos x$ is non-linear and has no simple solution.

(a) Use a Maclaurin series to approximate the right side of this equation with a linear function of x, valid when $x\approx0$.

(b) Solve this approximate differential equation with the initial condition $x(0)=0$.

(c) Do you expect your approximation to remain valid at late times? Explain, using the solution you found.

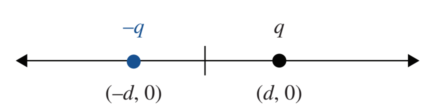
(d) Plot the approximate solution you found and the numerical solution to the original differential equation, from $t=0$ to $t=1$. Does the behavior match your prediction? Explain.

2.240 Use Maclaurin series to prove "Euler's Formula" which states that, for any real number x,

$$e^{ix}=\cos x+i\sin x$$

where i is an imaginary number, defined by the property $i^2=-1$.

2.241 The picture below shows an "electric dipole." Two equal and opposite charges sit on the x-axis.



The electric field on the x-axis due to the presence of these two charges is given by the equation

$$E=\frac{kq}{(x-d)^2}-\frac{kq}{(x+d)^2}\qquad(x>d)$$

This is very similar to the equation from the Motivating Exercise (Section 2.1), but in this case we are interested in points on the x-axis very far away from the crystal.

(a) Factor out kq/x^2 from the equation for E. What you are left with should only depend on the fraction d/x.

(b) If $x\gg d$ then d/x is small and you can expand the expression you just found

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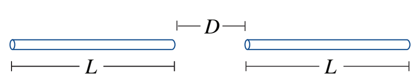
in a Maclaurin series for d/x. Find the first two non-vanishing terms of this expansion. (*Hint: You can make this much easier by using the binomial series.*)

(c) Using your Maclaurin series, argue that the electric field from a dipole drops off proportionally to $1/x^3$ at large distances.

2.242 Like π, e is an irrational number that has been calculated to large numbers of digits. (As of late 2010 the first trillion digits were known.) For this problem we'll let you get away with doing the first 10,000. Use the Maclaurin series for e^x with $x = 1$ to find successively better approximations of e. Keep adding terms up to and including the first term that is smaller than $10^{-10,000}$. Give as your final answer the 9997th through 10,000th digits. (Count the initial 2 as the first digit.)

2.243 Two rods, each of length L and charge per unit length λ, lie along the same line with a distance D between them. The electric force between these rods is

$$F = \frac{\lambda^2}{4\pi\epsilon_0} \ln\left(\frac{(L+D)^2}{D(2L+D)}\right)$$



(a) Rewrite this expression so L only appears in the combination L/D.

(b) Use a Maclaurin series in L/D to find an expression for the force valid when you pull the rods very far apart compared to their lengths. Keep only the first non-zero term.

(c) Rewrite $\lambda = Q/L$ where Q is the charge on each rod and simplify your answer. Explain why your answer makes physical sense in the limit $D \gg L$.

2.244 The "MIDI tuning standard" assigns a "midi note number" to each pitch: the note A440 is assigned $p = 69$, the A♯ a half-step above it is $p = 70$, and so on. This note number p is related to the frequency f of the sound (in Hz) by $p = 69 + (12/\ln 2)\ln(f/440)$. If a wind instrument plays A440, however, the actual frequency produced is $440\sqrt{T/T_0}$, where T is the temperature at which the instrument is being played and T_0 is the temperature at which the instrument was tuned.

(a) Write a first-order Taylor series for $p(T)$ about $T = T_0$.

(b) The formulas above assume temperature is measured in Kelvin. Suppose you tuned your flute at a comfortable room temperature of 290 K. Roughly how much would the pitch rise per degree of increase in T? Roughly how much would the temperature have to rise in order to increase your pitch from A to A♯? Use your Taylor series to answer both questions.

2.245 The relationship between the volume, pressure, and temperature of a gas can be written in the form $PV = nRT\left[1 + B(T)(n/V) + C(T)(n/V)^2 + \ldots\right]$ where the coefficients B and C are called the second and third "virial coefficients." (You may be familiar with the "ideal gas law" that results from $B(T) = C(T) = 0$.) Calculate the second and third virial coefficients for the Van der Waals equation of state $(P + an^2/V^2)(V - nb) = nRT$. *Hint*: Start by writing the equation in the form $PV = nRTf(n/V, T)$ and then expand f in a Maclaurin series in n/V.

2.246 **Exploration: Simple Harmonic Oscillations.** Oscillations occur in a wide variety of situations, ranging from atoms vibrating around their positions in a crystal to wrecking balls swinging back and forth to giant waves moving up and down on the surface of stars. In all of these situations there is an object (the atom, the wrecking ball, the fluid on the surface of the star, …) that experiences a force as a function of its position. In general these forces can be very complicated and can vary widely from one situation to another. In many situations, however, they can be well approximated by a simple equation.

(a) Using Newton's second law, write a differential equation for the position $x(t)$ of an object experiencing a force $F(x)$.

(b) Any oscillator moves back and forth across some equilibrium point. For simplicity you can always define that equilibrium point to be at $x = 0$, in which case it is natural to expand $F(x)$ in a Maclaurin series. Write a second-order Maclaurin series for $F(x)$ and plug this into the equation you wrote down in Part (a). The result should be a differential equation with d^2x/dt^2 on the left and three terms on the right.

(c) The force on an oscillator is always zero at the equilibrium point. Use that fact to eliminate one of the terms from your differential equation.

(d) Explain why for small amplitude oscillations one of the remaining terms will be much larger than the other one. Use that fact to eliminate one more term from your equation.

(e) In order for an object to oscillate it must have a "restoring force," which means that if $x > 0$ the force should be negative and if $x < 0$ the force should be positive. Use that fact to determine the sign of the constant in the one remaining term on the right-hand side of your differential equation.

(f) A simple harmonic oscillator is defined by the differential equation

$$\frac{d^2x}{dt^2} = -\omega^2 x$$

where ω is any (real) constant. Putting together everything you've done so far in this problem, explain why almost any oscillator can be well approximated by a simple harmonic oscillator for small amplitude oscillations. Write an equation for ω that depends on $F(x)$ and m.

(g) How could you have an oscillator that could *not* be well approximated by a simple harmonic oscillator for small amplitudes? In other words, write a force $F(x)$ that would describe an oscillator that would be an exception to the argument you just presented. Note that the answer is *not* to add damping because then the force F wouldn't just be a function of x.

2.247 Exploration: The Method of Power Series. We have seen that Taylor series can be used to help solve difficult differential equations by simplifying complicated functions. Taylor series can also be used in a more direct way to solve differential equations. In the "Method of Power Series" you assume a solution in the form of a power series and then solve for the coefficients. (This technique is explored further in Chapter 12.) In this problem you will solve $d^2y/dx^2 = -y$ with the conditions $y(0) = 0$ and $y'(0) = 1$ by plugging in the "guess" $y = c_0 + c_1x + c_2x^2 + c_3x^3 + \ldots$.

(a) Plug the initial condition $y(0) = 0$ into the "guess" and solve for c_0.

(b) Take the derivative of both sides of the guess and then use the initial condition $y'(0) = 1$ to solve for c_1.

(c) Now plug the guess into the differential equation you are solving. The resulting equation will set two different power series equal to each other.

(d) Your answer to Part (c) set two power series equal to each other. For two power series to be equal their constant terms must be equal. Write the resulting equation and solve it for c_2.

(e) Your answer to Part (c) set two power series equal to each other. For two power series to be equal their coefficients of x must be equal. Write the resulting equation and solve it for c_3.

(f) Following a similar process, solve for all coefficients up to c_7. Then write the solution to this differential equation as a Maclaurin series up to the seventh power.

(g) What function has that particular Maclaurin series? Does that function solve the given differential equation and initial conditions?

You probably knew the answer to that problem before you started. Next you will use the same technique to solve Airy's equation, which is used in optics to model the intensity of a rainbow and in quantum mechanics to represent a particle confined within a triangular potential well.

(h) Solve $\dfrac{d^2y}{dx^2} = xy$ with conditions $y(0) = 1$ and $y'(0) = 0$. Your solution will be in the form of a Maclaurin series up to the sixth power.

CHAPTER 3

Complex Numbers (Online)

3.6 Special Application: Electric Circuits

Figure 3.5 shows a circuit diagram with standard electrical symbols for a resistor, capacitor, and inductor. A more complicated circuit might have hundreds of these elements, each with a resistance R, a capacitance C, or an inductance L.

However simple or complicated the circuit, you provide a "stimulus" (a voltage $V(t)$ that might come for instance from a battery or outlet) and the circuit "response" is the resulting current $I(t)$. When you analyze the circuit, you determine what response it will have to a given stimulus.

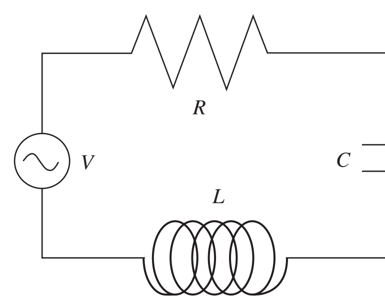


FIGURE 3.5 A simple RLC circuit.

In this section we will be looking at the response to a sinusoidal stimulus $V(t) = V_0 \sin(\omega t)$. This is a particularly important case because it models the voltage that comes out of a typical household outlet. Furthermore, more complicated functions can be built up as sums of sine waves (Problems 3.127–3.128), so a solution for a sinusoidal voltage turns out to be generally useful.

Before we go through the math, we're going to jump to the end and present most of the answer.

Response of an RLC circuit to a Sinusoidal Voltage

Given a voltage $V(t) = V_0 \sin(\omega t)$ the resulting current will be $I(t) = I_0 \sin(\omega t - \phi)$. That is, the current will oscillate with the same frequency ω as the voltage. However, the current will lag behind the voltage by a phase ϕ.

The current can also be expressed as $I(t) = A\sin(\omega t) + B\cos(\omega t)$ which may look like a more familiar solution to a differential equation. But the form $I(t) = I_0 \sin(\omega t - \phi)$ is mathematically equivalent (Problem 3.126) and lends itself to more direct physical interpretation. It is important to note that the "phase lag"[4] ϕ is measured in radians, not in seconds. A phase lag of zero means that the two oscillations are perfectly in sync; a phase lag of $\pi/2$ means that the current reaches its peak just as the voltage reaches zero.

In Figure 3.6, the period is 4 s (so $\omega = 2\pi/4$). The current lags the voltage by 1/3 of a second, but it is more useful to view the lag as $\pi/6$ radians or as 1/12 of a cycle.

But how do we compute the amplitude I_0 and phase lag ϕ of the response for any given circuit layout? This is where complex numbers come in.

[4]also called the "phase shift" or "phase difference".

Complex Impedance

The RLC circuit in Figure 3.5 comprises each of the basic circuit elements. The voltage drop[5] across a capacitor is proportional to the built-up charge ($V = Q/C$); the voltage drop across a resistor is proportional to the current ($V = IR$ or $V = Q'(t)R$); the voltage drop across an inductor is proportional to dI/dt ($V = LI'(t)$ or $V = LQ''(t)$). If we set the total voltage drop equal to the stimulus voltage $V(t)$ we get a differential equation for the charge Q. But it is preferable to work with the current I (which is easier to directly measure) so we take the derivative of both sides.

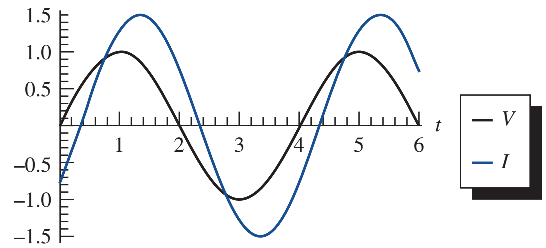


FIGURE 3.6 In this figure current lags voltage by $\pi/6$ radians. (A positive phase lag means current is to the right of voltage, so the voltage peaks first.)

$$LI''(t) + RI'(t) + \frac{1}{C}I = V'(t) \tag{3.6.1}$$

If we let $V(t) = V_0 \sin(\omega t)$ and solve this differential equation we can find everything we need to know about this particular circuit. But more complicated circuits have more complicated differential equations. So here is the easier approach. Because $V(t)$ is a sine, we can view it as the imaginary part of a complex exponential function $\mathbf{V}$. So Equation 3.6.1 becomes the imaginary part of the complex equation

$$L\mathbf{I}''(t) + R\mathbf{I}'(t) + \frac{1}{C}\mathbf{I} = \mathbf{V}'(t)$$

where $\mathbf{V} = V_0 e^{i\omega t}$. Since $V = Im(\mathbf{V})$ we know that the physical current I will be given by $I = Im(\mathbf{I})$.[6]

This equation is much easier to solve. We begin by guessing a solution of the form $\mathbf{I} = \mathbf{I_0} e^{i\omega t}$. (As we said above we are assuming that the current oscillates with the same frequency as the voltage. Mathematically or physically it's hard to imagine any other behavior, but as always the guess will prove itself by working.) When we plug in this guess and do a bit of algebra we end up here.

$$V_0 = \left(R + i\omega L - \frac{i}{\omega C}\right)\mathbf{I_0} \tag{3.6.2}$$

The quantity in parentheses is called the "impedance" Z, a complex number that represents the circuit layout of resistors, capacitors, and inductors. So the behavior of the circuit can be captured in a very simple-looking equation reminiscent of Ohm's law.

$$V_0 = \mathbf{I_0} Z \tag{3.6.3}$$

Because V_0 is real and Z is complex, we know that $\mathbf{I_0}$ must also be complex.

Equations 3.6.1 and 3.6.3 represent two very different approaches to analyzing a circuit.

Without complex numbers every circuit element looks mathematically different, as we saw before. A different arrangement of elements leads to a different differential equation, generally second order and not easy to solve.

[5]We use V for voltage; electrical engineers often use E.

[6]We could have considered a cosine for the potential and taken real parts instead of imaginary parts, but it's more common to use the imaginary part.

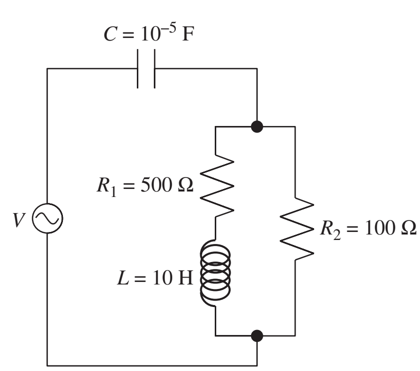
With complex numbers we approach RLC circuits very similarly to how we approach circuits that have only resistors: find the total impedance of all the circuit elements and write $V_0 = \mathbf{I_0} Z$. The resulting (complex) $\mathbf{I_0}$ tells us the (real) amplitude and phase lag of the current. Here are the steps we've left out of that brief description.

- How do you find the impedance? You can read the impedances of the individual elements directly from Equation 3.6.2 as R, $i\omega L$, and $-i/(\omega C)$. We combine elements just as we would combine resistors. Two elements "in series" (the current goes through one and then the other) add their impedances, so $Z_{\text{series}} = Z_1 + Z_2$. Two elements "in parallel" (the current goes through each element separately) add as $Z_{\text{parallel}} = \dfrac{1}{(1/Z_1) + (1/Z_2)}$.
- How do we find the amplitude of the current? Remember that $\mathbf{I}(t) = \mathbf{I_0} e^{i\omega t}$ and as we saw in Section 3.5 the amplitude of that function is the modulus of $\mathbf{I_0}$.
- Finally, what about the phase lag? Recall that the voltage is the imaginary part of $V_0 e^{i\omega t}$ and the current is the imaginary part of $\mathbf{I_0} e^{i\omega t}$. The exponentials are identical in the two expressions, so the phase difference between $\mathbf{V}$ and $\mathbf{I}$ is the phase difference between V_0 and $\mathbf{I_0}$. But $\mathbf{I_0} = V_0/Z$. Remembering how numbers divide on the complex plane, we conclude the phase lag is just the phase of Z.

The example below shows how we can predict the behavior of a circuit once all these pieces are in place.

EXAMPLE **A Complicated Circuit**

Question: If the circuit shown below is driven by a voltage $V = 3\sin(50t)$ find the amplitude of the resulting steady-state current and find the phase lag between the voltage and the current.



Solution:
We can find the equivalent impedance of this circuit by considering it to be made of two elements in series, the second of which is made up of two elements in parallel, one of which is made of two elements in series. That sounds messy, but using the rules for series and parallel circuit elements the total impedance is simply given by

$$Z = -\frac{i}{\omega C} + \frac{1}{\frac{1}{R_1 + i\omega L} + \frac{1}{R_2}}$$

Plugging in numbers gives $Z = 400 - 1800i$. Because $\mathbf{I} = \mathbf{V}/Z$ the amplitude of the current is given by $|\mathbf{V}|/|Z| = 3/1844 = 1.6 \times 10^{-3}$. The phase lag is the phase of Z, which in this case is $\tan^{-1}(-1800/400) = -1.4$ rad, or about $-80°$. That means the current leads the voltage by almost a quarter cycle, so when the voltage is at its peak the current has decreased nearly to zero.

And so what? If this circuit is part of an old-fashioned radio tuner, that might be perfectly fine. But if the source of this circuit is the power company, that $-80°$ phase lag is a disaster; it means you are getting almost none of the power you're paying for. See Problem 3.125.

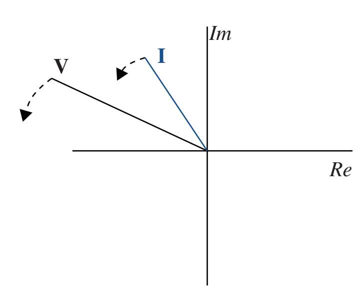
Note that the impedance of a resistor is just its resistance, but the impedance of an inductor or capacitor depends on the frequency of the voltage source. Physically this corresponds to the fact that a capacitor tends to impede low frequencies, an inductor impedes high frequencies, and a resistor is an equal opportunity impediment. Mathematically it reminds us that we are computing the response of a given circuit to a specific sinusoidal stimulus. A more complicated stimulus can be represented as a combination of different-frequency sine waves (a "Fourier series"); see Problems 3.127 and 3.128.

Stepping Back

We have arrived at a very general description of the "response" (the current) of a circuit to a sinusoidal "stimulus" (voltage source). For a circuit with voltage source $V = V_0 \sin(\omega t)$, the current will be $I = I_0 \sin(\omega t - \phi)$. To find the amplitude I_0 and the phase lag ϕ of the circuit response:

- Assign each circuit element an impedance of R, $i\omega L$, or $-i/(\omega C)$. Impedance represents how much a given element will obstruct the response of a circuit, but as you can see this depends on the frequency. Inductors tend to block high frequencies, capacitors block low frequencies, and resistors are equal opportunity obstructions.
- Add impedances in parallel or series using the same rules used for resistors to find the total impedance Z of the circuit.
- The amplitude of the oscillating current is given by $I_0 = V_0/|Z|$.
- The phase lag ϕ between voltage and current is the phase of Z.

The picture below shows the stimulus and response of a circuit with a phase lag of $\pi/6$ or thereabouts. Don't read anything into the relative lengths of the two lines, because $\mathbf{V}$ and $\mathbf{I}$ are measured in different units. What is important is that the two quantities will rotate around the complex plane with the same frequency ω and therefore keep the same phase lag between them. The real voltage and current are the imaginary parts of these complex quantities, so they will also oscillate with a constant phase lag between them.



When we draw $\mathbf{V}$ and $\mathbf{I}$ as points in the complex plane this way, those points are referred to as "phasors" because they are like vectors that show the relative phases of the two oscillations.

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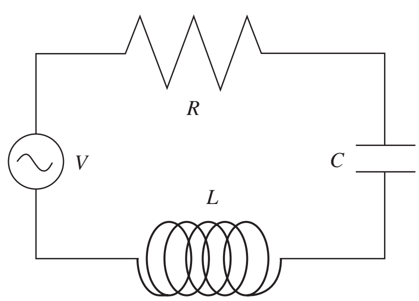
Frequently $e^{i\omega t}$ is left out of the phasor, so the phasors for **I** and **V** show their *initial* positions on the complex plane. It's understood that the actual, time-dependent quantities rotate in circles in the complex plane.

Note that our entire analysis is based on a *particular* solution to the differential equation. There is also a *complementary* solution that contains arbitrary constants and depends on initial conditions. In most cases, however, the complementary solution is "transient"—an exponential that quickly decays—leaving the particular solution as the "steady-state" or long-term solution.

3.6.1 Problems: Electric Circuits

3.119 Walk-Through: Analyzing a Circuit. The electric circuit shown below has voltage $V = 3\sin(500t)$, resistance 2000, capacitance 10^{-6}, and inductance 15 (all measured in SI units).



(a) Find the equivalent impedance of the resistor, capacitor, and inductor.

(b) Find the complex current **I**.

(c) Find the modulus of **I** to get the amplitude of the oscillating current.

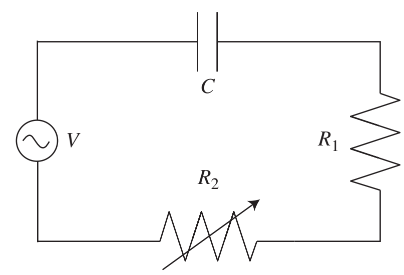
(d) Find the phase lag between the oscillating voltage and current by taking the phase of the impedance.

(e) Sketch the (real) current as a function of time. Include numbers on your plot that reflect the correct amplitude and frequency of the oscillation, and be sure to start at $t = 0$ with the correct phase. (Recall that V is a sine function, so the phase lag between V and I is minus the phase of I.)

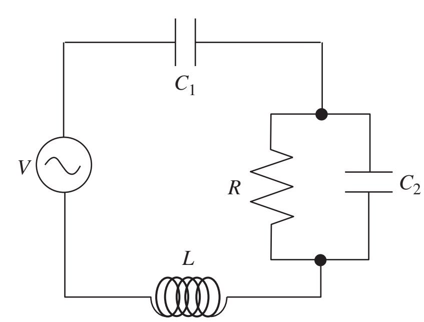
(f) How would your answers above change if the voltage was $V = 3\cos(500t)$?

In Problems 3.120–3.122 find the impedance of the indicated circuit and use that to find the amplitude and phase lag of the resulting current.

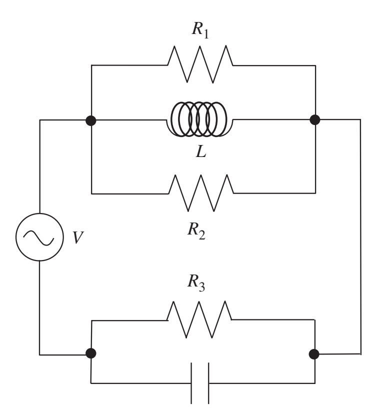
3.120 $V = 2\sin t$, $R_1 = 50$, $R_2 = 200$, $C = 10^{-6}$



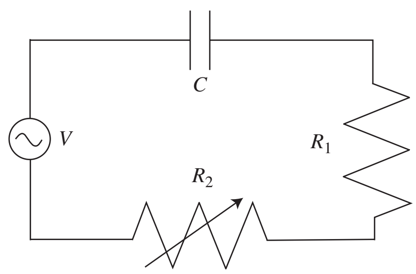
3.121 $V = 500\cos(400t)$, $R = 5000$, $C_1 = 3 \times 10^{-7}$, $C_2 = 10^{-6}$, $L = 25$



3.122 $V = \sin(t/10)$, $R_1 = 50$, $R_2 = 100$, $R_3 = 200$, $C = 10^{-7}$, $L = 5$.



3.123 In the circuit shown below $R_1 = 200\Omega$ and $C = 4 \times 10^{-8}F$. The resistor R_2 is a "rheostat," a resistor that can be tuned to different resistance values. The voltage oscillates at frequency $\omega = 120\pi$ sec^{-1}.

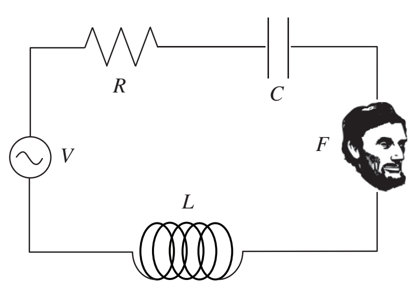


(a) How much does the current lag the voltage if $R_2 = 0$?

(b) What happens to the phase lag as $R_2 \to \infty$?

(c) What value of R_2 would you choose to make the current lead the voltage by $\pi/4$ (or, equivalently, lag the voltage by $7\pi/4$)?

3.124 Suppose someone were to invent a device called a "Feldor" whose voltage drop was proportional to $I''(t)$. The circuit shown below would obey the differential equation $FI'''(t) + LI''(t) + RI'(t) + (1/C)I = V'(t)$, where F is the "Feldance" of the Feldor. Find the impedance of a Feldor.



3.125 **The Power Factor.** The quantity VI (voltage times current) measures the power being delivered by a power source. When VI is negative power is actually being delivered *to* the source. If positive power alternates perfectly with negative power, the total "real power" delivered from the source to the load is zero.

(a) Consider first a circuit with only a power source and resistors: no capacitors or inductors.

  i. What is the phase lag of such a circuit?

  ii. For what fraction of the time is the power positive?

(b) Now consider a circuit with a phase lag of 180°. For what fraction of the time is the power positive?

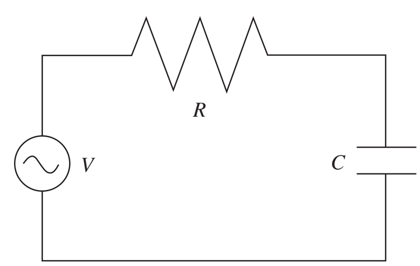
(c) Now consider a circuit with a phase lag of 90°. For what fraction of the time is the power positive?

(d) The "power factor" of a circuit is a unitless quantity defined as $\cos\phi$ where ϕ is the phase lag. Based on your answers above, describe the significance of the power factor. Why does the power company want this number to be as close to 1 as possible?

(e) In a circuit with a low power factor the energy is stored in the circuit and then fed back into the source. Where in the circuit (or "in what form") is the energy stored? (There are two important answers to this question. Discuss them both.)

3.126 Show that the differential equation solution $I(t) = A\sin(\omega t) + B\cos(\omega t)$ (where A and B are arbitrary constants) is equivalent to the solution $I(t) = I_0 \sin(\omega t - \phi)$ by finding I_0 and ϕ in terms of A and B. You can do this by looking up some trig identities, or you can use complex exponentials.

3.127 Consider the circuit shown below.



If the voltage source produces a voltage like $V = V_0 \sin(\omega t)$ then you know by now how to solve for the resulting current. In this problem, however, you will consider a voltage source $V = V_1 \sin(\omega_1 t) + V_2 \sin(\omega_2 t)$, and you'll solve for the current using the principle of linear superposition.

(a) Find the complex impedance of this circuit. (The answer will depend on ω, the frequency of the voltage source. Leave ω as a variable in this part; you'll fill in specific frequencies in the next parts.)

(b) Using the impedance you found in Part (a), find the complex current **I** that would result from a voltage source $V = V_1 \sin(\omega_1 t)$.

(c) Using the impedance you found in Part (a), find the complex current **I** that would result from the voltage source $V = V_2 \sin(\omega_2 t)$.

(d) Write the differential equation satisfied by **I** and show that the sum of the two currents you found is a solution to this differential equation.

(e) Write your solution $\mathbf{I}(t)$ for $R = 10^5\Omega$, $C = 10^{-6}$ F, $V_1 = 3$ V, $V_2 = 5$ V, $\omega_1 = 2\text{ s}^{-1}$, and $\omega_2 = 3\text{ s}^{-1}$. Plot $V(t)$ and the imaginary part of $\mathbf{I}(t)$ on the same plot. Include a range of times sufficient to see the behavior of the functions.

3.128 **Exploration: Other Driving Functions** *[This problem depends on Problem 3.127.]*. The circuit from Problem 3.127, with the impedance you calculated in Part 3.127(a), is driven by a different voltage source. This source produces a "square wave": $V = 1$ for $0 \le t < 1$, then $V = -1$ for $1 \le t < 2$, and this pattern repeats indefinitely with period 2. Such a function can be represented as a sum of an *infinite number* of sine waves, a "Fourier sine series." Finding the coefficients is a topic for another chapter, so here we just give them to you:

$$V(t) = \sum_{n=1}^{\infty} \frac{4}{n\pi}\sin(n\pi t) \text{ (odd } n \text{ only)}$$
$$= \frac{4}{\pi}\sin(\pi t) + \frac{4}{3\pi}\sin(3\pi t) + \frac{4}{5\pi}\sin(5\pi t) + \ldots$$

(As in Problem 3.127, just leave the resistance and capacitance as R and C until we tell you to put in numbers. Leaving them as letters for now will keep your equations more readable.)

(a) Plot the driving function $V(t)$ from $t = 0$ to $t = 6$. On the same plot show the first term of the series expansion: $(4/\pi)\sin(\pi t)$.

(b) Make two more plots, each showing the function $V(t)$ and a partial sum of its series expansion. On the first one put the sum through $n = 5$. On the second one put the sum through $n = 101$. Describe what happens to the partial sum as you include more terms.

(c) Write the complex solution $\mathbf{I}(t)$ that you get if you replace the voltage $V(t)$ with the first term in its series expansion (the $n = 1$ part of the series).

(d) Write the complex solution $\mathbf{I}(t)$ that you get if you replace the voltage $V(t)$ with the sum of the first two non-zero terms in its series expansion.

(e) Write the complex solution $\mathbf{I}(t)$ that you get if you replace the voltage $V(t)$ with the $\sin(n\pi t)$ term *(not the partial sum up to that term)* of its series expansion.

(f) Write an infinite series for the solution $\mathbf{I}(t)$ that you get using the complete infinite series expansion for $V(t)$.

(g) Now let $R = 10^5\Omega$ and $C = 10^{-6}$ F. Plot the partial sums of $Im(\mathbf{I}(t))$ for $n = 1$, $n = 11$, $n = 21$, $n = 31$, and $n = 101$, five plots in all. (Remember to plot partial sums, not individual terms.) Each of your plots should go from $t = 0$ to $t = 4$.

(h) Describe the behavior of the $n = 101$ plot. What does the current do at $t = 0$? What does it do shortly after that? What does it do at $t = 1$, and shortly after that? Describe what the capacitor is doing physically to produce the behavior you see. (If you skipped the computer part you can still do this part by *predicting* the behavior of $I(t)$ based on what you would expect the capacitor to do.)

3.7 Additional Problems

For Problems 3.129–3.137 find the real part, the imaginary part, the modulus, and the phase of the given number.

3.129 $p/i + (p+i)/(5-i)$ where p is a real number

3.130 $e^{3i}/(2+i)$

3.131 $(1-e^{2i})^{40}$

3.132 $3\cos(\pi/6) - 3i\sin(\pi/6)$

3.133 $3\sin(\pi/6) - 3i\cos(\pi/6)$

3.134 $3\cos(\pi/6) - 2i\sin(\pi/6)$

3.135 $\sqrt{e^{pi}}$ where p is a real number

3.136 i^i

3.137 $\ln(2-2i)$

3.138 Let $w = z^i$.

- **(a)** Write an expression for $\ln w$ as a function of $|z|$ and ϕ_z. Simplify as much as possible.
- **(b)** Exponentiate your expression for $\ln w$ to get an expression for w as a function of $|z|$ and ϕ_z.
- **(c)** Write $|w|$ and ϕ_w as functions of $|z|$ and ϕ_z, simplifying your answers as much as possible.

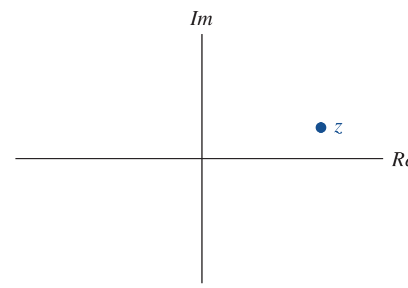
3.139

- **(a)** Write $\sin x$ in terms of complex exponentials. *Hint*: You can start by writing e^{ix} and e^{-ix} in terms of trig functions.
- **(b)** Write $\cos x$ in terms of complex exponentials.
- **(c)** Take the derivative of your complex exponential formula for $\sin x$. Explain why your result was predictable.
- **(d)** The "hyperbolic trig functions" are defined as $\sinh x = (1/2)(e^x - e^{-x})$, $\cosh x = (1/2)(e^x + e^{-x})$. Write expressions for $\sinh(ix)$ and $\cosh(ix)$ in terms of real trig functions.
- **(e)** Write expressions for $\sin(ix)$ and $\cos(ix)$ in terms of real hyperbolic trig functions.

3.140 Find the two solutions to the quadratic equation $4x^2 + 8x + 5 = 0$.

3.141 Solve for the real quantities r and s: $s^2 - 2s^2 i - 2r + ri = 9 - 18i$.

3.142 The figure shows a point z on the complex plane. Copy the figure. Then add the following points, and label them.

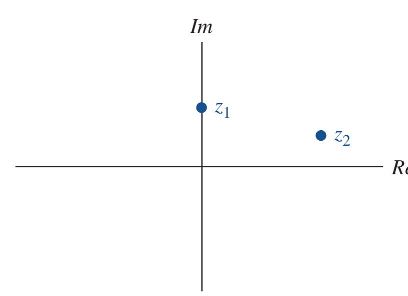


- **(a)** z^* (the complex conjugate of z)
- **(b)** $3z$
- **(c)** $z \times i$
- **(d)** $z \times 2e^{(\pi/6)i}$

3.143 Copy the drawing from Problem 3.142.

- **(a)** Draw all the points in the complex plane that have the same magnitude as the given point z. What shape does your drawing represent?
- **(b)** Draw all the points in the complex plane that have the same phase as the given point z. What shape does your drawing represent?

3.144 The drawing shows two complex numbers z_1 and z_2 on the complex plane. $|z_1| = 1$ and $|z_2|$ is unspecified (but greater than 1).



- **(a)** Is z_1 a real number (such as 5), an imaginary number (such as $3i$), or neither (a number $a + bi$ where $a \neq 0$ and $b \neq 0$)?
- **(b)** Is z_2 a real number, an imaginary number, or neither?

(c) Answer in words: if I take any complex number in the world (not necessarily z_2) and multiply it by z_1, what happens to its magnitude? What happens to its angle? Visually, what happens to the point?

(d) Copy the drawing into your homework. Then add to the drawing the labeled point $z_1 - z_2$.

(e) Add to your drawing the labeled point $z_1 z_2$.

3.145 If you specify a complex number with a free (real) variable, the result is generally a curve in the complex plane. For instance, the number $z = 3 + ki$ for $-2 \le k \le 2$ describes a vertical line segment four units long. Draw the curve described by each of the following functions. (Remember that in all cases k is a real number!)

(a) $z = k - i$ for $0 \le k \le 5$

(b) $z = 3k + ki$ for $0 \le k \le 5$

(c) $z = k + k^2 i$ for $-3 \le k \le 3$

(d) $z = 2e^{ik}$ for $0 \le k \le \pi/2$

(e) $z = ke^{i\pi/6}$ for $0 \le k \le 2$

(f) $z = ke^{ik}$ for $0 \le k \le \pi/2$

3.146 Prove that any complex number z and its complex conjugate z^* are the same distance from the origin on the complex plane.

3.147 You solve for the motion of a mass on a damped spring and one of the solutions you get is $x(t) = 3e^{(2-3i)t}$. How can you know without doing any more calculations that you made a mistake in deriving this solution?

3.148 You find the following particular solution describing the motion of an oscillator: $x(t) = (3 - 2i)e^{(-4+5i)t}$.

(a) What is the initial amplitude of the oscillations?

(b) What is the period?

(c) How long will it take before the amplitude drops to 1% of its initial value?

(d) Write a complex function $y(t)$ for an oscillator that is behaving exactly like the first one, but with oscillations $\pi/2$ behind it in phase.

3.149 A light beam moving in the z-direction is often represented by specifying the vector form of the electric field that forms part of the light beam.

$$\vec{E} = e^{i(kz-\omega t)}\left(E_{0x}e^{i\phi_x}\hat{i} + E_{0y}e^{i\phi_y}\hat{j}\right)$$

(Light waves are "transverse," meaning a light wave moving in the z-direction has no electric field component in the z-direction.) The electric field is actually the real or imaginary part of this expression, but for most purposes people work directly with the complex form.

(a) What is the wavelength of the light beam?

(b) What is the period of the light beam?

(c) Suppose $k = 2\text{m}^{-1}$, $\omega = 500\text{s}^{-1}$, $E_{0x} = 100$ N/C, $E_{0y} = 300$ N/C, $\phi_x = \pi$, $\phi_y = \pi/6$. How long will it be from the time the x-component peaks to the time the y-component peaks?

In Problems 3.150–3.153, find the general real-valued solution to the given differential equation.

3.150 $d^2x/dt^2 - 2(dx/dt) + 10x = 0$

3.151 $5(d^2y/dx^2) + 4(dy/dx) + 8y = 0$

3.152 $2(d^2r/dt^2) - 10(dr/dt) + 13r = 0$

3.153 $4(d^2x/dt^2) + 10(dx/dt) + 7x = 0$

3.154 The differential equation $d^2f/dt^2 + df/dt = \sin f$ is non-linear and has no simple solution.

(a) Assume $f(t)$ stays close to π. Replace the right side of the differential equation with the linear terms of its Taylor series about $f = \pi$. Your answer should be a linear differential equation for $f(t)$.

(b) Solve the resulting approximate differential equation.

(c) This approximation will only work at late times if $f(t)$ stays close to π. Based on your solution, is it valid to assume that if f starts out close to π with a small derivative, it will stay close to π at late times? Explain.

3.155 Evaluate $\int e^x \sin(ix)dx$. *Hint*: Begin by expressing $\sin(ix)$ as a sum of exponentials. See Section 3.4, Problem 3.90 if you're stuck.

3.156 An oscillating system obeys the differential equation $x''(t) + ax'(t) + x = 0$. Solve this equation for $a = 1$ and $a = 3$. In each case find a real-valued, general solution and sketch the basic shape of that solution $x(t)$.

3.157 An oscillator obeys the equation $x''(t) + ax'(t) + 2x = 0$.

(a) Find the general solution to this equation. Your answer will include the unknown constant a.

(b) For what values of a will the system oscillate?

(c) What will the system do if a is not in the range you just found?

3.158 A damped pendulum obeys the differential equation $\theta''(t) + 3\theta'(t) + 3\sin(\theta) = 0$. Find a linear differential equation that approximates this well for small amplitudes and find the general solution to that linear differential equation. Will the pendulum oscillate or simply decay?

3.159 Find the general real-valued solution to the equation $x''(t) + 2x'(t) + 5x = 3\sin(2t)$. Express your final answer as a real-valued function with two arbitrary constants. *Hint*: You will need to find a complementary solution and a particular solution.

CHAPTER 4

Partial Derivatives (Online)

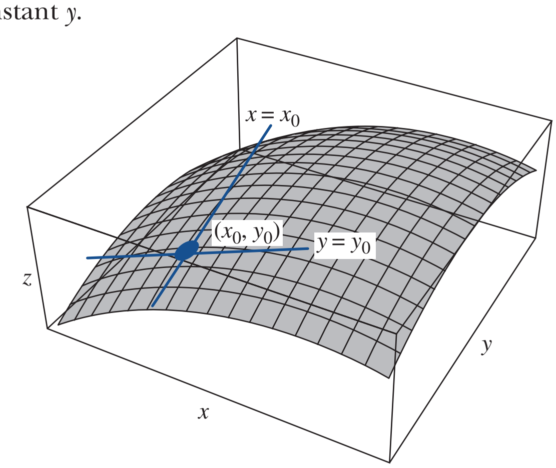
4.7 Tangent Plane Approximations and Power Series

It is often helpful to use a linear approximation to replace a complicated function $f(x)$ with a linear function that approximates f well when x is within a certain domain. If more accuracy is needed Taylor series can give higher order polynomial approximations. Such approximations were the main focus of Chapter 2.

In this section we apply a similar technique to multivariate functions, finding first a linear approximation (a plane), and then extending it to higher order terms.

4.7.1 Discovery Exercise: Tangent Plane Approximation

The drawing shows a function $z = f(x, y)$. Our goal is to find a plane that will approximate this function near the point (x_0, y_0, z_0): a tangent plane to the surface. The drawing does not show the tangent plane, but it does show two tangent lines at that point, one with a constant x and one with a constant y.



1. For a given function $f(x, y)$, how would we find the slope of the line labeled $y = y_0$? (Remember that this is the slope of the function in the x-direction, holding y constant.)
2. How would we find the slope of the line labeled $x = x_0$?
3. Recall that we are looking for a plane that we can use to approximate f. The equation for a plane can be written in the form $z = a(x - x_0) + b(y - y_0) + c$. Use this equation to answer the following questions:
   (a) At the point (x_0, y_0), what is the value of z?
   (b) What is the slope of z at that point as you move in the x-direction?
   (c) What is the slope of z at that point in the y-direction?

4. Find the values of a, b, and c for which the plane $z(x, y)$ has the same value, slope in the x-direction, and slope in the y-direction as $f(x, y)$ at the point (x_0, y_0).
See Check Yourself #24 in Appendix L
5. Once we have made the proper choice, will our plane also match the slopes of the original function in all *other* directions at that point? How do you know?

4.7.2 Explanation: Tangent Plane Approximations and Power Series

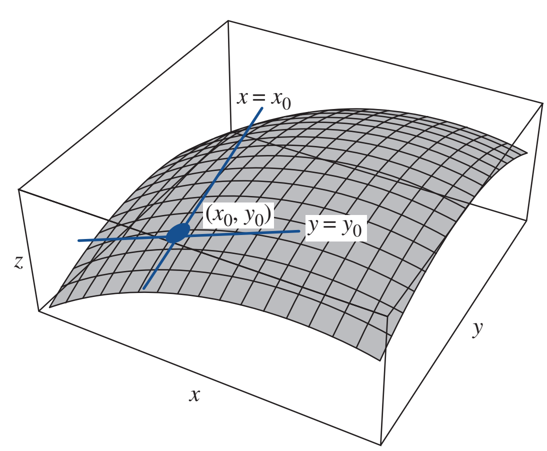
In Chapter 2 we found the tangent line to a curve at a given point. That's not a useless geometric exercise: the tangent line is useful because it serves as a *linear approximation* to the original function, and we can solve many important problems for linear functions that we cannot solve for more complicated functions. If a linear approximation is not sufficient, we can add more terms—a Taylor series—creating a higher order polynomial to approximate the function as accurately as necessary.

In this section we extend these ideas to multivariate functions. Our initial goal is to find the tangent plane to a surface. Once again, the real purpose of this exercise is to approximate a complicated function with something easier to work with. And once again, we will end with a formula that can be used to extend the approximation to higher order terms if necessary.

A Formula for the Tangent Plane

What is the definition of a tangent line to a curve? What makes it… tangent? Our answer is that the tangent line and the curve share a point, and they share the same derivative at that point. Based on that definition we can arrive quickly at a formula: the tangent line to $y = f(x)$ at the point (x_0, y_0) is $y = y_0 + f'(x_0)(x - x_0)$. The tangent line works as a good approximation to the original curve for values close to x_0 because both functions start at the same y-value and move up (or down) from there at the same rate.

A similar argument applies in higher dimensions. We begin with a definition: a tangent plane to the surface $z = f(x, y)$ at the point (x_0, y_0, z_0) must contain that point, and must match the original function at that point in both its partial derivatives. If the two functions share both their partial derivatives, then *all* their directional derivatives will be the same at that point. (Remember that $D_{\vec{u}}f = \vec{\nabla} f \cdot \vec{u}$ and $\vec{\nabla} f = \left(\partial f/\partial x\right)\hat{i} + \left(\partial f/\partial y\right)\hat{j}$.) Such a plane works as a good approximation for the original surface for points close to (x_0, y_0) because both functions start at the same z-value and, no matter which direction you travel in, they move up (or down) from there at the same rate.



These considerations are enough to arrive at a formula.

The Tangent Plane to a Surface

Given a surface S defined by a function $z = f(x, y)$ that is differentiable at the point (x_0, y_0), the tangent plane to S at (x_0, y_0) is given by the following formula.

$$z = f(x_0, y_0) + \left(\frac{\partial f}{\partial x}(x_0, y_0)\right)(x - x_0) + \left(\frac{\partial f}{\partial y}(x_0, y_0)\right)(y - y_0) \tag{4.7.1}$$

We present this formula with no derivation, although you may have arrived at something similar on your own if you worked through the Discovery Exercise (Section 4.7.1). As always, however, you shouldn't take our word for it. Convince yourself of the following facts.

- Equation 4.7.1 does in fact define a plane. (There are of course rigorous ways to prove this but you can see it intuitively by considering some possible values for the constants in the formula, which is everything on the right-hand side except x and y, and seeing what the function looks like.)
- The plane and the original function $f(x, y)$ intersect—have the same z-value—at (x_0, y_0).
- At that point, the plane and the original function also have the same $\partial z/\partial x$ and the same $\partial z/\partial y$.

If those conditions are satisfied, then we have found the tangent plane we are looking for.

EXAMPLE **Tangent Plane**

Problem:
Find the tangent plane to the function $f(x, y) = 3y + \ln(2x + y)$ at the point $(0, 1)$, and use it to approximate $f(0.1, 0.96)$.

Solution:
$f(0, 1) = 3$.
$\partial f/\partial x = 2/(2x + y)$, so $\partial f/\partial x(0, 1) = 2$.
$\partial f/\partial y = 3 + 1/(2x + y)$, so $\partial f/\partial y(0, 1) = 4$.
The formula for the tangent plane is therefore $z = 3 + 2x + 4(y - 1)$.
This formula gives $f(0.1, 0.96) \approx 3.04$. (The actual value is roughly 3.03.)

If a function depends on more than two variables, add a term for each variable. For example, the linear approximation to a function $f(x, y, z)$ about the point (x_0, y_0, z_0) is given by

$$f(x_0, y_0, z_0) + \left(\frac{\partial f}{\partial x}(x_0, y_0, z_0)\right)(x - x_0) + \left(\frac{\partial f}{\partial y}(x_0, y_0, z_0)\right)(y - y_0) + \left(\frac{\partial f}{\partial z}(x_0, y_0, z_0)\right)(z - z_0)$$

Linearizing Higher Order Differential Equations

As with single-variable linear approximations, one of the most important applications of multivariate linear approximations is to turn non-linear (and unsolvable) differential equations into linear ones that can actually be solved. In many cases the "variables" in the linear approximation are the dependent variable in the problem and its derivative(s).

EXAMPLE **Linearizing a Differential Equation**

Problem:
Find and solve a linear approximation to the differential equation

$$\ddot{x} = 1 - e^{3x+4\dot{x}}$$

(Recall that $\dot{x}$ means the derivative of x with respect to time.)

Solution:
The problem presents us with a function $\ddot{x}(x, \dot{x})$. If x and $\dot{x}$ are small then we can replace this function with a linear approximation around $(0, 0)$. Note how the following numbers all come directly from the differential equation itself.

$$\ddot{x}(0,0) = 0, (\partial\ddot{x}/\partial x)(0,0) = -3, \quad \text{and} \quad (\partial\ddot{x}/\partial\dot{x})(0,0) = -4, \quad \text{so} \quad 1 - e^{3x+4\dot{x}} \approx 0 - 3x - 4\dot{x}.$$

The equation $\ddot{x} = -4\dot{x} - 3x$ can be solved by guessing an exponential solution, which leads to $x(t) = Ae^{-3t} + Be^{-t}$. Of course, it's important to remember that this solution is only useful for small values of both x and $\dot{x}$! Fortunately this solution shows that if x and $\dot{x}$ start out small they will remain so since they will decay exponentially.

Higher Order Terms

A Taylor series begins with a linear approximation but adds higher order terms to match the second, third, and higher order derivatives of the function, providing a more accurate estimation tool. You can expand a function $f(x)$ into a Taylor series around the value $x = x_0$ with the formula:[3]

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \left(\frac{d^n f}{dx^n}(x_0) \right) \frac{1}{n!}(x - x_0)^n \\ &= f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2 + \frac{f'''(x_0)}{6}(x - x_0)^3 + \dots \end{aligned}$$

If you want to find the third-order term in the expansion of $\sqrt{x}$ around $x = 25$, this formula tells you to evaluate the third derivative of $\sqrt{x}$ at $x = 25$ and divide it by $3!$, and that gives you the coefficient of $(x - 25)^3$. In this fashion you can build a third-order polynomial that matches the original function's y-value and its first three derivatives at $x = x_0$. Note that the "0th derivative" of a function $f(x)$ is defined to be the function $f(x)$ itself (and recall that $0! = 1$), so the first term in this series is $f(x_0)$ as shown above.

The formula for a multivariate Taylor series looks similar.

The Taylor Series for a Multivariate Function

If a function $f(x, y)$ can be expanded into a polynomial around the point (x_0, y_0), then the formula is given by:

$$f(x,y) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left(\frac{\partial^{n+m} f}{\partial x^n \partial y^m}(x_0, y_0) \right) \frac{1}{n!m!}(x - x_0)^n (y - y_0)^m \tag{4.7.2}$$

To compute a Taylor polynomial of order 5, you write out all the terms for which $n + m \le 5$.

(The extension of this formula to functions of more than two variables is straightforward; see Problem 4.123.)

[3] Some people write the first term separately and start the series at $n = 1$, which avoids 0^0 appearing in the first term for $x = x_0$.

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In Problem 4.122 you will show that this formula makes a logical extension of our tangent plane. It reduces to Equation 4.7.1 in the case of a first-order approximation. For a second-order approximation, it matches the function $f(x, y)$ at the point (x_0, y_0) with the same z-value, the same (two) first derivatives, and the same (three) second derivatives. A third-order approximation matches all those plus all four third derivatives, and so on. In Problem 4.114 you'll go through part of the argument for why these requirements lead to this particular formula.

EXAMPLE **Multivariate Taylor Series**

Problem:
Find the second-order approximation to the function $z = 3y + \ln(2x + y)$ at the point $(0, 1)$, and use it to approximate $z(0.1, 0.96)$.

Solution:
First calculate the relevant derivatives, remembering that the 0th derivative is just the function itself.

$$\frac{\partial^0 z}{\partial x^0 \partial y^0} = z(x, y) = 3y + \ln(2x + y) \quad \text{so} \quad \frac{\partial^0 z}{\partial x^0 \partial y^0}(0, 1) = 3$$

$$\frac{\partial^1 z}{\partial x^1 \partial y^0} = \frac{\partial z}{\partial x} = \frac{2}{2x + y} \quad \text{so} \quad \frac{\partial^1 z}{\partial x^1 \partial y^0}(0, 1) = 2$$

$$\frac{\partial^1 z}{\partial x^0 \partial y^1} = \frac{\partial z}{\partial y} = 3 + \frac{1}{2x + y} \quad \text{so} \quad \frac{\partial^1 z}{\partial x^0 \partial y^1}(0, 1) = 4$$

$$\frac{\partial^2 z}{\partial x^2 \partial y^0} = \frac{\partial^2 z}{\partial x^2} = -4/(2x + y)^2 \quad \text{so} \quad \frac{\partial^2 z}{\partial x^2 \partial y^0}(0, 1) = -4$$

$$\frac{\partial^2 z}{\partial x^0 \partial y^2} = \frac{\partial^2 z}{\partial y^2} = -1/(2x + y)^2 \quad \text{so} \quad \frac{\partial^2 z}{\partial x^0 \partial y^2}(0, 1) = -1$$

$$\frac{\partial^2 z}{\partial x^1 \partial y^1} = \frac{\partial^2 z}{\partial x \partial y} = -2/(2x + y)^2 \quad \text{so} \quad \frac{\partial^2 z}{\partial x^1 \partial y^1}(0, 1) = -2$$

Plug this into the formula for a second-order Taylor series.

$$z(x, y) = 3 + 2x + 4(y - 1) - 2x^2 - (1/2)(y - 1)^2 - 2x(y - 1)$$

This formula puts $z(0.1, 0.96)$ at 3.027. (The actual value is roughly 3.028.)

As with Taylor series for one variable, you can find Taylor series for multivariate functions by multiplying other Taylor series, differentiating or integrating other Taylor series, or plugging in combinations of variables into them. This is shown in the example below and explored further in the problems.

EXAMPLE **Building a Complicated Taylor Series from Simpler Ones**

Problem:
Find the second-order Maclaurin series for $f(x, y) = e^x \sin(x + y)$.

Solution:
We can find the Maclaurin series for $\sin(x + y)$ by plugging $x + y$ into the series for sin:

$$\sin(x + y) = (x + y) - \frac{(x + y)^3}{6} + \ldots$$

Next we multiply this by the Maclaurin series for e^x, being careful to keep all terms up to the second order.

$$f(x, y) = \left[1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots\right]\left[(x + y) - \frac{(x + y)^3}{6} + \ldots\right] \approx x + y + x^2 + xy$$

You'll show in Problem 4.124 that you get the same answer using Equation 4.7.2.

4.7.3 Problems: Tangent Plane Approximations and Power Series

4.111 Let $f(x, y) = \sqrt{x + y^2}$, and let $g(x, y)$ be the tangent plane to $f(x, y)$ at the point $(40, 3)$.

(a) Find the formula for $g(x, y)$.

(b) Show that $f(40, 3) = g(40, 3)$ and $\vec{\nabla} f(40, 3) = \vec{\nabla} g(40, 3)$.

(c) Calculate $f(41, 2.9)$ and $g(41, 2.9)$.

(d) Calculate $f(50, 5)$ and $g(50, 5)$.

(e) In which case, Part (c) or (d), did $g(x, y)$ serve as a better approximation of $f(x, y)$? Why?

4.112 *[This problem depends on Problem 4.111.]* Let $h(x, y)$ be the second-order Taylor approximation to the function $f(x, y)$ at the point $(40, 3)$.

(a) Find the formula for $h(x, y)$.

(b) Show that at the point $(40, 3)$, $\partial^2 f/\partial x^2 = \partial^2 h/\partial x^2$ and $\partial^2 f/\partial x\, \partial y = \partial^2 h/\partial x\, \partial y$ and $\partial^2 f/\partial y^2 = \partial^2 h/\partial y^2$.

(c) Calculate $h(41, 2.9)$ and $h(43, 2.5)$.

(d) At both points, did g or h work better as an approximation for f?

4.113 One term in the Taylor series for a function $f(x, y)$ around $(0, 0)$ is

$$\left(\frac{\partial^5 f}{\partial x^2 \partial y^3}(0, 0)\right)\frac{1}{2! \times 3!} x^2 y^3$$

(a) Write down the term involving x^7 and y^4.

(b) Write down the term involving the same powers in a Taylor series around $(-3, \pi)$.

4.114 One term in the Taylor series for a function $f(x, y)$ around $(0, 0)$ is $C_{23} x^2 y^3$ where C_{23} is a constant.

(a) Find d^2/dx^2 of this term evaluated at $(0, 0)$.

(b) Find d^3/dx^3 of this term evaluated at $(0, 0)$.

(c) Find $d^2/(dx\, dy)$ of this term evaluated at $(0, 0)$.

(d) Find $d^6/(dx^3 dy^3)$ of this term evaluated at $(0, 0)$.

(e) We just asked you four different questions—four different derivatives of this function, all evaluated at $(0, 0)$. Write down and answer another such question. Your answer should not be zero. (*Hint*: there is only one correct question you can ask here!)

(f) The Taylor series and the function $f(x, y)$ should give the same answer for the derivative you wrote in Part (e). What value of C_{23} accomplishes this goal?

4.115 Find the tangent plane approximation to the function $f(x, y) = \sin(2x)\cos(3y)$

at the point $(\pi/6, \pi/6)$ and use it to approximate $f(1/2, 1/2)$.

4.116 Find the second-order approximation to the function $f(x, y) = \sin(2x)\cos(3y)$ at the point $(\pi/6, \pi/6)$ and use it to approximate $f(1/2, 1/2)$.

4.117 Find the tangent plane approximation to the function $z = x/y$ at the point $(6, 2, 3)$.

4.118 Find the second-order approximation to the function $z = x/y$ at the point $(6, 2, 3)$.

4.119 Find the fourth-order Taylor series approximation for $\sin(x + y^2)$ around $(0, 0)$. (*Hint*: There's a quick and easy way to do this. Just be sure that you toss out all terms above the fourth order.)

4.120 It is possible to do this entire problem without using Equation 4.7.2. (The second part can come quickly from the first, and the third from the second.)

(a) Find the third-order Taylor series approximation for $\sin(x + y)$ around $(0, 0)$.

(b) Find the third-order Taylor series approximation for $\sin(x + y)$ around $(0, \pi)$.

(c) Find the second-order Taylor series approximation for $\cos(x + y)$ around $(0, \pi)$.

4.121 **(a)** Find the third-order Taylor series approximation for e^{x+2y} around $(0, 0)$.

(b) Take $\partial/\partial x$ of your answer to part (a). The result is the second-order Taylor series approximation for what function?

(c) Take $\partial/\partial y$ of your answer to part (a). The result is the second-order Taylor series approximation for what function?

4.122 Write all the terms of Equation 4.7.2 for which $n + m \le 1$—in other words a first-order series. Show that this results in Equation 4.7.1, the tangent plane approximation.

4.123 Equation 4.7.2 gives the formula for the Taylor series of a function of two variables $f(x, y)$.

(a) By extending this formula, write the formula for a Taylor series of a three-variable function: $f(x, y, z) = \ldots$.

(b) Use your formula to calculate the first-order- and second-order Maclaurin series for the function $f(x, y, z) = x^2 + ye^{5z}$.

(c) Use your first-order- and second-order expansions to approximate $f(0.01, 0.02, -0.01)$. As a check on your formula, your answers should both be close to the correct value and your second-order one should be closer than the first-order one.

4.124 Find the second-order Maclaurin series for $f(x, y) = e^x \sin(x + y)$ by plugging it into Equation 4.7.2 and verify that you get the same answer we derived for it by easier methods in the Explanation (Section 4.7.2).

4.125 Suppose an object A is moving with a velocity v_{AB} relative to an object B, and B is moving with a velocity v_{BC} (in the same direction) relative to an object C. According to special relativity, the velocity of A with respect to C is:

$$v_{AC} = \frac{v_{BC} + v_{AB}}{1 + v_{BC}v_{AB}/c^2}$$

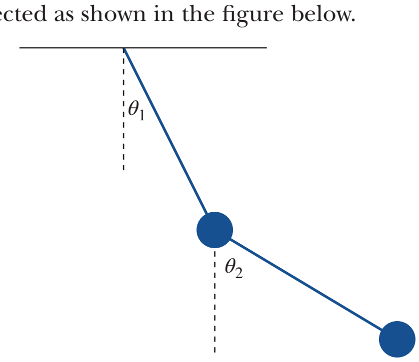
where c, the speed of light, is a constant.

(a) Find the linear approximation to v_{AC} when both velocities are much smaller than c. Explain why your answer makes sense physically.

(b) Find the second-order approximation to v_{AC} when both velocities are close to the speed of light. Use your approximation to confirm that, as both velocities approach c, v_{AC} also approaches c (not $2c$ as classical mechanics would predict).

4.126 Find an approximate general solution to the differential equation $d^2x/dt^2 = (1 + x + \dot{x})/(1 + x - \dot{x})$ using a linear approximation valid when x and $\dot{x}$ are both close to 0.

4.127 Two coupled pendulums of length L are connected as shown in the figure below.



The equations describing this system are

$$2\ddot{\theta}_1 + \ddot{\theta}_2 \cos\left(\theta_1 - \theta_2\right) + \dot{\theta}_2^2 \sin\left(\theta_1 - \theta_2\right) + 2\frac{g}{L}\sin\theta_1 = 0 \qquad (4.7.3)$$

$$\ddot{\theta}_2 + \ddot{\theta}_1 \cos\left(\theta_1 - \theta_2\right) - \dot{\theta}_1^2 \sin\left(\theta_1 - \theta_2\right) + \frac{g}{L}\sin\theta_2 = 0 \qquad (4.7.4)$$

These equations have no solution in terms of simple functions. If you assume the amplitude of oscillations is small, however, then you can find approximate solutions.

(a) The first equation begins with a function of θ_1, θ_2, $\dot{\theta}_2$, $\ddot{\theta}_1$, and $\ddot{\theta}_2$. Write the linear approximation for that five-variable function.

(b) Do the same for the second equation (with a slightly different list of variables) and then write the two resulting simpler differential equations.

The differential equations you just wrote do have relatively simple solutions, which describe the motion of these pendulums for small oscillations. One such solution takes the following form.

$$\theta_1 = Ae^{it\sqrt{\left(2+\sqrt{2}\right)(g/L)}} \qquad (4.7.5)$$
$$\theta_2 = Be^{it\sqrt{\left(2+\sqrt{2}\right)(g/L)}}$$

(c) Plug this solution into your linear approximation to Equation 4.7.3 and solve for A in terms of B. Plug all the numbers into a calculator and express your answer in the form $A =$<*a number*>B.

(d) Repeat Part (c) for your approximation to Equation 4.7.4 and verify that you get the same relationship between A and B. This tells you that for any two numbers A and B with the relationship you found, Equation 4.7.5 is a solution to this pair of differential equations.

(e) If the motion of the coupled pendulum is described by this solution and the upper pendulum is oscillating with an amplitude of 5°, what will be the amplitude of oscillation of the lower pendulum?

4.128 Generate plots of the function $z = \sin(x^2 y)$ in the range $-2 \le x \le 2$, $-2 \le y \le 2$ and of its power series at different orders. What order do you need to go to before the power series plot looks nearly identical to the plot of the actual function?

4.10 Special Application: Thermodynamics

A sealed canister is filled with gas. A thermometer allows you to constantly monitor the temperature of the gas, a manometer lets you monitor its pressure, and a heating coil allows you to add controlled amounts of energy to it. In one experiment you tighten the lid of the container and slowly add energy until the temperature of the gas has gone up by 5 K. You record the amount of energy you added and the change in pressure that resulted. In a second experiment the lid is a piston that can slide up or down freely, allowing the gas to expand or contract. The piston is still sealed so no gas can enter or leave, and it's insulated so the only cooling or heating comes from the coil that you control. This time as you add energy you find that the gas expands, pushing the piston up, and the pressure of the gas remains constant. You also find that you need to add more energy to the gas to raise its temperature by 5 K than you had needed when the lid was locked in place.

These experiments fall within the domain of "thermodynamics," which deals with the flow of energy between systems. It is one of the fields that most heavily uses partial derivatives and differentials. In Problem 4.178 you'll come back to these two experiments and calculate the changes in pressure, volume and energy in both cases. In order to get there, you'll need some of the central formulas of thermodynamics:

1. The "first law of thermodynamics," which can be rewritten as the "thermodynamic identity," addresses the ways in which energy transfers into and out of a system.
2. "Heat capacity" addresses the change in temperature that results from such a transfer of energy.

The first law, and the application of heat capacity to a system, are universal. To figure out the heat capacity of a specific system, you need more information about that system. This brings us to our last topic:

3. The "ideal gas law" and "equipartition theorem" describe specific systems in enough detail to allow you to figure out their heat capacities in many cases. Although these laws are not universal, they apply in a broad variety of important real-world situations.

The First Law of Thermodynamics and the Thermodynamic Identity

A brick lying on the ground has an "internal energy" U due to the motions of its molecules and the forces between them. A brick falling from a roof has a "total energy" E which is the sum of internal, kinetic, and potential energies. In this section we will consider containers of gas at rest. They may expand or contract, but they won't move, so their only energy changes will be in their internal energy. There are two ways a system's environment can change the system's internal energy. Heat (Q) is the spontaneous transfer of energy from a hot object to a cold object. Work (W) is essentially any other transfer of energy, which can include mechanical work (pushing or pulling the system), electrical work (running a current through it), and more.[5] The relationship between energy, heat, and work is expressed in the "first law of thermodynamics":

$$dU = Q + W \quad \textit{the first law of thermodynamics} \tag{4.10.1}$$

We don't usually set a differential equal to a normal quantity, but Q and W are not normal variables. Q is the *heat entering the system*; it is not an energy, but the increase in energy due to one specific cause. (Some texts use ΔQ and dQ for normal and infinitesimal flows of heat respectively, but that seems to imply "a change in heat" which is not completely accurate; rather, heat itself is a change in energy.) W is also a change, the energy added to the system

[5] We are assuming that each system maintains a constant number of particles, which rules out energy exchange by direct transfer of particles from one system to another.

due to all other causes. (Some texts define W as the work *done by* the system, and therefore write $dU = Q - W$.) We will use Q and W without a prior Δ or d, and you will need to know from context whether we are referring to a regular change in energy or an infinitesimal one.

In this section we will consider the relatively simple case of a gas in a closed container, and we will only consider work done by compressing or expanding the gas. You will show in Problem 4.189 that the work done on a gas when it is compressed by a small amount dV is $-P\,dV$, where P is the pressure of the gas and V is its volume. The sign is negative because positive work is done on the system when dV is negative. The heat entering a system can similarly be written as $T\,dS$ where T is the temperature and S is the "entropy."[6] That expression can be derived from a more fundamental definition of entropy having to do with the microscopic properties of the system, but for our purposes you can think of $dS = Q/T$ as the definition of entropy. (It's how entropy was first defined.) Putting all this together gives the "thermodynamic identity," which (among its other virtues) looks more like a good equation with differentials should.

$$dU = T\,dS - P\,dV \quad \textit{the thermodynamic identity} \tag{4.10.2}$$

Heat Capacity

When you add energy to a system you generally increase its temperature. The amount of heat required per unit increase in temperature is the "heat capacity" (C) of the system. This definition can be written as $C = Q/dT$. Using $dS = Q/T$ (from above) we get:

$$C = T\frac{dS}{dT} \tag{4.10.3}$$

In this form, however, the definition of heat capacity is ambiguous because the entropy depends on all three of the state variables T, P, and V. How fast entropy changes with respect to temperature depends on what is happening to the other two variables at the same time. The simplest possibility is to hold the volume constant $(dV = 0)$, so Equation 4.10.2 becomes $dU = T\,dS$. Putting that together with Equation 4.10.3 and the chain rule,

$$C_V = \left(\frac{\partial U}{\partial T}\right)_V \quad \textit{heat capacity at constant volume}$$

Recall that the subscript V on this partial derivative means V is the variable you are holding constant as you differentiate with respect to T. We'll discuss this issue in more detail below.

Physically, this is the heat capacity of a system assuming there is no work being done on or by the system $(P\,dV = 0)$. While C_V is relatively simple to calculate for many systems, it is not usually the heat capacity of interest. That's because when you heat something it tends to expand, which causes it to do work on its environment: it is the pressure, rather than the volume, that stays constant. Most tabulated values of heat capacity refer to "heat capacity at constant pressure." In Problem 4.198 you will show that

$$C_P = \left(\frac{\partial U}{\partial T}\right)_P + P\left(\frac{\partial V}{\partial T}\right)_P \quad \textit{heat capacity at constant pressure} \tag{4.10.4}$$

Since some of the energy you put in as heat is going into work on the environment, the heat you need to add to get a certain increase in temperature is greater, so C_P is always larger than C_V.

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[6]This section will contain many formulas with temperature in them. Those formulas only work if temperature is measured in Kelvin, or some other scale where $T = 0$ means the absolute zero of temperature. If you think about the familiar ideal gas law $PV = nRT$ (which we discuss below), it should be clear that if a gas is at 0° Fahrenheit that doesn't mean P or V is zero. The ideal gas law, like most thermodynamic formulas, simply isn't true for temperatures in Fahrenheit or Celsius.

Ideal Gases

To make calculations about a particular fluid, you need to know the relationships between properties such as pressure, volume, temperature, and energy. Different relationships lead to different behavior. Here we address only one case, an "ideal gas." But this case is not just a textbook oversimplification: more complicated systems can often be approximated by the ideal gas law in cases of low densities, such as the densities typically found at room temperature and pressure, so these equations serve as a useful model for most gases an engineer is likely to encounter. You'll consider some other systems in the problems.

We will need two equations to model an ideal gas. The first, the "equation of state," relates pressure, volume, and temperature:

$$PV = nRT \quad \textit{the ideal gas equation}$$

Here n is the number of moles (number of molecules divided by a constant called "Avogadro's number") and $R = 8.3\,\text{J}/(\text{mol K})$ is a constant.[7]

The second equation, which applies to ideal gases and a variety of other systems, relates the internal energy to the temperature:

$$U = \frac{f}{2}nRT \quad \textit{the equipartition theorem}$$

Here f is the number of "degrees of freedom," which essentially means how many ways the molecules can move. A monatomic molecule such as helium has three degrees of freedom because it can move in three independent directions. A diatomic molecule such as hydrogen can also move in three directions, but in addition it can rotate around two independent axes, so it has five degrees of freedom. (Technically you could consider other degrees of freedom such as vibrations or rotations about the long axis of a diatomic molecule, but for quantum mechanical reasons those motions cannot be excited at room temperature for most gases.)

The equation of state and the equipartition theorem allow you to predict measurable quantities. For example, the heat capacity at constant volume of an ideal gas is

$$C_V = \left(\frac{\partial U}{\partial T}\right)_V = \frac{f}{2}nR \tag{4.10.5}$$

For a container with five moles of helium gas, C_V is thus $20.8\,J/K$.

Another look at $(\partial f/\partial x)_y$

Throughout this section, and throughout thermodynamics more generally, frequent use is made of the notation $(\partial f/\partial x)_y$, meaning the partial derivative of f with respect to x, holding y constant. To consider in more detail what such a derivative means, we turn briefly to a non-thermodynamic example from basic Geometry.

The area of a right triangle can be written[8] as $A = (bc^2 - b^3)/(2a)$, where a and b are the legs and c is the hypotenuse. If the side lengths are changing, then the chain rule gives us $dA/dt = (\partial A/\partial a)(da/dt) + (\partial A/\partial b)(db/dt) + (\partial A/\partial c)(dc/dt)$, which becomes:

$$\frac{dA}{dt} = \left(\frac{-(bc^2 - b^3)}{2a^2}\right)\left(\frac{da}{dt}\right) + \left(\frac{c^2 - 3b^2}{2a}\right)\left(\frac{db}{dt}\right) + \left(\frac{bc}{a}\right)\left(\frac{dc}{dt}\right) \tag{4.10.6}$$

[7] Physicists often write the ideal gas equation in terms of the number of molecules rather than the number of moles: $PV = Nk_BT$. "Boltzmann's constant" k_B is just R divided by Avogadro's number.

[8] You can easily confirm this formula for yourself. Your next question might be "Who would write it that way, and why?" We would, because we're writing a math book. So there.

We're about to say some unflattering things about this equation, but first a word of reassurance: if we tell you the side lengths of a right triangle and how fast those lengths are changing, Equation 4.10.6 will correctly tell you how fast the area is growing. We are not about to take back everything we've promised about the chain rule.

But we are about to point out that $(\partial A/\partial a)$ is a fictitious quantity.

You know by now that $(\partial A/\partial a)$ means "Find how much A changes if you change a while holding b and c constant." But you also know that you *cannot possibly* change a while holding b and c constant! The three variables are related by the Pythagorean theorem: $a^2 + b^2 = c^2$. If you change one side, one or both of the others must change. (You can't have a 3.01-4-5 right triangle.) So $\partial A/\partial a$ is a *useful* quantity, as a step toward finding a total dA, but it is not *physically meaningful* by itself. In fact, as you will show in Problem 4.191, different forms of the area formula lead to completely different values for $\partial A/\partial a$.

EXAMPLE **Same Partial Derivative, Different Answers**

Question: The function $f(x, y) = x + y$ is defined on the domain $y = x$. Find $\partial f/\partial x$.

One solution:
If we take the function as given, $f(x + y) = x + y$, then clearly $\partial f/\partial x = 1$.

Some other solutions:
Because this function is subject to a constraint, we can use the equation $y = x$ to rewrite the function. If we write it as $f(x, y) = 2y$ then $\partial f/\partial x = 0$. And if we rewrite it as $f(x, y) = 2x$ then $\partial f/\partial x = 2$.

Why did we get three answers for one question? Because the question involves a useful fiction. You cannot "change x while holding y constant" while also maintaining the constraint $y = x$.

So why did you write a whole chapter about partial derivatives if they don't mean anything? Two reasons. First, sometimes they do mean something. If x and y were truly independent, then $\partial f/\partial x$ would be a real and meaningful quantity. And as we will see below, even when there is a constraint, we can frame partial derivatives in a perfectly meaningful way by carefully specifying what stays constant and what doesn't.

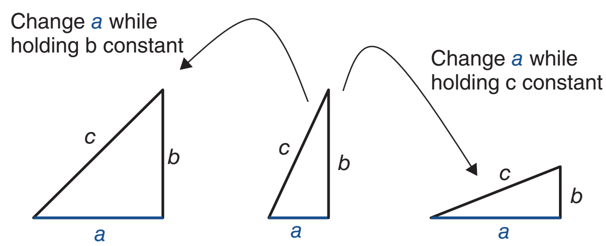
But the second reason is even more important: as we stressed above, the chain rule still works! In this example, if you write $df/dt = (\partial f/\partial x)(dx/dt) + (\partial f/\partial y)(dy/dt)$ you will get the right answer, $df/dt = 2(dx/dt)$, no matter what form you use.

In short, there are two kinds of partial derivatives: the ones that are physically meaningful (and have one unique answer), and the ones that are not physically meaningful (and may have different answers). Both kinds can be used to find correct total derivatives.

Thermodynamics is rife with constrained multivariate functions. For instance, $\partial U/\partial T$ falls into the "useful but not physical" category because you can't change T while holding P and V constant. But we can get physically meaningful derivatives by specifying one variable to hold constant, and allowing the others to change as they must. Consider how we can apply this strategy to our triangle.

- $\partial A/\partial a$, as discussed above, means "see what happens to A if you change a while holding b and c constant." It can be a useful step on the road to finding a total dA, but it has no intrinsic meaning, and its value depends on the specific form of your $A(a, b, c)$ function.

- $(\partial A/\partial a)_b$ asks about the change in area if you change a while holding b constant. This is real: the change in a causes a change in c, and the area of the triangle changes in response to both of these changes. Mathematically, you would find this quantity by solving $a^2 + b^2 = c^2$ for c and then plugging in to find $A(a, b) = (1/2)ab$ before evaluating the partial derivative.
- $(\partial A/\partial a)_c$ is also real: you change a while holding c constant, which causes a change in b, and the area of the triangle changes in response to both of these changes. Mathematically, you would find this quantity by solving $a^2 + b^2 = c^2$ for b and then plugging in to find $A(a, c) = (1/2)a\sqrt{c^2 - a^2}$ before taking the derivative.



Please don't think that we are saying "partial derivatives are physically meaningful only when they have parentheses." The message is quite different: "Partial derivatives are physically meaningful when they represent a possible change." For instance, if $T(x, y, z)$ represents the temperature in the room, then $\partial T/\partial x$ (which implicitly means "holding y and z constant") is perfectly meaningful, since x, y, and z are all independent. But if our function is confined to the plane $3x + 2y + 5z = 7$ then a change in x must be accompanied by a change in either y or z. In that case, $\partial T/\partial x$ would be helpful only as part of a total dT, but $(\partial T/\partial x)_y$ would mean more than that.

Perhaps surprisingly, this distinction between "only useful" and "actually physical" partial derivatives can be important in how you use them in equations. As an example, suppose that three quantities E, F, and G are related by:

$$dE = dF + F\,dG \qquad \textit{given} \tag{4.10.7}$$

You can take this at face value as a statement about small changes: "If F and G each changes by a small amount, then here is how much E will change." If all three variables depend on time, then you can also divide both sides by dt:

$$\frac{dE}{dt} = \frac{dF}{dt} + F\frac{dG}{dt} \qquad \textit{follows from Equation 4.10.7}$$

This is now a statement about *rates* of change: "If F and G are changing this fast right now, then here is how fast E is changing."

But what if E, F, and G are all functions of x and y? Since we have stressed that dx is a meaningful (and manipulable) variable and ∂x is not, you should be suspicious if we assert this.

$$\left(\frac{\partial E}{\partial x}\right)_y = \left(\frac{\partial F}{\partial x}\right)_y + F\left(\frac{\partial G}{\partial x}\right)_y \qquad \textit{Does this follow from Equation 4.10.7?} \tag{4.10.8}$$

Does Equation 4.10.7 imply Equation 4.10.8? If x and y are independent, so all of these partial derivatives have unique physical values, then the answer is yes. If x and y are related by some external constraint, however, then you can change the values of these partials simply by rewriting your functions as we did for $\partial A/\partial a$ above, and Equation 4.10.8 doesn't follow from Equation 4.10.7. You'll see an example where you are not allowed to do this division in

Problem 4.196, and in Problem 4.198 you'll apply a valid use of "dividing by ∂x" to derive an important thermodynamic equation.

Thermodynamics makes frequent use of this notation and this trick, generally without explanation. So let's be clear: you cannot get from Equation 4.10.7 to Equation 4.10.8 by "dividing both sides by ∂x" (there's no such thing), or by taking derivatives with the chain rule. You can't get there by any mathematical step, because it is not always true. But if x and y are independent of each other, then changing x while holding y constant will lead to real values of dE, dF, and dG, and under those circumstances the leap to Equation 4.10.8 is safe.

4.10.1 Problems: Thermodynamics

4.178 A canister contains 5 moles of hydrogen gas at 300 K and 10^5 Pa (the SI unit of pressure). You may consider the hydrogen to be an ideal gas.

(a) How much thermal energy does the hydrogen contain?

(b) How much more thermal energy would the hydrogen contain if it were at 350 K?

(c) Use Equation 4.10.5 to find C_V for the hydrogen and use that to calculate how much heat you have to add to the gas to raise it from 300 K to 350 K at constant volume. Verify that your answer matches the energy difference you found in Part (b).

(d) How much would the pressure of the gas increase as you heated it at constant volume?

(e) Use Equation 4.10.4 to find C_P for the hydrogen.

(f) Use your answer to Part (e) to calculate how much heat would be required to raise the gas from 300 K to 350 K at constant pressure.

(g) Use the first law of thermodynamics and your answers to the previous parts to calculate how much work is done on or by the gas as you heat it from 300 K to 350 K at constant pressure.

(h) Recall that we define W to be positive when work is done *on* the system and negative when it is done *by* the system. Based on the sign you found for the work, is work being done on the hydrogen by its surroundings, or on the surroundings by the hydrogen? Based on that, is the hydrogen gas expanding or contracting as you heat it? (*Hint*: Don't forget to use common sense and experience. If your answer to this question contradicts what you would physically expect, go back and see if you made a sign error.)

(i) Using $W = -P\,dV$, find the amount by which the volume of the gas increased or decreased as you heated it at a constant pressure of 10^{-5} Pa.

4.179 **(a)** What is $(\partial U/\partial V)_S$? (*Hint*: If you spend more than 30 s on this problem you're making it harder than it needs to be.)

(b) Briefly describe an experiment you could perform to vary V while holding S constant. *Hint*: look at the definition of entropy.

4.180 For a system that obeys the equipartition theorem we could have written the definition of C_V as a total derivative, dU/dT. Explain why this is equivalent to the definition we gave for systems that obey equipartition, but not necessarily for other systems.

For Problems 4.181–4.185 you should assume all gases are ideal. At normal temperatures and pressures this is usually a good approximation. Pay attention to how many degrees of freedom f the gas in each problem has.

4.181 An "isothermal" process is one that takes place at a constant temperature. Assume a container with n moles of helium at volume V and pressure P is being expanded isothermally at a rate dV/dt.

(a) At what rate is the internal energy of the helium changing? (*Hint*: This requires no calculations.)

(b) Using the thermodynamic identity and your answer to Part (a), find the rate of change of the helium's entropy.

4.182 An "adiabatic" process is one in which no heat enters or leaves the system. Assume a container with n moles of helium at volume V and pressure P is being expanded adiabatically at a rate dV/dt.

(a) At what rate is the entropy of the helium changing? (*Hint*: This requires no calculations.)

(b) Using the thermodynamic identity and your answer to Part (a), find the rate of change of the helium's energy.

(c) Is the helium's temperature remaining constant, increasing, or decreasing?

4.183 An "isobaric" process is one that takes place at constant pressure. Assume a container with n moles of helium at volume V and pressure P is being expanded isobarically at a rate dV/dt.

(a) At what rate is the temperature of the helium changing?

(b) At what rate is the energy of the helium changing?

(c) Using the thermodynamic identity and your answer to Part (b), find the rate of change of the helium's entropy.

(d) Is heat entering the system, leaving the system, or neither?

4.184 A container with a movable piston that can allow it to expand and contract contains n moles of helium. The container walls are thin enough that the helium remains at a constant room temperature T. You slowly compress the container so that the volume goes from V_0 to V_f.

(a) Write an expression for $P(V)$, the pressure as a function of volume while the helium is being compressed.

(b) Recall that a *small* amount of compression $-dV$ requires an amount of work $-P\,dV$ to be done on the system. To find the total work done in going from V_0 to V_f take an integral to add up all the infinitesimal amounts of work done along the way.

(c) Did the sign of your answer to Part (b) come out the way you would expect? Explain.

(d) How much did the energy of the helium change during the process? (*Hint*: This should be trivial to answer.)

(e) How much heat entered or left the system during the process?

(f) Find the change in entropy of the helium during the compression.

4.185 Methane is a "polyatomic" molecule, meaning it has more than two atoms.

(a) A polyatomic molecule can rotate about any of the three axes. How many degrees of freedom f does it have?

(b) How much heat is required to raise 30 moles of methane gas from 300 K to 350 K at constant pressure?

4.186 **(a)** What is C_P for an ideal gas with f degrees of freedom?

(b) Is the expression you just found for C_P larger than or smaller than Equation 4.10.5 for C_V?

(c) Explain why your answer to Part (b) makes sense physically.

4.187 The ideal gas approximation assumes that the molecules of a gas don't interact with each other. At high densities, an approximation that takes into account some molecular interactions is the van der Waals equation of state:

$$PV + \frac{an^2}{V} - nbP - \frac{abn^3}{V^2} = nRT$$

The energy of a van der Waals gas is:

$$U = \frac{fnRT}{2} - \frac{an^2}{V}$$

(a) Calculate C_V for a van der Waals gas.

(b) From the energy equation you can conclude that $(\partial U/\partial T)_P = fnR/2 + (an^2/V^2)(\partial V/\partial T)_P$. Use implicit differentiation and the van der Waals equation of state to find $(\partial V/\partial T)_P$ and thus derive an expression for C_P for a van der Waals gas.

4.188 Electromagnetic radiation can be considered a gas of particles called "photons." The gas is ideal (not just approximately ideal, like normal gases), but instead of the usual equipartition theorem it obeys the relation $U = 3nRT$. Derive C_V and C_P for n moles of photons.

4.189 Consider a container of gas with a movable piston. Suppose the gas is allowed to expand in such a way that the piston moves by a distance L. As it expands the gas exerts a force on the piston given by the pressure P of the gas times the cross-sectional area A of the piston. Recall from introductory mechanics that the mechanical work you do on an object is the force you exert on it times the distance it moves (assuming they are in the same direction).

(a) Show that the work done by the gas on the piston is $P\,dV$.

(b) Argue using Newton's third law that the work done by the piston on the gas is $-P\,dV$.

4.190 The Explanation above discussed the different meanings of $\partial A/\partial a$, $(\partial A/\partial a)_b$, and $(\partial A/\partial a)_c$ in a right triangle. Give a similar discussion of the different meanings of $\partial S/\partial T$, $(\partial S/\partial T)_V$, and $(\partial S/\partial T)_P$ where S, the entropy, depends on temperature, pressure, and volume, which are in turn constrained by an equation of state such as the ideal gas law.

4.191 In the Explanation above, we analyzed a right triangle using the admittedly perverse (but correct!) area formula $A = (bc^2 - b^3)/(2a)$. In this problem you will replicate our analysis based on the more conventional $A = (1/2)ab$.

(a) Find $\partial A/\partial a$, $\partial A/\partial b$, and $\partial A/\partial c$ and use them to write a general expression for dA/dt.

(b) Based on the Pythagorean $c^2 = a^2 + b^2$, write a formula for dc/dt based on a, b, c, da/dt, and db/dt.

(c) If $a = 3$, $b = 4$, $c = 5$, $da/dt = -2$, and $db/dt = 6$, find dc/dt.

(d) Plug the numbers from Part (c) into your formula from Part (a). Show that your $\partial A/\partial a$ is not the same as ours, but our final dA/dt answers are the same.

4.192 Consider the function $f = x^2 + yz$ where $2x - yz^2 = 3$.

(a) Using the function in the form given above, find $\partial f/\partial x$.

(b) Rewrite f as a function of x and z only. When you take the derivative of the resulting equation with respect to x, you will find $(\partial f/\partial x)_z$.

(c) Find $(\partial f/\partial x)_y$.

4.193 The entropy of a monatomic ideal gas is $S = C + nR\ln(V) + (3/2)nR\ln T$ where C is a constant.

(a) Calculate $(\partial S/\partial T)_V$.

(b) Calculate $(\partial S/\partial T)_P$. (*Hint*: Start by using the ideal gas law to eliminate V from the equation for S.)

(c) Show that you can rewrite the entropy of an ideal gas as either $C + nR\ln((1/2)V + (1/2)nRT/P) + (3/2)nR\ln T$ or $C + nR\ln((1/3)V + (2/3)nRT/P) + (3/2)nR\ln T$.

(d) Using the two expressions for entropy in Part (c), calculate $(\partial S/\partial T)$ holding P and V constant. Prove that your two answers are *not* equivalent.

(e) We seem to have a problem. If you do an experiment where you change T while holding P and V constant, and measure the resulting change in S, you cannot possibly get two different results. So how can a series of valid mathematical steps lead to two different values of $\partial S/\partial T$?

4.194 A light bulb has a constant resistance R. A battery supplies a voltage V across it, which causes a current $I = V/R$ to flow through it. The power emitted by the light bulb (in the form of light and heat) is $P = IV$. The voltage, and thus the current, are changing with time.

(a) Draw the dependency tree for the power in this arrangement.

(b) Write the chain rule for dP/dt.

(c) Using the equations $P = IV$ and $I = V/R$, calculate dP/dt as a function of V and dV/dt.

(d) Redo Parts (a)–(c) starting from the equations $P = I^2R$ and $I = V/R$.

(e) Note that $\partial P/\partial I$ came out differently in your two calculations, but in both cases led to the same dP/dt. Why must dP/dt come out the same no matter how you calculate it?

4.195 Consider a function f defined everywhere on a plane. We use ρ and ϕ for the polar coordinates on the plane.

(a) The derivative $(\partial f/\partial x)_y$ looks for a change in f when you advance x by a small amount while holding y constant. Draw a picture of a point (x, y). Then draw a small line segment from that point that allows x to change but holds y constant. Label dx on your drawing.

(b) The derivative $(\partial f/\partial x)_\rho$ looks for a change in f when you advance x by a small amount while holding ρ constant. Draw a picture of a point (x, y) and a small line segment from that point that allows x to change but holds ρ constant. Label dx on your drawing.

(c) The derivative $(\partial f/\partial x)_\phi$ looks for a change in f when you advance x by a small amount while holding ϕ constant. Draw a picture of a point (x, y) and a small line segment from that point that allows x to change but holds ϕ constant. Label dx on your drawing.

Now we consider the specific function $f = y$ at the point $(4, 3)$.

(d) Calculate $(\partial f/\partial x)_y$ at the given point.

(e) Rewrite f as a function of x and ρ. Using this form, calculate $(\partial f/\partial x)_\rho$ at the given point.

(f) Rewrite f as a function of x and ϕ. Using this form, calculate $(\partial f/\partial x)_\phi$ at the given point.

4.196 In this problem you will prove by example that you cannot generally divide both sides of an equation by ∂x when x and y are not independent. Consider three quantities $f(x,y) = 3x^2y + 2$, $a(x,y) = 2x^4$, and $b(x,y) = y^2$, where $y = x^2$.

(a) Use the chain rule to calculate df/dx, da/dx, and db/dx, and verify that $df/dx = da/dx + db/dx$. Because these are total derivatives you can multiply both sides of the equation by dx and conclude that $df = da + db$.

(b) Using the forms given in the problem for f, a, and b, show that the equation $(\partial f/\partial x) = (\partial a/\partial x) + (\partial b/\partial x)$ is false.

4.197 Consider a function of x and y, which are themselves related by $y = x^2$.

(a) Let $a_1 = x + xy^2$.
  - **i.** Calculate $\partial a_1/\partial x$ and $\partial a_1/\partial y$.
  - **ii.** Use the chain rule to write a formula for da_1 as a function of x, y, dx, and dy.
  - **iii.** Now plug in $y = x^2$ to find da_1 as a function of x and dx only.

(b) Let $a_2 = \sqrt{y} + x^3y$.
  - **i.** Calculate $\partial a_2/\partial x$ and $\partial a_2/\partial y$.
  - **ii.** Use the chain rule to write a formula for da_2 as a function of x, y, dx, and dy.
  - **iii.** Now plug in $y = x^2$ to find da_2 as a function of x and dx only.

(c) Show that $a_1 = a_2$. (Assume $x > 0$.)

(d) What was the same in these two examples, and what was different?

4.198 **(a)** Derive Equation 4.10.4, starting from the thermodynamic identity.

(b) Equation 4.10.4 looks like the thermodynamic identity divided by ∂T, but in general dividing by a partial is not legal. Why is it OK in this case?

4.199 The "enthalpy" H of a system is defined as $H = U + PV$.

(a) Express the differential dH in terms of T, S, P, and V and their differentials. In other words write a formula for dH without U in it.

(b) Using your formula for dH, show that for any process done at constant pressure the change in enthalpy equals the amount of heat that enters your system. (Many reactions occur at constant pressure because they are open to the atmosphere. Chemists often refer to tables listing the enthalpy of gases in different states to figure out how much heat will enter or leave when they undergo certain reactions.)

4.200 **Maxwell Relations** The thermodynamic identity can be used to derive non-obvious relationships between certain derivatives.

(a) What is

$$\left(\frac{\partial U}{\partial S}\right)_V$$

Hint: If you spend more than 30 s on this problem you're making it harder than it needs to be.

(b) Take the derivative $(\partial/\partial V)_S$ of your answer to Part (a) to get an expression for $(\partial^2 U/\partial V\partial S)$. Your answer should be in the form of a *first* derivative.

(c) Derive a similar expression for $(\partial^2 U/\partial S\partial V)$. Using the equality of mixed partial derivatives, write an equation relating two first derivatives. This equation is known as a "Maxwell relation."

(d) Describe experiments that you could perform to directly measure each of the two derivatives that you just said should be equal to each other. When you think of these derivatives as descriptions of physical processes in this way it is *far* from obvious that these two quantities would be equal.

4.201 *[This problem depends on Problems 4.199 and 4.200.]* Use the formula you derived for dH in Problem 4.199 to derive another Maxwell relation similar to the one you derived in Problem 4.200.

4.11 Additional Problems

For Problems 4.202–4.205 find $\partial f/\partial x$, $\partial^2 f/\partial x\partial y$, $\vec{\nabla} f$, and $D_{\vec{u}} f$.

4.202 $f(x,y) = x - y + e^{xy}$ and $\vec{u} = \hat{i} - \hat{j}$

4.203 $f(x,y) = x^2y^3$ and $\vec{u} = \hat{j}$

4.204 $f(x,y) = ax + by^2 + c\sin(ey)$ and $\vec{u} = \hat{i} + 3\hat{j}$

4.205 $f(x,y) = x/(x+y)$ and $\vec{u}$ has magnitude 2 and direction 20° clockwise from the positive y-axis.

For Problems 4.206–4.208 say which of the given functions is a solution to the given partial differential equation. There may be more than one correct answer. Assume any letters other than f, x, and t represent constants.

4.206 $\partial f/\partial t = -\partial f/\partial x$

- **(a)** $f(x,t) = x^2t^{-2}$
- **(b)** $f(x,t) = e^{x-t}$
- **(c)** $f(x,t) = \cos^2(t-x)$
- **(d)** $f(x,t) = (x+t)^2$

4.207 $\partial^2 f/\partial t^2 = c^2\partial^2 f/\partial x^2 - (c/\sqrt{2})\left(\partial^2 f/\partial x\partial t\right)$

- **(a)** $f(x,t) = c$
- **(b)** $f(x,t) = e^{\sqrt{2}ax+act}$
- **(c)** $f(x,t) = \sin(x+\sqrt{2}ct)$
- **(d)** $f(x,t) = xt$

4.208 $\partial^2 f/\partial t^2 = (x^2/t^2)\partial^2 f/\partial x^2$

- **(a)** $f(x,t) = c$
- **(b)** $f(x,t) = x^3t^3$
- **(c)** $f(x,t) = x^2t^3$
- **(d)** $f(x,t) = \cos(ax) + \sin(bt)$

4.209 Chemical Kinetics The "Arrhenius equation" tells us that the rate k of a chemical reaction is given by $k = Ae^{-E_a/RT}$ where T is the temperature and E_a is the activation energy (the energy that must be overcome for the reaction to occur). A and R are positive constants.

- **(a)** Compute $\partial k/\partial T$. Based on the sign of your answer, you should be able to conclude that "If you increase the temperature, the reaction rate increases."
- **(b)** Compute $\partial k/\partial E_a$. Then write a sentence, similar to the quoted sentence in Part (a), explaining what the sign of your answer means.
- **(c)** Compute $\partial^2 k/\partial T^2$. How high does E_a have to be in order to make $\partial^2 k/\partial T^2$ positive? What does it tell you about the system when it is positive?
- **(d)** Compute $\partial^2 k/\partial T\partial E_a$. How *low* does E_a have to be in order to make $\partial^2 k/\partial T\partial E_a$ positive? What does it tell you about the system when it is positive? *Hint*: You can answer the verbal part two different ways, depending on whether you think of your second derivative as $\partial^2 k/\partial T\partial E_a$ or $\partial^2 k/\partial E_a\partial T$.

4.210 The Wave Equation

- **(a)** Verify that $f(x,t) = \sin(x+ct) + \ln(x-ct)$ is a solution to the wave equation $\left(\partial^2 f/\partial t^2\right) = c^2\left(\partial^2 f/\partial x^2\right)$.
- **(b)** Next show more generally that $f(x,t) = a(x+ct) + b(x-ct)$ is a solution, where a and b can be any functions whatsoever.

4.211 The current I in an RLC circuit depends on the applied voltage V, the resistance R, the inductance L, and the capacitance C. For an alternating current with a variable inductor, both V and L depend on time t. The resistance depends on temperature T which, in turn, depends on time. The capacitance does not depend on time or temperature.

- **(a)** Draw the dependency tree for I.
- **(b)** Write a formula for dI/dt.

4.212 Your company's profit P depends on the number of items you sell (N), the price you charge per item (I), and your costs (C). The number of items you sell depends on the price you charge, and your costs depend on the number of items you sell. (For simplicity we'll assume you make exactly as many as you sell.)

- **(a)** Draw the dependency diagram for your profit and write an expression for dP/dI.
- **(b)** One simple $N(I)$ function is $N = ae^{-bI}$ where a and b are constants. Show that this function makes sense at the low and high extremes. Give the units on the two constants.
- **(c)** Write plausible functions for the other dependencies in the problem and explain why they make sense. (These may be simpler than the possibility we proposed for $N(I)$.) Your answers may contain unknown constants. Assuming P, I, and C have units of dollars and N is unitless, specify the units of any constants you introduce.

(d) Using the functions you just wrote evaluate your expression for dP/dI. Your answer should contain all of the variables given in the problem as well as any constants you introduced in your function. Make sure each term in your answer has correct units.

4.213 The traffic density c on interstate highway I-95 depends on position x and time t, and your position on the highway depends on time.

(a) Write an expression for dc/dt, the traffic density you see as you drive.

(b) Your expression for dc/dt should contain $\partial c/\partial t$ in it. Explain what each of these two derivatives means.

(c) Describe a circumstance where you would expect $dc/dt > 0$ and $\partial c/\partial t < 0$. Don't use any mathematical terms like "with respect to" or "rate of change" in your answer. Just describe what's happening on I-95 and what your car is doing.

4.214 Consider the curve defined by the equation $\sin y = \cos x$.

(a) Use implicit differentiation to find dy/dx as a function of x and y.

(b) Use the identity $\sin^2 y + \cos^2 y = 1$ and the original equation $\sin y = \cos x$ to express dy/dx as a function of x only.

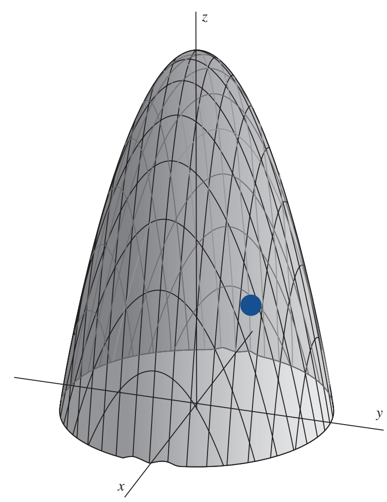
(c) Use the original equation to find $y(0)$. (There are infinitely many possibilities here—feel free to just give the simplest one.)

(d) Use your answers to Parts (b) and (c) to write the equation for this curve in a simpler form. Then confirm that your simplified equation satisfies the original relationship $\sin y = \cos x$.

4.215 The drawing shows the paraboloid $z = 50 - x^2 - y^2$ and shows the point $(4, 4, 18)$ on that paraboloid. "Which way does the gradient point at this location?" is an ill-defined question, because it depends on what function we use to represent the surface. We are going to ask this question for two different functions and find two different answers. Note that in *neither* case will the gradient point "up the hill" as many students expect.

(a) First consider the function $z(x, y) = 50 - x^2 - y^2$. Calculate the gradient of this function at the point $(4, 4, 18)$.

(b) Explain visually why the gradient pointed the way it did.



(c) If we picked a different point on this paraboloid, explain how we could determine visually (with no calculations) which way the gradient would point.

(d) Now consider the function $f(x, y, z) = x^2 + y^2 + z$, with the paraboloid representing one level surface of f. Calculate the gradient of this function at the point $(4, 4, 18)$.

(e) Explain visually why the gradient pointed the way it did.

(f) If we picked a different point on this paraboloid, explain how we could determine visually (with no calculations) which way the gradient would point.

4.216 Consider the curve $y + xe^y = 1$.

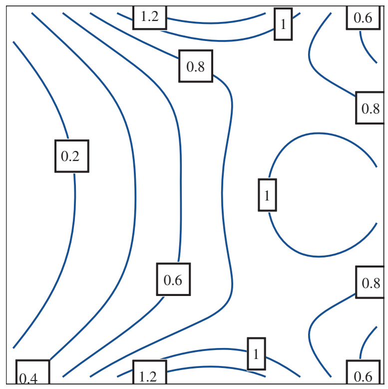
(a) Find the slope of this curve at the point $(1, 0)$.

(b) Find the gradient of the function $f(x, y) = y + xe^y$ at the point $(1, 0)$.

(c) Find the angle between the gradient vector $\vec{\nabla} f(1, 0)$ and the tangent line to the curve at that point. Explain why your answer makes sense.

(d) Find the concavity of the curve at the point $(1, 0)$.

4.217 The plot below shows the contour lines of a function $z(x, y)$. Copy this plot and add to it vectors at a variety of points (at least five, in different parts of the plot) showing the gradient at those points. Your vectors won't be precise, but they should all point in the correct direction and they should be larger in places with big gradients than they are in places with small gradients. (Pay careful attention to the numbers on the contours so you can tell where the function is increasing or decreasing.)



(If you want to print the picture from a computer instead of copying it, make a contour plot of $z(x, y) = e^{-(x-1)^2-y^2} + 0.3e^{-x^2+y^2}$ in the range $-1.2 \le x \le 1.2$, $-1.2 \le y \le 1.2$.)

4.218 Meteorologists study the "pressure gradient," the gradient of the air pressure in the atmosphere, to predict winds.

(a) Does the "pressure gradient force" point in the direction of the pressure gradient, or opposite that direction? Why?

(b) Suppose the air pressure in a local area is modeled by the equation $p(x, y, z) = x/(5 \ln z)$. Write a unit vector in the direction of the pressure gradient force at the point $(2, 2, e)$.

(c) Compute the directional derivative of $p(x, y, z)$ in the positive y-direction. Explain why your result makes sense based on the pressure function.

4.219 Consider the function $f(x, y) = x/(2x + y + 1)$.

(a) Create the tangent plane to this function at the origin, and use it to approximate $f(0.01, 0.01)$.

(b) Create the second-order Taylor series to this function at the origin, and use it to approximate $f(0.01, 0.01)$.

(c) Calculate the actual value of $f(0.01, 0.01)$. Which approximation was closer?

(d) Create the tangent plane to this function at the point $(1, 1, 1/4)$, and use it to approximate $f(1.01, 0.9)$.

(e) Create the second-order Taylor series to this function at the point $(1, 1, 1/4)$, and use it to approximate $f(1.01, 0.9)$.

(f) Calculate the actual value of $f(1.01, 0.9)$. Which approximation was closer?

Problems 4.220–4.222 deal with "linear programming," meaning optimization where the objective function and all of the constraints are linear.

4.220 You are a manager at the Nezzer Chocolate Factory, responsible for the production of hand-crafted artistic chocolate bunnies and eggs. To make a Nezzer Egg requires 5 hours of labor and \$3 dollars of combined labor and material cost, and you sell them for \$5 dollars each (a \$2 profit). A Nezzer Bunny requires 3 hours and \$5 dollars and sells for \$6 (a \$1 profit). If you make g eggs and b bunnies your profit is $P = 2g + b$. Your total budget per day is \$200, and you have 250 hours of labor available to you each day.

(a) Draw the region in the g-b-plane that is allowed by these constraints. (*Hint*: Remember to consider the least possible number of eggs and bunnies you can make.)

(b) How many bunnies and eggs should you make to maximize your profit?

(c) Now suppose the market price of bunnies goes up to \$11 (so the profit is now \$6), and all the other numbers remain the same. How many eggs and bunnies should you produce?

(d) You should have found that with the price of bunnies at \$11 you could maximize your profits by making *only* bunnies. What is the minimum market price for bunnies that makes it optimal to produce all bunnies?

4.221 Your chemical plant has three reactors that use Unobtanium to make Gloppity-Glop. Reactor 1 uses 50 kg of Unobtanium and takes 2 kilowatt-hours to make a liter of Gloppity-Glop. Reactor 2 uses 20 kg of Unobtanium and takes 5 kilowatt-hours to make one liter. Reactor 3 only uses 10 kg of Unobtanium but it takes 10 kilowatt-hours per liter. (The reactors can make fractional numbers of liters of Gloppity-Glop.) You have 1000 kg of Unobtanium and enough fuel to produce 200 kilowatt-hours of energy.

(a) You want to make as much Gloppity-Glop as you can with your resources. Write the function you are trying to maximize. It should depend on the amounts R_1, R_2, and R_3 you produce from the reactors.

(b) Write the constraints on these variables. They should all be in the form of inequalities. In addition to the constraints described in the problem, there is also the requirement that none of these variables can be less than zero. Explain why.

(c) If you were to make a plot with R_1, R_2, and R_3 as the axes the constraints would define an allowed region on that plot. Show that the maximum value of Gloppity-Glop you can produce cannot occur in the interior of that region, which means it must be on the boundary. (Because you have three variables it's probably more trouble than it's worth to try to draw this region.)

(d) To test the boundaries consider each one in turn. First, assume that you use all 1000 kg of Unobtanium. This should turn one of your constraints into an equality. Use that constraint to eliminate R_3. Your remaining constraint is still an inequality and the constraint $R_3 \geq 0$ now becomes another inequality involving R_1 and R_2. Write these constraints and draw the allowed region in the $R_1 - R_2$-plane. Find the maximum amount of Gloppity-Glop you can make using all 1000 kg of Unobtanium.

(e) Repeat the process to find the maximum amount of Gloppity-Glop you can make using all 200 kilowatt-hours.

(f) How many liters of Gloppity-Glop should you make in each reactor, and how much total Gloppity-Glop can you make?

4.222 *[This problem depends on Problem 4.221.]* All possible solutions to Problem 4.221 could be plotted as points in a 3D space of R_1, R_2, and R_3. Each of the five constraints is a plane in that space, and between them they form the boundaries of the allowed region. (They form boundaries because they are all *inequalities*. If one of the constraints had been an *equation* such as $5R_1 - 2R_2 + 3R_3 = 7$, that would have forced the solution to lie on that surface instead of lying in a region bounded by it.) Because the objective function and the constraints are all linear functions, the optimal solution must lie on a vertex point where different boundary planes intersect.

(a) One vertex point can be described like this. "At the vertex of $R_2 = 0$, $R_3 = 0$, and $50R_1 + 20R_2 + 10R_3 = 1000$, Reactor 1 is used until it consumes all the Unobtanium. It makes $R_1 = 20$ L of Gloppity-Glop using 40 kilowatt-hours." List all the other possible vertices and give a similar description for each. Note in particular which vertices represent impossible solutions—that is, solutions that violate the constraints of the problem. (The example we gave is possible.)

(b) If you went through the same process you used in Problem 4.221 with an energy limit of 20 kilowatt-hours instead of 200 kilowatt-hours, you would find that the optimal solution used all of the available energy, but did *not* use all 1000 kg of available Unobtanium. Based on your answer to Part (a), how many reactors are used in that solution? (You can go through the whole process to find the solution, but you don't need to do that to answer this question.)

(c) Suppose another reaction used two chemicals, Unobtanium and Wonderflonium, to make Gloppity-Glopp. Once again you have three reactors, each of which uses a specified amount of Unobtanium, a specified amount of Wonderflonium, and a specified amount of energy to produce a liter of Gloppity-Glopp. For this reaction there are six constraints: none of the reactor outputs can be negative and you can't exceed the available amounts of Unobtanium, Wonderflonium, or energy. If the optimal solution involved using all of the energy and all of the Wonderflonium, but not using all of the Unobtanium, how many reactors would have to be idle in that solution?

4.223 The National Weather Service uses the following formula to calculate "wind chill," meaning the effective temperature you feel on a windy day: $W = 35.74 + 0.6215T - 35.75v^{0.16} + 0.4275Tv^{0.16}$. Here T is the temperature in Fahrenheit and v is the wind speed in mph.[9] This formula is only valid for $-50 < T < 50$ and $v > 3$.

(a) Find $\partial W / \partial v$.

(b) Explain how the formula you just found shows you that this formula for W cannot possibly be accurate for temperatures above 85° F.

4.224 In special relativity the Lorentz transformations give the position and time of an event in one reference frame in terms of its position and time in another: $x' = \gamma(x - vt)$,

[9] See for example http://www.nws.noaa.gov/os/windchill/index.shtml. The authors would like to thank Courtney Lannert for suggesting this problem.

$t' = \gamma(t - vx/c^2)$. Here v is the velocity of the primed frame relative to the unprimed frame, $\gamma = 1/\sqrt{1 - v^2/c^2}$, and c is the speed of light. Suppose an object is moving at speed $u = dx/dt$ in the unprimed frame and you want to know the velocity an observer in the primed frame will measure.

(a) Using the Lorentz transformations, write expressions for dx' and dt' in terms of dx and dt. (Both v and c are constant.)

(b) Use those expressions to find dx'/dt'. Write your final answer only in terms of u, v, and c, and simplify as much as possible.

(c) The formula you just derived is sometimes called the "velocity addition" rule in special relativity. Evaluate it in the limiting cases $uv \ll c^2$ and $u = c$ and explain why your answer makes sense in both cases.

4.225 Exploration: The Earth's Equatorial Bulge If the Earth were a perfect sphere, it would create a gravitational potential of $-GM/r$ where M is the mass of the Earth and r is distance from the Earth's center $(r > R)$. The Earth actually bulges around the equator, however, so a more accurate expression for its potential is

$$U = -\frac{GM}{r} + \frac{GMR^2 J_2}{2r^3}\left(3\sin^2 l - 1\right)$$

Here R is the Earth's radius at the equator, J_2 is a measure of "ellipticity" (how far its shape is from a perfect sphere), and l is latitude: 0 at the equator and $\pi/2$ at the poles.

(a) Find a second-order Taylor series for this potential about a point on the equator $(r = R,\ l = 0)$.

(b) The gravitational force is given by $\vec{F} = -\vec{\nabla} U$, but the formula we gave for gradient in this chapter only applies to perpendicular distance variables, not angles (like latitude). To convert to a more amenable coordinate system, rewrite your Taylor series for U using the substitution $y = Rl$, where y is distance north from the equator.

(c) For $y << R$, we can treat $\hat{r}$ as a unit vector perpendicular to $\hat{y}$ and therefore write $\vec{\nabla} U = (\partial U/\partial y)\hat{y} + (\partial U/\partial r)\hat{r}$. Based on this approximation, find the gravitational force $\vec{F}(y, r)$. Simplify your answer as much as possible.

(d) For the Earth the ellipticity J_2 is roughly 10^{-3}. What direction does the Earth's gravitational field point at a latitude 5 degrees north of the equator? Express your answer as a difference between the actual direction you get from your formula for $\vec{F}$ and the direction it would be if the Earth were spherical (straight down). (*Hint*: when it's time to calculate the angle of difference, you will once again make the approximation that $\hat{r}$ is perpendicular to $\hat{y}$.)

CHAPTER 5

Integrals in Two or More Dimensions (Online)

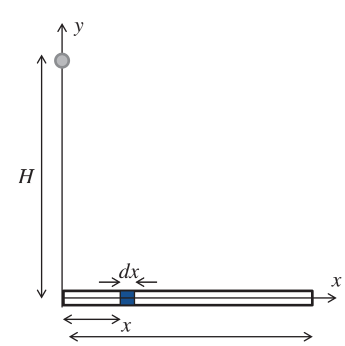
5.11 Special Application: Gravitational Forces

In the 17th century Isaac Newton proposed that every object exerts a gravitational force on every other object. The magnitude of this force is $Gm_1 m_2/r^2$ where G is a constant, m_1 and m_2 are the masses of the two objects, and r is the distance between them. The direction of the force on each object is directly toward the other one. Using this law, Newton was able to derive the orbits of the planets around the sun and the moon around the Earth, as long as he pretended that these bodies were all infinitesimal point masses. Unfortunately, they aren't. So Newton had to figure out how his new gravitational law would apply to an entire sphere.

You may be able to guess where this story is going … Newton broke the sphere up into lots of tiny pieces, calculated the force exerted by each piece, and then added them all up. In the limit where the pieces became infinitesimally small, this gave the exact answer. In short, he invented calculus.

This section will discuss the calculation of gravitational forces from extended objects. The procedure will be just like the other integration problems in this chapter with one new complication: force is a vector, so when we add the forces exerted by all the tiny pieces making up an object, we will need to add them component by component.[2] To illustrate how that works, we'll start with a much simpler problem, the gravitational field from a thin rod.

Say a uniform rod of mass M extends from the origin to the point $(L, 0)$. We want to find the gravitational force exerted by this rod on a mass m at the point $(0, H)$.



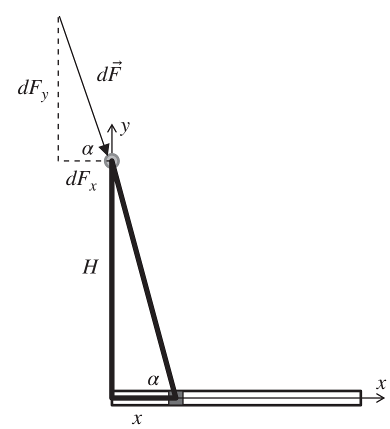
The picture above shows a slice at position x with thickness dx. The distance from the slice to the mass m is $r = \sqrt{x^2 + H^2}$. Because the rod has uniform density M/L the mass

[2]That's why people often calculate gravitational potential instead of gravitational force.

of the slice is $dm = (M/L)dx$, so the gravitational force exerted by that thin slice on the mass m is

$$dF = \frac{Gm\,dm}{r^2} = \frac{GmM}{L}\frac{dx}{x^2+H^2}$$

This is the *magnitude* of the force exerted by our dx slice. Integrating means adding up the contributions of all the slices, but to add the vectors $d\vec{F}$ we need to break them into components. The picture below shows the direction of $d\vec{F}$, with an angle labeled α, shown in two places on the figure.

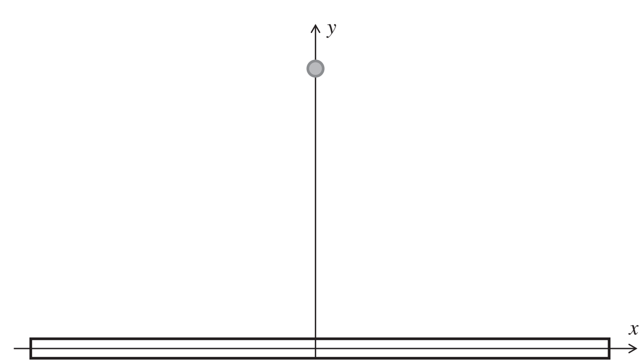


From this picture $dF_x = dF\cos\alpha = (x/r)dF$. The total x-component of the force of the rod on the mass is the sum of all the individual x-components:

$$F_x = \int_0^L \frac{Gm}{L}\frac{x}{\left(x^2+H^2\right)^{3/2}}dx = \frac{GmM}{L}\left[\frac{1}{H} - \frac{1}{\sqrt{L^2+H^2}}\right]$$

If you're wondering where that power of (3/2) came from, remember that dF has a $1/r^2$ in it and $\cos(\alpha)$ has a $(1/r)$ in it, so dF_x has $1/r^3$, which is $\left(x^2+y^2\right)^{-3/2}$. The terms in parentheses have units of one over distance, so the whole expression for F_x has units of GmM over distance squared, which is correct. We'll leave the y-component for you to work out in Problem 5.291.

In many situations you can save yourself some trouble by using the symmetry of the problem to eliminate some components. For example, suppose the rod had stretched from $(-L, 0)$ to $(L, 0)$ and the mass was still at $(0, H)$.



Just looking at the picture we can see that the mass is pulled equally to the left and right, so we can skip F_x and just calculate F_y.

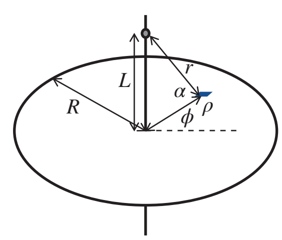
For 2D and 3D objects the process is the same, with a multiple integral to sum the force from all the differential boxes.

EXAMPLE Gravitational Force from a Circle

Question: A flat disk of radius R has density $\sigma = k\rho^2$, where ρ is distance from the center of the disk. Find the gravitational force exerted by the disk on a mass m along its central axis, a distance L from the center of the disk.

Answer: First we need to choose a coordinate system. Remember that the integration is just over the disk, so this will be a 2D integral. We choose our axes so that the disk is in the xy-plane and the z-axis goes through the center, and we choose polar coordinates.

The disk and mass are shown below, along with a small box at polar coordinates (ρ, ϕ). The mass of the small box is $dm = \sigma dA = (k\rho^2)(\rho d\rho d\phi)$ and the distance from the box to the mass is $r = \sqrt{\rho^2 + L^2}$.



By symmetry we know the force of the disk will point straight down, so we only need to find the z-component. The force points directly along the line labeled r, so $dF_z = -dF \sin\alpha = -dF(L/r)$, where α is the angle labeled in the picture. Putting everything together

$$dF_z = -\frac{Gmdm}{r^2}\frac{L}{r} = -Gmk\frac{\rho^3}{\left(\rho^2 + L^2\right)^{3/2}}d\rho d\phi$$

$$F_z = -Gmk\int_0^R\int_0^{2\pi}\frac{\rho^3}{\left(\rho^2 + L^2\right)^{3/2}}d\phi d\rho$$

The ϕ integral is trivial and just gives a factor of 2π. You can approach the ρ integral with a trig substitution, or just pop it into a computer, to get:

$$\vec{F} = -\frac{2\pi Gmk}{3}\left(2L^3 + \left(R^2 - 2L^2\right)\sqrt{R^2 + L^2}\right)\hat{k}$$

We'll leave it to you to use the density function $\sigma = k\rho^2$ to find the units of k and then check the units of this answer.

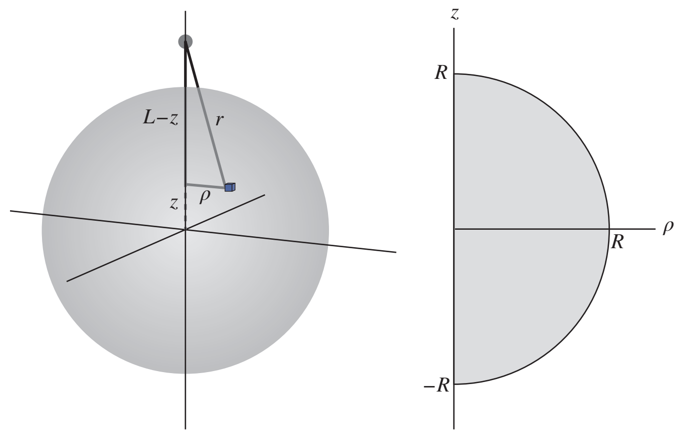
We are now at last ready to return to the problem that began, not only this section and this chapter, but in many ways the history of modern mathematics.

EXAMPLE

Newton's Problem, or The Gravitational Force from a Sphere

Question: Find the gravitational force exerted by a uniform sphere of mass M and radius R on a mass m at a distance L from the center of the sphere, where $L > R$.

Answer: For this problem it's not obvious what the right coordinate system is. The limits of integration are easy in spherical coordinates, but the distance from a box at coordinates (r, θ, ϕ) to the mass m is messy in this coordinate system. Conversely, that distance is easiest to express in Cartesian coordinates, but the limits of integration are messy. Cylindrical coordinates split the difference: medium messiness in both the distance formula and the limits of integration. We'll set it up it here in cylindrical coordinates, and you will do it in the problems in the other two coordinate systems.



We've defined the z-axis to go from the center of the sphere to the mass m, so the mass is at $(0, 0, L)$. The distance from a small box at (ρ, ϕ, z) to the mass is $r = \sqrt{\rho^2 + (L - z)^2}$. The density of the sphere is M/V where $V = (4/3)\pi R^3$, so the mass of the small box is $3M/\left(4\pi R^3\right)\rho d\rho d\phi dz$. By symmetry we only need the z-component of the force, which is given by $dF_z = -dF(L - z)/r$.

For the limits of integration, ϕ is easiest as usual: 0 to 2π. That leaves the 2D shape shown above, where the curved edge is the semicircle with the equation $\rho^2 + z^2 = R^2$. So ρ goes from 0 to $\sqrt{R^2 - z^2}$ and z from $-R$ to R. We now have all the pieces in place:

$$F_z = -\frac{3GmM}{4\pi R^3}\int_{-R}^{R}\int_0^{\sqrt{R^2-z^2}}\int_0^{2\pi}\frac{\rho(L-z)}{\left(\rho^2+(L-z)^2\right)^{3/2}}\,d\phi\,d\rho\,dz$$

After you evaluate the ϕ and ρ integrals, you end up here:

$$\frac{3GMm}{2R^3}\int_{-R}^{R}\left(\frac{L-z}{\left[R^2-z^2+(L-z)^2\right]^{\frac{1}{2}}}-1\right)dz$$

You'll use integration by parts in Problem 5.293 to show that this comes out to $-GMm/L^2$. Of course you can also check this with a computer. (In later chapters you'll learn easier ways to calculate gravitational forces.)

5.11.1 Problems: Gravitational Forces

5.282 **Walk-Through: Finding Gravitational Force.** A uniform square of mass M extends from $(-L, -L)$ to (L, L). A point mass m is at position $(0, 2L)$.

(a) Draw the square and the point mass. Include in the square a differential box. Label the position (x, y) of the box and the distance r from the box to the mass m.

(b) Find the mass dm of the tiny box and the distance r from the box to the mass.

(c) Find the magnitude of the gravitational force dF exerted by the tiny box on the mass m.

(d) Explain how we can know that the total gravitational force of the square on the mass will be in the $-y$-direction.

(e) Calculate the y-component of the gravitational force dF that you found.

(f) Set up a double integral over the square to add up the contributions from all the differential boxes.

(g) Evaluate the double integral to find the total gravitational force $\vec{F}$ exerted by the square on the mass m. Express your answer in unit vector notation and check that it has correct units.

In Problems 5.283–5.289 find the gravitational force exerted by a uniform object of mass M with the specified shape on a point mass m at the specified location. Remember to use symmetry so you don't calculate more components of $\vec{F}$ than you need to.

5.283 A thin rod of length L on a mass a distance w from the end of the rod (along the same line as the rod).

5.284 A thin rod stretching from $(-L, 0)$ to $(L, 0)$ on a mass at point $(0, H)$.

5.285 A square in the xy-plane with corners at $(-L, -L, 0)$ and $(L, L, 0)$ on a mass at the point $(0, 0, H)$.

5.286 A uniform disk of radius R on a mass along its central axis at a distance H from the center of the disk.

5.287 A disk of radius $R = 0.5$ m and mass $M = 5$ kg on a 2 kg mass a distance R from the edge of the disk (but in the same plane).

5.288 A 5000 kg cube of edge length $L = 10$ m on a 1 kg mass attached to one of its corners.

5.289 A right circular cylinder of height H and radius R on a mass along its central axis at a distance L above the bottom of the cylinder. (Assume $L > H$ so the mass m is outside the cylinder.)

5.290 A thin rod of uniformly distributed mass M stretches from $(0, 0)$ to $(0, L)$. You are going to find its gravitational force on a mass m at a point (x, y) where x and y are both larger than L. (*Hint*: Be careful not to name a variable of integration y, since y already means one of the coordinates of the mass m in this problem.)

(a) Write integrals that represents the components of the force.

(b) Calculate the y-component of the force.

(c) Calculate the x-component of the force.

(d) Calculate the magnitude of the force and take its limit as $L \to 0$. Explain why your answer makes sense.

5.291 In the Explanation we calculated the x-component of the gravitational force exerted by a uniform rod of mass M from the origin to $(L, 0)$ on a mass m at $(0, H)$. Find the y-component of that force.

5.292 A thin rod goes from $(-1, 0)$ to $(1, 0)$ with density $\lambda = e^x$. Find the gravitational force of this rod on a rod of mass 1 at position $(0, 3)$. Assume all numbers are in SI units, and just leave G in your answer.

5.293 In the example "Newton's Problem" we set up an integral to calculate the gravitational force exerted by a uniform sphere, using cylindrical coordinates. In this problem you will finish that calculation, using integration by parts.

(a) The second term in parentheses is just the number 1. Evaluate that part of the integral.

(b) Focusing on the first term in the parentheses, the denominator contains $(L - z)^2$. Expand that out and simplify as much as possible. You should end up with a linear function of z under the square root.

(c) The numerator is $L - z$. Split this into two fractions and evaluate the part where the numerator is L using u-substitution. (Remember that $L > R$ so $\sqrt{(R-L)^2} = L - R$.)

(d) The one thing you have left to integrate is of the form $z/\sqrt{a+bz}$, where a and b are functions of R and L (and therefore constants). Integrate this using integration by parts with $u = z$ and $dv = dz/\sqrt{a+bz}$.

(e) Put all your answers together to find the gravitational field from a sphere.

5.294 In the example "Newton's Problem" we set up an integral to calculate the gravitational force exerted by a uniform sphere using cylindrical coordinates. Set up triple integrals to calculate the same quantity in Cartesian and spherical coordinates. (You do not need to evaluate the integrals. Presumably you know what the result would be.)

5.295 A short, straight wire of length ds carrying a current I produces the following magnetic field at a nearby point P.

$$d\vec{B} = \frac{\mu_0 I}{4\pi}\frac{d\vec{s} \times \vec{r}}{r^3}$$

Here $d\vec{s}$ is a vector of length ds that points in the direction of the current flow and $\vec{r}$ goes from the wire to P. We use r for the magnitude of $\vec{r}$. A circular wire of radius R is in the xy-plane, centered on the origin, carrying a counterclockwise current I. Find the magnetic field this wire produces at the point $(0, 0, H)$.

5.296 Exploration: Two Extended Objects

A uniform cube mass M_1 has two opposite corners at $(0, 0, 0)$ and (L, L, L). Another uniform cube of mass M_2 has opposite corners at $(0, 0, 2L)$ and $(L, L, 3L)$. In this problem you will find the gravitational force of cube 1 on cube 2.

(a) Draw the two cubes and draw a tiny box inside each one. Remember that you can't use the same label for two different variables, so you should call the position of one box (x, y, z) and the other one (x', y', z').

(b) Use Newton's law for the gravitational force between point particles to find the force exerted by the tiny box in cube 1 on the tiny box in cube 2.

By symmetry, the total force of cube 1 on cube 2 must be in the z-direction, so we will henceforth ignore the x- and y-components.

(c) Find the z-component of the force you found in Part (b).

(d) Set up, but do not yet evaluate, an integral for the z-component of the force of all of cube 1 on all of cube 2.

(e) That sixth-order integral is hard to evaluate, even for some computer algebra programs. As it stands now you can't evaluate it numerically because the limits have L in them. Set $L = 1$ and evaluate the integral numerically. The result for F_z should still include G, M_1, and M_2. Using that result and the requirement that the answer have units of force, find F_z in terms of G, M_1, M_2, and L.

5.12 Additional Problems

5.297 Region R is the rectangle bordered by the lines $x = 5$, $y = 3$, and the coordinate axes.

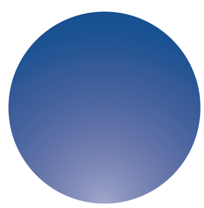
(a) If Region R has a density given by $\sigma = 10$, what is its total mass?

(b) If Region R has a density given by $\sigma = 10x$, what is its total mass?

(c) If Region R has a density given by $\sigma = 10y$, what is its total mass?

(d) If Region R has a density given by $\sigma = 10xy$, what is its total mass?

5.298 Uh-oh, my printer is running out of ink! I tried to print a solid blue circle, but the further the paper went in, the less ink I had, and the circle came out looking like this.



Glancing at it I notice that the ink is very thick on top, and gradually paler as the circle progresses. Measuring carefully I discover that the circle has a 5 inch radius and that the density of ink is given by approximately 2^h dots per square inch, where h is the height above the center of the circle in inches. (So at the top of the circle, $h = 5$; at the bottom, $h = -5$.)

(a) Calculate the density of dots at the very bottom of the circle. If the entire circle had that density, how many dots would it have?

(b) Draw a differential slice of the circle. Of course, your slice must have more or less a uniform density!

(c) Compute the area of your slice, as a function of h and dh only.

(d) Compute the number of dots in your slice.

(e) Set up an integral to represent the number of dots in the entire circle.

(f) Evaluate the integral. Give your answer rounded to the nearest integer.

5.299 Region R is bounded by the line $y = (L - x)/2$ and the x- and y-axes. Within this region is a metal whose density is given by ky^2. Both L and k are constants. Find the total mass of this metal.

For Problems 5.300–5.311 evaluate the given integrals. (Remember that unless there is a symbol you should be able to solve the problem using no technique more advanced than u-substitution or integration by parts. The first step is often choosing, or changing, the coordinate system.)

5.300 $\int_0^H \int_z^{2z} \int_{y-z}^{y+z} \left(x^2 - y^2\right)\, dx\, dy\, dz$

5.301 $\int_1^3 \int_{x^2}^9 x/(x^2 + y)^2\, dy\, dx$

5.302 $\int_{-5}^5 \int_{-\sqrt{25-y^2}}^0 (x^2 + y^2)^{3/2}\, dx\, dy$

5.303 $\int_{-H}^H \int_0^z \int_0^{\rho/z} d\phi\, d\rho\, dz$

5.304 $\int_{R_1}^{R_2} \int_0^{\cos^{-1}(r/R_2)} \int_0^{\pi} r^2 \sin\theta\, d\phi\, d\theta\, dr$

5.305 The integral of x^3 in the region in the first quadrant bounded by the lines $y = x$ and $x = 1/2$ and the curve $y = \sin(\pi x^2/2)$.

5.306 The integral of z^2 in the region bounded by the xy-plane, the cylinder $x^2 + y^2 = R^2$, and the paraboloid $z = H + \left(x^2 + y^2\right)/H$.

5.307 The integral of ρ (distance from the z-axis) over a sphere of radius R centered on the origin.

5.308 The integral of $r\cos^2\theta$ (where r and θ are the usual spherical coordinates) in the region bounded by the xy-plane and the cone $(z-1)^2 = x^2 + y^2$.

5.309 The integral of r (distance from the origin) over a cylinder of radius R and height H with its base centered on the origin.

5.310 The integral of the field $\vec{A} = 3\hat{i} + 5\hat{j}$ along…

(a) A straight line from $(0, 0)$ to $(6, 10)$.

(b) A straight line from $(0, 0)$ to $(10, -6)$.

(c) A straight line from $(0, 0)$ to $(-6, -10)$.

(d) A straight line from $(0, 0)$ to $(6, 0)$.

5.311 The integral of the field $\vec{B} = (x + y)\hat{i} + (xy)\hat{j}$ along…

(a) The curve $y = e^x$ from $x = 0$ to $x = 1$.

(b) The curve $x = \cos(2t)$, $y = 2t$ from $t = 0$ to $t = \pi$.

For Problems 5.312–5.317 indicate what region the integral is over. You can do this with a drawing, a description, or both, as long as you make it clear exactly what the region is. (Your description *cannot* simply describe the range of the coordinates. It has to use words, such as "a sphere of radius R centered on the origin" or "a cone with vertex at the origin, height H, and radius R on top.") You can tell what coordinate system each one is in by the names of the integration variables, e.g. x, y, and z for Cartesian coordinates and so on. Then evaluate the integral to find the volume of the region.

5.312 $\int_0^L \int_0^L \int_0^L dx\, dy\, dz$

5.313 $\int_{-R}^{R} \int_{-\sqrt{R^2-z^2}}^{\sqrt{R^2-z^2}} \int_{-\sqrt{R^2-y^2-z^2}}^{\sqrt{R^2-y^2-z^2}} dx\, dy\, dz$

5.314 $\int_0^H \int_0^R \int_{\pi/2}^{\pi} \rho\, d\phi\, d\rho\, dz$

5.315 $\int_{-H}^{H} \int_0^{R+kz^2} \int_0^{2\pi} \rho\, d\phi\, d\rho\, dz$

5.316 $\int_0^R \int_0^{\pi/2} \int_0^{\pi/2} r^2 \sin\theta\, d\phi\, d\theta\, dr$

5.317 $\int_0^R \int_0^{\cos^{-1}(r/R)} \int_0^{2\pi} r^2 \sin\theta\, d\phi\, d\theta\, dr$

For Problems 5.318–5.322 you will be given the shape of an object. If no mass density is specified assume the object is uniform with total mass M. If no charge density is specified assume it is uniform with total density Q. Coordinate names such as ϕ and r have their usual meanings, and all other letters represent positive constants. For each object:

(a) If a mass density is specified, find the total mass of the object.

(b) If a charge density is specified, find the total charge of the object.

(c) Find the center of mass of the object (all components).

(d) Find the moment of inertia of the object about the z-axis.

(e) Find the electric potential produced by the object at the origin.

5.318 A triangle in the xy-plane is bounded by the y-axis, the line $y = 2$, and the line $y = 2x$. It has mass density axy and charge density bxy^2.

5.319 A sphere of radius R centered on the origin with mass density $ar^2 \sin\theta$ and charge density $b/(r + R)$. (The trig identity $\sin^2\theta = (1/2)[1 - \cos(2\theta)]$ may help with the integration here.)

5.320 The $x > 0$ half of a right circular cylinder of radius R centered on the z-axis and going from $z = -H$ to $z = H$. (For the potential set up the triple integral and evaluate two of the three integrations, but leave the answer as a single integral. All of the other integrals should be straightforward to evaluate.)

5.321 A four-sided pyramid whose base is a square going from $(-L, -L, 0)$ to $(L, L, 0)$ and whose vertex is at $(0, 0, H)$. The charge density is βr where r is (as usual) distance from the origin.

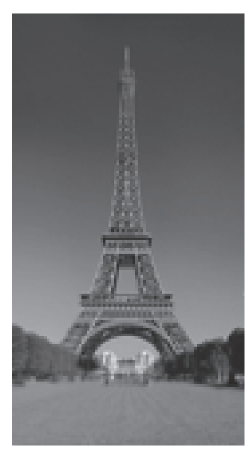
5.322 A sphere of radius R centered on the point $(0, 0, R)$.

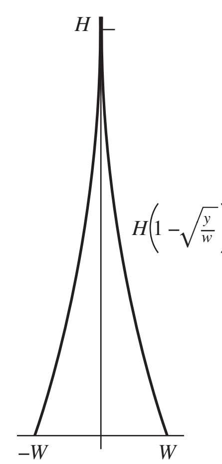
5.323 The Eiffel tower (shown below) has a square cross section at each height.

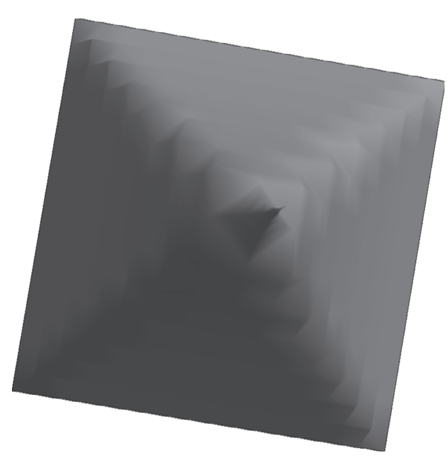
The surfaces on the sides are complicated, but can be well approximated by these two formulas.

$$\text{Front side: } z = H\left(1 - \sqrt{\frac{x}{W}}\right)$$

$$\text{Right side: } z = H\left(1 - \sqrt{\frac{y}{W}}\right)$$







The height H is about 320 m and W (which is half the width at the bottom) is about 62 m. Find the volume of the structure. (*Hint*: You'll make your life easier if you just use H and W until you get your final answer. Then you can easily check units, and then you can plug in numbers.)

5.324 The equations $x = v\cos u$, $y = v\sin u$, $z = u$ for $0 \le u \le 2\pi$ and $-1 \le v \le 1$ describe a "helicoid."

(a) To see what this shape looks like on the xy-plane, substitute $u = 0$ into the equations for x and y. (Remember that u can take any value in $[0, 2\pi]$.) Draw the resulting shape on a set of xy-axes labeled $z = 0$.

(b) Substitute $u = \pi/6$ into the equations for x and y. Draw the resulting shape on a set of xy-axes labeled $z = \pi/6$.

(c) Do a similar drawing for representative u-values between 0 and 2π.

(d) Describe and/or draw this helicoid.

(e) Evaluate the surface integral of the function $\vec{f} = y\hat{i} + x\hat{j} - (x/\cos z)\hat{k}$ through this helicoid.

5.325 *[This problem depends on Problem 5.324.]* Problem 5.324 examined one particular helicoid. A more general formula is $x = av\cos u$, $y = av\sin u$, $z = bu$. How do the constants a and b change the helicoid?

5.326 The "impulse" exerted by a constant force on an object is equal to the force multiplied by the amount of time it acts. A particle experiences an exponentially growing force $F = ke^{t/\tau}$ from time $t = 0$ to time $t = 2\tau$. Find the total impulse on the particle.

5.327 The region bounded by the curves $x^2 + y^2 = 20$ and $x = y^2$ is filled with a charge density $\sigma = x(y + 1)$. Find the total charge in this region.

(a) Sketch the two curves and the region between them.

(b) Find the x- and y-coordinates of the points of intersection of the two curves and label them on your graph.

(c) Your first impulse, looking at the region, might be to integrate over the top half only, and then double the answer to get the entire region. Explain why you cannot do so in this case.

(d) The easiest (and therefore best) way to set up this problem is with horizontal slices. The lower limit on x is given by one simple formula for all slices, and the upper limit on x is given by one other simple formula for all slices. Write the double integral this way and evaluate it to find the total charge.

(e) Vertical slices in this case require breaking the region up into two subregions. The left-hand region is bounded below by $y = -\sqrt{x}$ and above by $y = \sqrt{x}$; the right-hand region by different bounds. Set up the integrals to find and sum the total charges of these two regions separately.

(f) Evaluate the integrals of your two regions and add them. You should of course get the same answer you found with horizontal slices!

5.328 The surface of a flat rock reflects different amounts of light in different places. The surface is a circle of radius R centered on the origin and the fraction of light reflected from each point is $k\rho$, where ρ is distance from the center of the circle. If light is shining uniformly on the entire surface, what fraction of that light is reflected?

5.329 A cylindrical tank of radius R and height H is filled with air and methane gas. The fraction of the total gas mixture that is methane gas is proportional to distance from the central axis. What fraction of the gas in the cylinder is methane? (The answer will contain an unknown constant of proportionality.)

5.330 A cylinder of radius 2 m and height 1.5 m is filled with a gas. The density of the gas drops exponentially with distance from the central axis, going from 2.5 kg/m^3 on the central axis to 1/5 that density at the outer edge.

(a) Write the density of the gas in Cartesian coordinates. If you use any letters (other than the coordinates and e) you should specify what their numerical values are.

(b) Set up a Cartesian integral for the moment of inertia of the cylinder about its central axis.

(c) Evaluate the integral numerically to find the moment of inertia of the gas.

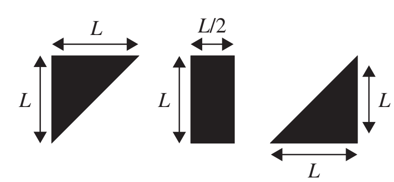
(d) Repeat Parts (a)–(c) in cylindrical coordinates. You should be able to verify that the integral appears different in the two coordinate systems, but gives the same answer.

5.331 The curve $x = \rho\cos(\omega t)$, $y = \rho\sin(\omega t)$, $z = H - vt$ is a vertical spiral. An object slides down this spiral from $t = 0$ until $t = H/v$ under the influence of gravity, which exerts a force of $\vec{F} = -mg\hat{k}$. Assume every letter in those formulas besides x, y, z, and t is a positive constant.

(a) Explain how you know that forces other than gravity must also be at work here, even though the problem does not mention them.

(b) How much work does gravity do on the object during this journey? (Even if you know the answer from physics, derive it with a line integral.)

5.332 The pressure of a fluid is the force per unit area that it exerts on its boundaries. In a tank of water the pressure increases linearly with depth. Suppose an aquarium tank has three viewing windows, two triangles and one square, as shown below. All three have the same area.[3]



(a) Which window has the greatest force on it from the water pressure? Which one has the least force on it? Explain how you know without doing any calculations.

(b) Now do the calculations to find the force on each window. Your answers will contain unspecified constants for the pressure at the top of the windows and the rate of increase of pressure with depth.

5.333 **Exploration: Gaussian Integrals**
The function $f(x) = e^{-x^2}$ is called a "Gaussian function." It has no indefinite integral in terms of elementary functions.[4] The mathematician Gauss figured out a clever trick, however, for evaluating the *definite* integral $\int_{-\infty}^{\infty} e^{-x^2}\,dx$.

We start by defining a constant that's equal to the number we are looking for: $S = \int_{-\infty}^{\infty} e^{-x^2}\,dx$. (Not much progress so far, but we have to start somewhere.) Therefore, $S^2 = \left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)\left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)$.

(a) The next step is to replace every x in the second integral with y. Explain briefly how we know that this does not change the answer.

(b) Combine the two integrals in your expression for S^2 into one double integral. (Section 3.2 explained "separable" integrals; run that process in reverse.)

(c) Use the laws of exponents to combine the two exponential functions into one.

(d) Looking at your limits of integration, what 2D region are you integrating over?

(e) Rewrite your double integral in polar coordinates. This involves rewriting the integrand in terms of ρ and ϕ and also replacing $dx\,dy$ with $\rho\,d\rho\,d\phi$. Use your answer to Part (d) to figure out the limits of integration in polar coordinates.

(f) You started with an integral that was impossible to evaluate. In polar coordinates you should now have one that is simple to evaluate. Do so and find the value of S^2. Then take the square root to get S, the number you were originally looking for.

(g) Use a computer to plot $\int_{-s}^{s} e^{-x^2}\,dx$ as a function of s, from $s = 0$ to $s = 10$. You should see the answer rapidly approach the value of S you computed in Part (f).

(h) What is $\int_0^{\infty} e^{-x^2}\,dx$? (Your answer should be exact, not approximate.)

(i) Explain why you cannot use this trick to calculate $\int_0^1 e^{-x^2}\,dx$.

[3] We'd like to thank Drew Guswa for suggesting this problem to us.

[4] Mathematicians use this phrase a lot. What it really means is that there is no normal function whose derivative is e^{-x^2}, but it comes up a lot, so people created a function whose *definition* is $g(x) = \int_0^x e^{-t^2}\,dt$. So we can't technically say that the antiderivative of the Gaussian doesn't exist, only that it cannot be expressed in terms of other functions.

CHAPTER 6

Linear Algebra I (Online)

6.10 Additional Problems

6.220 The "main diagonal" of a matrix goes from the upper left to the lower right. The "trace" of a square matrix is defined as the sum of the elements on the main diagonal. Write an expression in summation notation for the trace of an $N \times N$ matrix **M**. Your answer should be in terms of the elements M_{ij} of the matrix.

6.221 The "Kronecker delta" δ_{ij} is defined as 1 when $i = j$ and 0 when $i \neq j$. Using the Kronecker delta as the definition of the elements of a 3×3 matrix, write that matrix in standard form. You should recognize the matrix you write down.

6.222 Suppose a typical hive of honeybees contains 100,000 workers and 2000 drones, and a typical colony of carpenter ants contains 1500 workers and D_A drones.

(a) Write a matrix to convert from "number of bee hives and number of ant colonies" to "number of workers and number of drones."

(b) Write a matrix to convert from "number of workers and number of drones" to "number of bee hives and number of ant colonies."

(c) For what value of D_A is it impossible to answer Part (b)?

(d) Explain why your answer to Part (c) makes sense without making any reference to matrices or linear algebra.

6.223 Vector $\vec{A} = -\hat{\imath} - \hat{\jmath}$ and vector $\vec{B} = 2\hat{\imath}$.

(a) Do vectors $\vec{A}$ and $\vec{B}$ form a basis for all vectors on the xy-plane? How do you know?

(b) Draw the vectors $3\vec{A} - \vec{B}$ and $-\vec{A} + 3\vec{B}$.

(c) The vector $\vec{X} = a\vec{A} + b\vec{B}$ can also be represented as $x\hat{\imath} + y\hat{\jmath}$. Write a matrix to convert from a and b to x and y.

(d) Write a matrix to convert from x and y to a and b.

(e) Use your matrix from Part (d) to convert the matrix $3\hat{\imath} + 4\hat{\jmath}$ into the $\vec{A}\vec{B}$ representation.

(f) Show graphically that your answer to Part (e) does in fact add up to $3\hat{\imath} + 4\hat{\jmath}$.

6.224 In this problem you will prove that the determinant of the identity matrix **I** is 1 in any number of dimensions.

(a) Prove that $|\mathbf{I}| = 1$ for a 2×2 matrix.

(b) Prove that if $|\mathbf{I}| = 1$ for an $n \times n$ matrix, then $|\mathbf{I}| = 1$ for an $(n+1) \times (n+1)$ matrix. *Hint*: write out what **I** looks like!

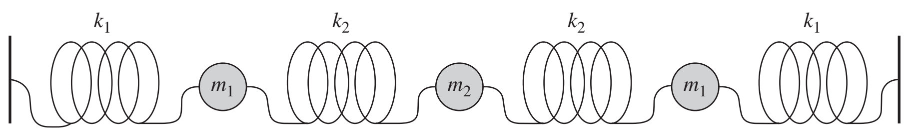
(c) Explain this result in terms of the effect this matrix has in transforming shapes.

6.225 You cannot use a matrix to convert Cartesian coordinates to spherical coordinates because the transformation is not linear. You can, however, use a matrix to convert a vector defined at a given point from the $\hat{\imath}\hat{\jmath}\hat{k}$ basis to the $\hat{r}\hat{\theta}\hat{\phi}$ basis.

(a) At the point $(1, 0, 1)$ (in Cartesian coordinates) $\hat{r} = (1/\sqrt{2})(\hat{\imath} + \hat{k})$, $\hat{\theta} = (1/\sqrt{2})(\hat{\imath} - \hat{k})$, and $\hat{\phi} = \hat{\jmath}$. Write a matrix for converting a vector defined at that point from Cartesian to spherical coordinates.

(b) Write a matrix for converting a vector defined at the point $(1, 1, 0)$ from Cartesian to spherical coordinates.

(c) Write a matrix for converting a vector defined at the point (x, y, z) from Cartesian to cylindrical coordinates. (We switched to cylindrical for the general case because it's easier than spherical.)



6.226 The image above shows three coupled oscillators. Take $m_1 = 1$ kg, $m_2 = 2$ kg, $k_1 = 3$ N/m, and $k_2 = 2$ N/m.

(a) Write differential equations for the three positions, find the normal modes of the system, and write the general solution. (This will require finding the eigenvalues and eigenvectors of a 3×3 matrix. That's normally cumbersome but in this case the characteristic equation includes one expression that you can easily factor out of every term.)

(b) One of the normal modes represents a motion in which you pull all three balls to the right (or left), pulling the middle one twice as far as the outer two, and then let go. They will then oscillate that way forever, always moving to the right and left together, with the middle one oscillating with twice as large an amplitude as the other two. Using that description as a guide, write similar descriptions for what the other two normal modes physically represent.

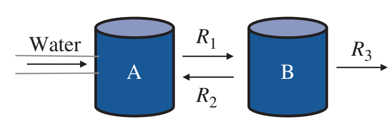
6.227 The image below shows four coupled oscillators. Take $k = 6$ N/m, $m_1 = 3$ kg, and $m_2 = 2$ kg.

You should be able to write down their equations of motion and solve them to find the normal modes (with a computer to help you find eigenvectors and eigenvalues), but in this problem we want to focus on the physical interpretation instead, so we're just going to give the eigenvectors of the transformation matrix.

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \\ -3 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ -1 \\ 1 \\ 2 \end{pmatrix}$$

The first of these eigenvectors describes a normal mode in which you pull all four balls to the right (or left), pulling the middle two slightly farther than the outer two, and then let go. They will then oscillate that way forever, always moving to the right and left together, but with the middle ones oscillating with a larger amplitude than the outer two. Using that description as a guide, write similar descriptions for what the other three eigenvectors physically represent.

6.228 Two tanks contain a mixture of water and frobscottle. Let A and B be the amounts of frobscottle in each tank, and assume the tanks are well mixed so the concentration is uniform throughout each tank (but may be different in the two tanks). Throughout the problem we will measure all amounts in millions of gallons. In those units each tank has a volume of 1. Pipes carry fluid back and forth between each tank, and away from the second tank into the river. Clean water is pumped into the first tank to keep the volume constant.



Note that A and B represent amounts of frobscottle in each tank, while the R variables represents rates at which mixed amounts of water and frobscottle are pumped out of the tanks.

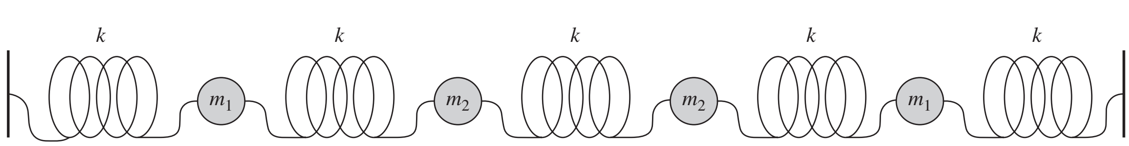
(a) Assuming the volume in each tank remains constant, express R_3 in terms of R_1 and R_2.

(b) Write a pair of coupled differential equations for A and B.

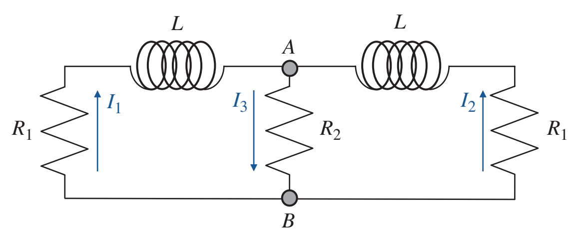
(c) Find the normal modes of the system.

(d) Let $R_1 = 9$, $R_2 = 4$, $A(0) = 3$, and $B(0) = 3$. Solve for $A(t)$ and $B(t)$.

(e) Both of the normal modes you found should be valid mathematical solutions to the equations, and you showed in the previous part that you can make physical solutions out of combinations of them, but it is not physically possible for the system to be entirely in one of the normal modes. Which one, and why not?



6.229 The picture below shows a circuit with inductors and resistors.



(a) The total current entering any given point must equal the total current leaving that point. Use that fact to write I_3 in terms of I_1 and I_2.

(b) The voltage drop across a resistor is IR, where I is the current through the resistor. The voltage drop across an inductor is $(dI/dt)L$. Write equations that express the fact that the voltage drop from A to B is the same whether you go along the left, middle, or right path. Rearrange your answers to get a pair of coupled differential equations for dI_1/dt and dI_2/dt in terms of I_1 and I_2.

(c) Find the normal modes of the system.

(d) In one of the normal modes the current on both sides of the circuit is equal, so at some point in time a current I is flowing in one direction (up or down) on the left, an equal current I is flowing in that same direction on the right, and a current $2I$ is flowing the opposite way in the middle. All of these currents die off exponentially, with the outer two always equal and the middle one always twice as large and opposite in direction. Using this description as a guide, describe the physical state represented by the other normal mode.

(e) Let $L = 10^{-7}$ H, $R_1 = 20\ \Omega$, and $R_2 = 10\ \Omega$, and assume that initially $I_1 = 3$ A and $I_2 = 2$ A. Find $I_1(t)$ and $I_2(t)$.

6.230 **Exploration: A Quantum Mechanical Well.** Many quantum mechanics problems start by solving Schrödinger's equation:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x)\psi = E\psi$$

The potential function $V(x)$ is specified in the problem (just as classical dynamics problems begin by specifying a force). In this problem you will solve Schrödinger's equation for the potential function:

$$V = \begin{cases} V_0 & x < 0 \\ 0 & 0 \le x \le a \\ V_0 & x > a \end{cases}$$

where m, $\hbar$, V_0, a and E are positive constants, and (this is very important) $E < V_0$. The value of ψ is generally complex, but in this problem make all your solutions real. The boundary condition is that $\lim_{x\to\pm\infty} \psi(x) = 0$.

(a) Begin by writing and solving Schrödinger's equation in the rightmost region. Your general solution will have two arbitrary constants, but the boundary condition at $x \to \infty$ will eliminate one of them.

(b) Repeat Part (a) for the leftmost region. Once again, your final solution will have only one arbitrary constant (but not the *same* arbitrary constant as in the first solution).

(c) Write and solve Schrödinger's equation in the middle region. This time you will be left with two arbitrary constants.

You now have four arbitrary constants: one on the left, one on the right, and two in the middle. But now we introduce a postulate of quantum mechanics: both $\psi(x)$ and $d\psi/dx$ must be continuous. So $\psi(a)$ calculated from Part (a) must agree with $\psi(a)$ calculated from Part (c) and so on.

(d) The requirement of continuity imposes four different restrictions, two on each boundary. Write the equations that represent those restrictions. *Hint*: you can save some writing if you define two new constants: $\alpha = \sqrt{2m(V_0 - E)}/\hbar$ and $\beta = \sqrt{2mE}/\hbar$.

(e) You should now have four homogeneous linear equations for four arbitrary constants. One solution is the trivial one where they are all zero, but that can't represent a physical state, so the only possible physical states are ones for which there are other solutions. Write an equation that must be satisfied in order for those nontrivial solutions to exist. This will require taking a 4×4 determinant, but it will have enough zeros in it that expansion by minors will not be too time-consuming to do by hand. Simplify the equation as much as possible (leaving it in terms of α and β). This equation describes the physically possible energies E for a particle in a finite potential well, although it cannot be analytically solved for E.

(f) At what step would your answers first have begun to look different if E were greater than V_0?

6.231 *[This problem depends on Problem 6.230.]* The mass of an electron is 9.1×10^{-31} kg and the constant $\hbar$ is 6.63×10^{-34} J·s. Consider an electron in a potential well with $V_0 = 10^{-18}$ J and $a = 10^{-8}$ m.

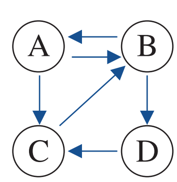
(a) Write the function $f(E)$ that must equal 0 at a physically allowed value of the energy.

(b) Plot that function from $E = 0$ to $E = V_0$. How many allowed values of energy are there for the electron in this well?

(c) Numerically find the value of the lowest allowed energy. (It must be positive.) Using these numbers your answer will come out in Joules. Convert them to the more standard particle physics energy unit of "electron volts": 1 eV $= 1.6 \times 10^{-19}$ J.

6.232 Exploration: Google PageRank

One of the techniques Google uses to select search results is "PageRank," invented by Google founders Sergey Brin and Larry Page (for whom it is named).[9] The basic idea is that each web page is given a rank based on what other pages link to it. The higher a page's rank is, the more rank it confers on other pages that it links to. To illustrate how the system works, consider the following web of four pages.



Initially each page is given a ranking of 1 over the number of pages, so in this case they each start with 1/4. In each iteration, each page distributes its current rank equally among all the other pages that it links to. For example, page A links to pages B and C, so it gives them each half of its current rank, or 1/8. Page C only links to page B, so it gives its entire current rank to B. After the first iteration, page B has a new rank of 3/8, the sum of the ranks it inherited from A and C. (It doesn't matter if page A links to page B once or twenty times; the algorithm only counts *whether* one page links to another. It also ignores any links from a page to itself.)

(a) Write the column vector $\mathbf{r}_1$ representing the rankings of the four pages after one iteration. Write the vector $\mathbf{r}_2$ giving their rankings after two iterations.

(b) Write the matrix $\mathbf{L}$ that you multiply by $\mathbf{r}_i$ to get $\mathbf{r}_{i+1}$.

(c) In the limit of infinitely many iterations, you approach a vector $\mathbf{r}$ that is no longer changing. From this you can conclude that $\mathbf{r}$ is an eigenvector of $\mathbf{L}$. What is its eigenvalue?

(d) Find the solution $\mathbf{r}$ for this particular web.

PageRank has a simple interpretation. If a user starts on a random page and randomly follows links, the rank of a given page after i iterations is the probability that he will be on that page after following i links. In practice, however, users sometimes jump to a new random page rather than following links. If d is the probability of a user following a link, and $1 - d$ is the probability of the user jumping to a new random page, then the probability of landing on the nth page after i iterations is given by:

$$r_i(n) = \frac{1-d}{N} + d \sum_{m=1}^{N} r_{i-1}(m) L_{mn}$$

Here $r_i(n)$ is the rank of page n after i iterations, and N is the total number of pages. That may sound complicated, but it's just what you did above. If a page links to seven other pages, then each iteration it gives 1/7 of its rank to each of those pages. The difference is that now it gives $d/7$ of its rank to each of those pages, and each page also receives a rank $(1-d)/N$ for the chance that the user jumped to that page randomly instead of following a link. This formula can be written in matrix form, using $\mathbf{1}$ for a column matrix where all the entries are 1.

$$\mathbf{r}_i = \left(\frac{1-d}{N}\right)\mathbf{1} + d\,\mathbf{L}\,\mathbf{r}_{i-1} \qquad (6.10.1)$$

The probability d is called the "damping factor."

(e) Show that in the case $d = 1$ Equation 6.10.1 reduces to the simpler formula you were using above. What assumption does $d = 1$ represent?

(f) Once again the steady-state solution is the one where $\mathbf{r}$ doesn't change from one iteration to the next. Write a matrix

[9] Sergey Brin and Lawrence Page. 1998. The anatomy of a large-scale hypertextual Web search engine. Comput. Netw. ISDN Syst. 30, 1–7 (April 1998), 107–117.

equation for **r** and solve it. (In this part you're considering a general web, not the particular four-page example given above.) *Big hint*: Solving for **r** will require bringing both the **r** terms to one side of the equation. To combine those two terms you'll need to insert an identity matrix **I** in front of one of them. Finally, to get **r** by itself you'll multiply both sides by the inverse of the matrix that multiplies it.

(g) What would you expect the rankings to approach in the limit $d \to 0$ and why? What would you expect them to approach in the limit $d \to 1$ and why?

6.233 *[This problem depends on Problem 6.232.]* Unless otherwise specified, everything in this problem refers to the four-page web given in Problem 6.232.

(a) Find the solution **r** for the four-page web given above, using a damping factor $d = 0.85$ (which is the value recommended by Brin and Page). How do the numbers look different from the undamped solution you found above, and why do these differences make sense in light of your answers to Part (g) of Problem 6.232?

(b) Using the simple algorithm with no damping factor, calculate the first 100 iterations of $\mathbf{r}_i$. Make a plot showing each of the four ranks as a function of i. For reference, put horizontal lines on your plot representing the steady-state values **r**.

(c) Repeat Part (b) using a damping factor $d = 0.5$. (The horizontal lines in this plot should reflect the damped solution.)

(d) Calculate the steady-state solution **r** for 100 values of d from 0 to 1. (You may find it easier to include values close to 1, but not $d = 1$ itself.) Make a plot showing the steady-state solution for each of the four ranks as a function of d. Explain what your plot looks like and why it makes sense.

6.234 Exploration: Cramer's Rule
Cramer's rule is a method for solving n linear equations with n unknowns.

$$\mathbf{M}\mathbf{x} = \mathbf{b} \tag{6.10.2}$$

Here **x** is a column vector made up of the unknowns x_1, x_2, etc., and **M** and **b** are a square matrix and a column matrix respectively. Let $\mathbf{N_i}$ be a matrix formed by replacing the i^{th} column of **M** with **b**. Cramer's rule says $x_i = |\mathbf{N}_i|/|\mathbf{M}|$.

(a) Use Cramer's rule to solve the equations $x_1 + 2x_2 = 3$, $2x_1 - x_2 = -1$. Verify that your answers work.

In the rest of the problem you'll show why Cramer's rule works. For definiteness, we'll let **M** be 3×3 and we'll derive Cramer's rule for finding x_2.

(b) If $\mathbf{M}^{-1}$ is the inverse of **M**, what is $\mathbf{M}^{-1}$ times the first column of **M**? Your answer should be a column vector.

(c) Multiply both sides of Equation 6.10.2 *on the left* with $\mathbf{M}^{-1}$ to find $\mathbf{M}^{-1}\mathbf{b}$. Once again your answer should be a column vector.

(d) Using your previous answers, what is $\mathbf{M}^{-1}\mathbf{N}_2$? Your answer, of course, should be a square matrix.

(e) Take the determinant of the square matrix you just wrote. Write your answer as an equation: $|\mathbf{M}^{-1}\mathbf{N}_2|$ equals such-and-such.

(f) Rewrite the equation you just wrote to derive Cramer's rule for this specific case. Explicitly state what properties of determinants you are using.

6.235 In Section 6.6 we asserted that only square matrices can be inverted.

(a) The definition of an inverse matrix requires that $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$ *and* $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$. Explain why it's only possible for both of these to be valid if **A** is a square matrix.

To see why this result makes sense we'll consider two non-square matrix transformations.

(b) As our first example, suppose you have 3 molecules of water (H_2O), 2 molecules of hydrogen peroxide (H_2O_2), and 4 molecules of hydrogen (H_2). How many atoms of hydrogen and oxygen do you have?

(c) Continuing with that example, now suppose you have 8 atoms of hydrogen and 4 atoms of oxygen. Write two possibilities for how many of each of those molecules you might have.

(d) Write a matrix for converting from molecules of water, hydrogen peroxide, and hydrogen to atoms of hydrogen and oxygen. Explain using your answer to Part (c) why this matrix can't be inverted.

(e) As our second example let's say you have 2 molecules of glucose ($C_6H_{12}O_6$) and 3 molecules of ethanol (C_2H_6O).[10]

[10] We really don't want to know what you have them for.

How many atoms of carbon, hydrogen, and oxygen do you have?

(f) Now suppose you have 12 atoms each of carbon, hydrogen, and oxygen. Prove that no combination of these two molecules can account for those numbers of atoms. This is true even if you allow fractional and negative numbers of molecules. *Hint*: set this up as a set of equations for the number of atoms as a function of G and E, the numbers of molecules.

(g) Write a matrix for converting from molecules of glucose and ethanol to atoms of carbon, hydrogen, and oxygen. Explain using your answer to Part (f) why this matrix can't be inverted.

(h) Explain in general why 2×3 and 3×2 matrices can't be inverted. Your answer should not be in terms of equations, but of the kinds of transformations these perform. Note that the answer is different for 2×3 and 3×2.

CHAPTER 7

Linear Algebra II (Online)

7.4 Row Reduction

Given a system of n linear equations with n unknowns, a determinant of zero tells you "this system has either no solutions or infinitely many solutions," but it doesn't tell you which situation you're in. A non-zero determinant means "this system has one unique solution," but it doesn't tell you what that solution is. Moreover, calculating determinants for very large matrices is computationally expensive. In this section we introduce a technique that fills in those holes: it tells you the solution, or it tells you that the equations are inconsistent, or it tells you that the equations are linearly dependent, and it's more efficient than finding a determinant for large systems of equations. In that sense, this section finishes the job that Section 6.7 started.

But in another sense, this section is independent of the rest of what we've been doing with linear algebra. In this section you will not multiply matrices. You will not find their inverses, determinants, or eigenvalues. The matrix manipulation rules in this section are unrelated to the rules we presented throughout Chapter 6. This is simply another way you can use grids of numbers to help you solve linear equations.

7.4.1 Explanation: Row Reduction

Solving Linear Equations by Elimination

You may have learned at some point to solve simultaneous linear equations using a technique called "elimination" or "addition and subtraction." You are always allowed to add or subtract two equations; in this technique you do so to isolate variables. We will demonstrate this process on the following three equations. The variables are x, y, and z; we use the letters A, B, and C to refer to *equations*.

$$\begin{aligned} A_0 &: \; x - y + 2z = 10 \\ B_0 &: \; 3x - 2z = 11 \\ C_0 &: \; 2x + 4y - 6z = -6 \end{aligned} \tag{7.4.1}$$

First we eliminate x from the bottom two equations by subtracting $3A_0$ from B_0 and $2A_0$ from C_0.

$$\begin{aligned} A_1 &: \; x - y + 2z = 10 \\ B_1 &: \; 3y - 8z = -19 \\ C_1 &: \; 6y - 10z = -26 \end{aligned}$$

Next we eliminate y from the third equation by subtracting $2B_1$ from it.

$$\begin{aligned} A_2 &: \; x - y + 2z = 10 \\ B_2 &: \; 3y - 8z = -19 \\ C_2 &: \; 6z = 12 \end{aligned}$$

Dividing C_2 by 6 gives $z = 2$. You can plug this into B_2 to get $y = -1$ and plug both of these into A_2 to get $x = 5$.

Row Reduction

In Chapter 6 we wrote the "matrix of coefficients" for n equations with n unknowns. That was a square matrix because it contained the coefficients of all the variables but ignored the constant terms. When we give the constants their own column, the resulting "augmented matrix" represents the entire system of equations.

$$\begin{pmatrix} 1 & -1 & 2 & 10 \\ 3 & 0 & -2 & 11 \\ 2 & 4 & -6 & -6 \end{pmatrix} \tag{7.4.2}$$

Equation 7.4.2 is perfectly equivalent to Equations 7.4.1; we just don't bother mentioning the variables because we know where they go. For instance, the row $\begin{pmatrix} 1 & -1 & 2 & 10 \end{pmatrix}$ represents the equation $x - y + 2z = 10$. Below we demonstrate "row reduction" on Equation 7.4.2; your job is to follow how this technique is line-by-line identical to the "elimination" example above. For instance, where we previously subtracted three times the first equation from the second equation, we now subtract three times the first *row* from the second row.

EXAMPLE **Row Reduction**

Question: Solve the three equations represented by Equation 7.4.2.

Solution:
Subtract three times the first row from the second row, and also subtract two times the first row from the third. This turns the first number in the last two rows into zero; that is, it eliminates x from the second and third equations.

$$\begin{pmatrix} 1 & -1 & 2 & 10 \\ 0 & 3 & -8 & -19 \\ 0 & 6 & -10 & -26 \end{pmatrix}$$

Subtract two times the second row from the third one to get 0 in the first *two* spots of the third row.

$$\begin{pmatrix} 1 & -1 & 2 & 10 \\ 0 & 3 & -8 & -19 \\ 0 & 0 & 6 & 12 \end{pmatrix}$$

Divide the third row by 6 to get (0 0 1 2), which stands for $z = 2$. Add eight times that to the second row to get $\begin{pmatrix} 0 & 3 & 0 & -3 \end{pmatrix}$ and then divide that by 3 to get $y = -1$. Finally add the second row minus two times the third row to the first one, and our augmented matrix now looks like this.

$$\begin{pmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{pmatrix} \tag{7.4.3}$$

We interpret this matrix as $x = 5$, $y = -1$, and $z = 2$: the same result we found above.

Now that we've done it, what did we do? Every step in row reduction consists of one of the three operations described below.

Allowed Operations in Row Reduction

1. Add a multiple of one row to another row.
2. Multiply a row by a non-zero constant.
3. Switch two rows.

These are called the "elementary operations" on a matrix.

(Switching rows is never necessary, but in some cases it makes the calculations easier.)

The goal is to keep performing elementary operations until the matrix looks something like Equation 7.4.3: a square identity matrix with an additional column on the right. Such a matrix gives all the answers directly, such as $x = 5$ and so on.

Row Reduction with Dependent or Inconsistent Equations

In Chapter 6 we said that if the determinant of the matrix of coefficients is non-zero, the equations have a unique solution. If the determinant is zero, the equations are either linearly dependent (infinitely many solutions) or inconsistent (no solutions). In either case row reduction provides more information.

As an example, consider the following equations:

$$\begin{aligned} x - 2y + 5z &= 2 \\ 3x + y &= 9 \\ x - 9y + 20z &= -1 \end{aligned}$$

We begin merrily row reducing.

$$\begin{pmatrix} 1 & -2 & 5 & 2 \\ 3 & 1 & 0 & 9 \\ 1 & -9 & 20 & -1 \end{pmatrix}$$ Write an augmented matrix to represent the equations.

$$\begin{pmatrix} 1 & -2 & 5 & 2 \\ 0 & 7 & -15 & 3 \\ 0 & -7 & 15 & -3 \end{pmatrix}$$ Subtract three times the first row from the second, and subtract the first row from the third.

$$\begin{pmatrix} 1 & -2 & 5 & 2 \\ 0 & 7 & -15 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$ Add the second row to the third, and… uh-oh.

We've just found that our original equations are equivalent to the three equations $x - 2y + 5z = 2$, $7y - 15z = 3$, and $0 = 0$. That last row tells us that our original equations were linearly dependent; we thought we had three constraints, but we really only had two. If we want to find the values of x, y, and z, we're going to need more information.

If we had ended up with $0 = 7$ instead of $0 = 0$, that would have told us something quite different. Zero never equals seven, even on a very bad day. Such a result would indicate that our original equations were inconsistent: they had no solution.

Rank of a Matrix

We have now seen enough about row reduction to classify a set of equations into one of three categories: "linearly dependent" (an infinite number of solutions), "inconsistent" (no solution), or "has a unique solution" (which row reduction then gives us). But there is more to know, and sometimes it's important to know it.

Suppose you start with twenty equations with twenty unknowns and discover they are linearly dependent. That means you cannot solve for all twenty variables because there is no unique solution. What you can do is specify some of the variables—either arbitrarily, or based on conditions outside your twenty equations—and then solve for the rest. But how many? Perhaps you can specify one variable and then solve for the other nineteen. On the other hand, perhaps you need to specify thirteen of the variables before you can find the remaining seven. This important distinction is captured in the "rank" of the augmented matrix.

Definition and Use: Rank of a Matrix

The "rank" of a matrix is its number of linearly independent rows.

If an augmented matrix of rank R represents a set of equations, you can solve those equations to find R of the variables in terms of the remaining variables.

In our above example, if your augmented matrix turned out to be rank twelve, you could solve for twelve of the variables in terms of the remaining eight.

Now you know what to do with the rank of a matrix, but how do you find it? The answer is, once again, row reduction. You keep manipulating until the matrix is in "row echelon form."

Definition and Use: Row Echelon Form

The "leading zeroes" in a matrix row are the zeroes on the left before any non-zero term. For instance the row (0 0 5 0) has two leading zeroes.

To see if a matrix is in "row echelon form" start at the top and look at each row in succession.

1. Each row must have more leading zeroes than the previous row until you reach a row that is all zeroes.
2. If you do reach a row that is all zeroes, all the rows below it must also have all zeroes. (In other words the all-zero rows are clustered at the bottom.)

When a matrix is in row echelon form, the number of rows that are *not* all zeroes is the rank of the matrix.

In the particular case of n equations with n unknowns, the augmented matrix has dimensions $n \times (n+1)$. In row echelon form each row must have more leading zeroes than the previous row, so the bottom row must have at least $n-1$ leading zeroes. If that bottom row does not consist of all zeroes then your matrix is rank n and your equations are not linearly dependent. (They may or may not be inconsistent.)

EXAMPLE **Rank of a Matrix**

$$\mathbf{A} = \begin{pmatrix} 1 & 5 & -3 & 2 & 0 & 7 \\ 0 & 2 & \pi & 4 & 9 & 0 \\ 0 & -6 & -3\pi & -12 & -27 & 0 \\ 0 & 0 & 0 & 4 & 8 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} 1 & 5 & -3 & 2 & 0 & 7 \\ 0 & 2 & \pi & 4 & 9 & 0 \\ 0 & 0 & 0 & -12 & -27 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Matrix **A** is not in row echelon form, because the third row does not have more leading zeroes than the second. More row reduction is required before you can determine the rank of this matrix.

Matrix **B** is in row echelon form. It has three non-zero rows, indicating a rank three matrix. If this matrix represents a set of five equations, you could solve for three of the variables in terms of the other two.

Summary: Rank of a Matrix

What does the rank of a matrix tell you about a system of equations? Before we answer that, remember that there are two ways to represent a system of equations in a matrix. The "matrix of coefficients" includes all the terms that involve the variables. The "augmented matrix" adds an additional column for the constant terms in the equations. We will consider both matrices in the following discussion—don't get them confused!

For n equations with n unknowns here are the possibilities.

I. *Row reduction produces a unique solution.*
- The equations are linearly independent and consistent.
- The matrix of coefficients has a non-zero determinant.
- The matrix of coefficients and the augmented matrix are both of rank n.

II. *Row reduction produces a row with all zeroes except the last column.*
- The equations are inconsistent (no solution).
- The matrix of coefficients has a zero determinant.
- The rank of the matrix of coefficients is less than n.
- The rank of the augmented matrix is larger than the rank of the matrix of coefficients.

III. *Row reduction produces m rows with all zeroes (but none with all zeroes except the last column).*
- The equations are linearly dependent (infinitely many solutions). Specifically, only $n - m$ of the equations are linearly independent.
- The matrix of coefficients has a zero determinant.
- The matrix of coefficients and the augmented matrix both have rank $n - m$.
- You need to specify m of the variables before you can solve for the remaining $n - m$ of them.

What if the augmented matrix has a lower rank than the matrix of coefficients? Take a moment to convince yourself that it's impossible; if this did happen it would mean that you had a matrix with all linearly independent rows, but adding a column to it made those rows linearly dependent.

Finally, these results can be generalized to n linear equations with u unknowns. Let M be the matrix of coefficients and A the augmented matrix, and let R_M and R_A be their ranks. The only possibilities are:

- If $R_A > R_M$ the equations are inconsistent.
- If $R_A = R_M = u$ the equations have one unique solution.

- If $R_A = R_M < u$ the equations have infinitely many solutions and can only be solved for R_A of the unknowns in terms of the remaining ones.

In the last case listed above the R_A unknowns that you solve for are called the "dependent variables" and the remaining ones that you have to specify are called the "free variables." You're guaranteed in this case that there is at least one set of R_A variables that you can designate as dependent, but you cannot always do that for any set of R_A variables that you choose.

7.4.2 Problems: Row Reduction

7.85 Walk-Through: Row Reduction. Consider the following equations:

$$\begin{aligned} 2x + 3y + z + t &= 2 \\ -2x - y + 2z + 4t &= -21 \\ 8x + 20y + 7z + 4t &= 10 \\ 2x + 15y + 10z + 11t &= -34 \end{aligned}$$

(a) Write the augmented matrix for this set of equations. (This will be a 4×5 matrix.)

(b) Replace the second row with the sum of the first and second rows.

(c) Replace the third row with the third row minus four times the first row.

(d) The previous two steps eliminated the x coefficient from the second and third equations. Take a similar step to eliminate the x coefficient from the fourth equation.

(e) You have now used the first row of the matrix to make the first column 0 in all the *other* rows. In a similar way, use the (new) second row of the matrix to make the second column zero in the third and fourth rows.

(f) Use the third row to make the third column zero in the fourth row.

(g) Explain how you can determine that your matrix is now in row echelon form.

(h) Based on your result, identify the rank of the original matrix. Identify the original equations as inconsistent, linearly dependent, or having a unique solution. If they are linearly dependent, specify how many variables you could solve for. If there is a unique solution, find it.

7.86 *[This problem depends on Problem 7.85.]* For these two variations of Problem 7.85, determine if the equations are linearly independent (and find the rank of the matrix), inconsistent, or have a unique solution (and find it).

(a) Solve the system of equations from Problem 7.85 changing the constant term on the right of the first equation from 2 to 1.

(b) Solve the system of equations from Problem 7.85 again, with the constant term in the first equation back to its original 2, but this time changing the coefficient of t in the fourth equation from 11 to 24.

For Problems 7.87–7.98 row reduce the augmented matrix for the given set of equations. Give the rank of the augmented matrix and say whether the equations are inconsistent, linearly dependent, or neither. If they are linearly dependent, how many variables could you solve for? Give the unique solution to the equations if there is one.

7.87 $x + y = 3,\ x - y = 2$

7.88 $x + 2y + 3z = 0,\ x - 2y - 3z = 0,\ 2x + 4y = 0$

7.89 $x + 2y + 3z = 0,\ x - 2y - 3z = 0,\ 3x - 2y - 3z = 0$

7.90 $x + y + z = 2,\ x - 2y + 3z = -1,\ 3x - 3y + 4z = 3$

7.91 $x - 2y + z = 3,\ x + z = 4,\ x + 4y + z = 2$

7.92 $x + 2y - z = 2,\ -2x - 4y + 2z = -4,\ 3x + 6y - 3z = 6$

7.93 $2x - y + 3z + t = -2,\ 6x + y + 4z - 2t = -30,\ 4x + 6y + 3t = 36,\ -10x + y - 2z - 4t = 2$

7.94 $x - z + 2t = -1,\ x - y + z - t = 3,\ 3x - 2z = 4,\ 2x - 3y - z + t = 1$

7.95 $x + y + 6z + 7t = 21,\ 4x - 5y + 8z + t = -22,\ 9x + 8z + t = -17,\ x + y + 6z + 7t = 21$

7.96 $4a - 7c + 4d = -11,\ 4a + 7b - 6c + 6d = 22,\ 4a + 7b + 9c - 4d = 17,\ 8a + 7b + 2c = 9$

7.97 $-a - 3b - c + 6d = 2,\ 9a + 6b - 3c - 8d = -22,\ 2a + 6b - 3c + 5d = 25,\ 7a - 5d = -31$

7.98 $a - b + c - d = 1,\ 2a - b - c + 2d = -2,\ 3a - 2b + d = -1,\ 4a - 3b + c = 0$

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[5] To preserve his anonymity we will refer to him only as "Dad."

7.99 *True story*: A chemical engineering professor we know[5] was once creating a homework problem involving a combustion reactor in which methane (CH_4) and ethane (C_2H_6) react with oxygen (O_2) to form carbon monoxide (CO), carbon dioxide (CO_2), and water (H_2O).

$$CH_4 + \frac{3}{2}O_2 \rightarrow CO + 2H_2O$$
$$CH_4 + 2O_2 \rightarrow CO_2 + 2H_2O$$
$$C_2H_6 + \frac{5}{2}O_2 \rightarrow 2CO + 3H_2O$$
$$C_2H_6 + \frac{7}{2}O_2 \rightarrow 2CO_2 + 3H_2O$$

This reaction involves six molecular species. The initial concentrations of these species would be given as part of the problem. Of the six final concentrations, some would be given as "measured values" and the students would calculate the rest. According to the method he was using, the number of measured values he would have to give equals the number of species minus the number of reactions. In this example he would specify two final concentrations (6–4) and the students would calculate the other four. But his calculations did not lead to consistent answers. In this problem you will see why, and how many final concentrations he needed to specify in order to make his problem work.

(a) Let A represent the final concentration of CH_4, B the final concentration of C_2H_6, and C, D, E, and F the final concentrations of O_2, CO, CO_2, and H_2O, respectively. Write an algebraic equation to represent each chemical reaction by replacing each chemical species with the symbol for its concentration and the arrow with an equal sign. For example, the first reaction equation would be $A + (3/2)C = D + 2F$.

(b) Consider A and B to be "constants" and the remaining concentrations as variables. (We could just as well have chosen any other two variables.) Rearrange your equations into standard form, with the variables on the left and the constants on the right. Simplify the equations so that all coefficients are integers.

(c) Suppose you determined, using either a determinant or row reduction, that your four equations have a unique solution. What would that tell you about the reaction? That is, what would you measure and what could you then calculate?

(d) Write an augmented matrix for your four equations and reduce it to row echelon form.

(e) Mathematically—forget about chemistry for a moment—what does your final matrix tell you about the equations you started with?

(f) Now let's return to our chemical engineer. He will have to measure the concentrations of some of his species, and then he will be able to calculate the rest. How many does he need to measure, and how many can he then calculate?

(g) We found in this case that the rows of the augmented matrix were linearly dependent, and that in turn told us something about the reaction. What different lesson would we draw if the rows of the augmented matrix were inconsistent?

7.100 *[This problem depends on Problem 7.99.]* The "water gas shift" process can be described in terms of the following three reactions.

$$H_2O + CO \rightarrow CO_2 + H_2$$
$$H_2O + H \rightarrow H_2 + OH$$
$$OH + CO \rightarrow CO_2 + H$$

Assuming the initial concentrations of all six species are given, how many of the final concentrations would you need to measure before the rest would be determined?

7.101 *[This problem depends on Problem 7.99.]* An experiment involves the following five reactions. The initial concentrations of all six species are known. You are going to measure as many final concentrations as you have to, and then calculate the remaining concentrations based on those measurements. How many concentrations do you have to measure, and how many can you then calculate?

$$2N_2O_5 \rightarrow 2N_2O_4 + O_2$$
$$N_2O_5 \rightarrow NO_2 + NO_3$$
$$NO_3 \rightarrow O_2 + NO$$
$$NO + N_2O_5 \rightarrow 3NO_2$$
$$2NO_2 \rightarrow N_2O_4$$

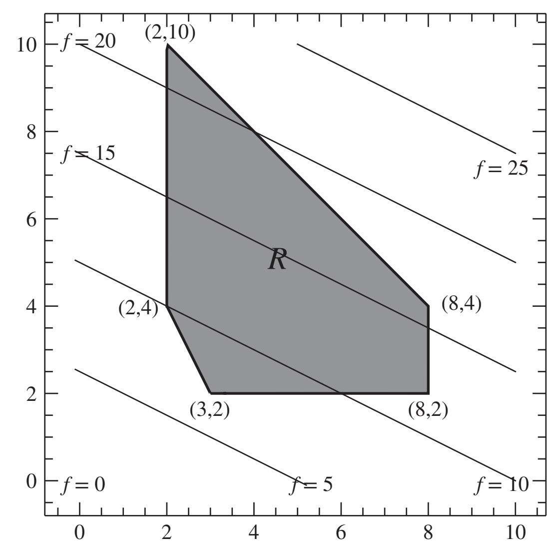
You may have a computer do the row reduction; your job is to set up the matrix and then interpret the results.

7.5 Linear Programming and the Simplex Method

"Linear Programming" means optimizing a linear function subject to linear constraints. In principle this is about as easy as an optimization problem can get, but many problems involve so many variables that brute force methods are impractical. In this section we describe the "simplex algorithm," a variant of row reduction that can handle problems with thousands of variables or more.

7.5.1 Discovery Exercise: Linear Programming and the Simplex Method

The picture shows a closed region R bounded by five lines. (The term "closed" indicates that the boundary lines are part of the region.) The picture also shows five contour lines of the function $f(x, y) = x + 2y$.



1. What point in region R has the largest possible value of $f(x, y)$?
2. What point in region R has the smallest possible value of $f(x, y)$?

See Check Yourself #47 in Appendix L

3. Redraw region R and then draw in four contour lines of the function $g(x, y) = y - x$.
4. What point in region R has the largest value of g? The smallest value?

You should have found that all four extrema occurred at the vertices of region R. When we know in advance that this is going to be true, we can search for minima and maxima by looking only at the vertices.

5. Let $h(x) = y + 2x$. The minimum of $h(x)$ within region R occurs at an infinite number of points. What points are those? Could we still find the maximum and minimum if we only looked at the vertices?
6. In many optimization problems the maximum or minimum is found inside the region, or along a border. Why did all the extrema in this exercise occur at the vertices? *Hint*: Your answer will involve a particular property of the functions $f(x)$, $g(x)$, and $h(x)$, and also a property of the region R.

7.5.2 Explanation: Linear Programming and the Simplex Method

Chapter 4 presented two different methods of optimization, one based on the gradient and one on Lagrange multipliers. These techniques find critical points, either in the interior of the allowed region or on its boundaries. If you want to find a global maximum or minimum you evaluate the function at all of those critical points and at the endpoints of the boundaries.

In this section we will consider linear functions with linear constraints. There are never any critical points, so all you need to do is check the endpoints of the boundaries—the vertices. But when the problem involves hundreds or thousands of variables, an efficient algorithm for optimizing a function without checking every vertex becomes essential. Of course our sample problems will involve only a few variables. But if you imagine scheduling airplane flights or laying out a factory floor, you can see the importance of highly scalable algorithms.

A Sample Problem: Two Machines

Your company has two machines. The cheap machine produces 2 items per hour and costs \$4 per hour to run. The fancy machine produces 4 items per hour and costs \$6 per hour to run. You can't run either machine for more than 40 hours per week and your total budget for running the machines is \$300. How many hours per week should you run each machine in order to maximize the number of items you produce?

As in any optimization problem you have an "objective function" that you want to maximize (or minimize) subject to constraints on the variables. In this case the objective function is the number of items you produce per week. The constraints include the three stated explicitly above, plus the implicit constraint that you can't run a machine for a negative amount of time.

$$\begin{aligned} \text{objective function: } & f(F,C) = 4F + 2C \\ \text{constraints: } & F \le 40; \quad C \le 40; \quad 6F + 4C \le 300; \quad F, C \ge 0 \end{aligned} \tag{7.5.1}$$

What makes this a "linear programming" problem is that the objective function and the constraints are all based on linear functions. In this problem the constraints are all inequalities but they can also be equations as long as they are linear.

Very small problems can often be solved by inspection. In this example the fancy machine is more efficient so you'll want to run it for 40 hours and use the rest of your budget running the cheap machine. But things get complicated once you throw in maintenance costs, shipping times, different types of workers, raw materials needed, and so on. In this section we describe the "simplex algorithm," the most commonly used method for solving linear programming problems.

A Picture and Some Words

The independent variables in Equations 7.5.1 represent machine hours, but we can think of them as coordinates that define a two-dimensional space. The five constraints define a region in that space.

The Discovery Exercise (Section 7.5.1) was designed to convince you of the following fact.

Very Important Fact

If your objective is a linear function, and your constraints are linear functions that define a closed bounded region, then the minimum and maximum values of your objective function within the region occur at vertices of the boundary.

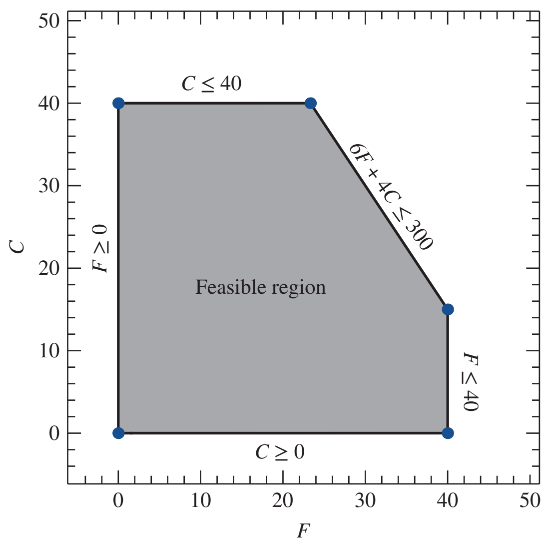


FIGURE 7.5 The "feasible region" for the machine problem. The blue dots at the vertices indicate "basic feasible solutions."

A "feasible solution" of Equations 7.5.1 is any point in the shaded region of Figure 7.5; that is, any combination of the independent variables that satisfies all the constraints in the problem. A "basic feasible solution" is any vertex of the boundary of the region.

The important fact above suggests an optimization technique based on basic feasible solutions. Find all the vertices—that is, all the points of intersection of the constraints—and evaluate the objective function at each one. The highest value you find is the maximum value of the function within the feasible region.

That's absolutely true, but it's not practical. Consider the region bounded by the constraints $x + 2y + 10z \leq 20$, $8x + y + 4z \leq 12$, $2x + 6y - z \leq 6$, $3y + z \leq 3$, $x \geq 0$, $y \geq 0$, $z \geq 0$. We'll get you started by telling you that one of the vertices is the origin. Now you find the rest.

We trust you see the difficulty. With three variables each constraint is a plane, the feasible region is a polyhedron, and each vertex is the intersection of three constraints. With a hundred variables and a thousand constraints … well, we need a systematic approach. Each step of the simplex method moves from one vertex to another vertex at which the objective function is higher. In cases where the origin is one of the vertices, you start there and step from vertex to vertex until you reach the maximum. In cases where the origin is not one of the vertices we add a "first phase" that finds a vertex to use as a starting point.

Before we present the method, we need to make a few more points about constraints.

- A constraint that is an *inequality* forms a boundary of the feasible region, and there is no limit to how many you can have. For instance, the feasible region for two variables with 17 inequality constraints would be a 17-sided polygon.
- A constraint that is an *equation* reduces the dimensionality of the problem by one. If you have three variables related by a linear equation, the feasible region lies on the plane defined by that equation. You could if you wished solve that equation to eliminate one variable and thus have a two-variable problem with only inequality constraints.
- If your objective function is parallel to a constraint then the maximum or minimum may occur along that constraint. (Imagine trying to minimize the function $f(F, C) = C$ in Figure 7.5.) It still must be true that the extremum occurs at a vertex, but it can occur at more than one vertex and everywhere on the boundary connecting them.

- If there is no combination of variables that satisfies the constraints the problem is "infeasible." (Imagine trying to satisfy $y < x + 3$ and $y > 2x + 7$, remembering that both x and y must be positive.) If there is a feasible region but the function has no maximum or minimum within it then the problem is "unbounded," which can only occur if the feasible region is infinite. (See Problem 7.118.) All this looks simple in a 2D picture, but faced with a set of inequalities it may not be obvious if they bound a finite region, an infinite region, or no region at all. The simplex method, in addition to finding maxima and minima, helps us properly classify the region defined by the constraints.

One Problem Written in Four Ways

The method we present here involves a gradual process of rewriting the initial problem. The objective function and the constraints will generally go through four distinct forms, two of which have names.

1. The original form in which they are presented (such as Equations 7.5.1 above).
2. The "normal form."
3. The "restricted normal form."
4. And finally, the form that we actually want.

We're going to begin at the end, so you know what we're looking for. The following is a maximization problem presented in Form 4. (This is a new problem; we'll come back to our machinery example later.)

$$\begin{array}{ccccccccc} & & & & f(x_1, x_2, x_3, x_4) = -2x_3 - 7x_4 + 20 & & & & \\ x_1 & + & & + & 7x_3 & + & 13x_4 & = & 20 \\ & & x_2 & - & 2x_3 & + & x_4 & = & 8 \\ & & & & & & x_1, x_2, x_3, x_4 \ge 0 & & \end{array} \tag{7.5.2}$$

Once your problem looks like Equations 7.5.2, you're done. Here's why.

The last constraint says none of the variables can be negative. Based on that you should be able to convince yourself that this particular objective function cannot possibly be bigger than 20. So if the objective function can reach 20 exactly—that is, if $x_3 = x_4 = 0$ is compatible with the other constraints—then we have our maximum. Plug that in and the first two constraints immediately become $x_1 = 20$ and $x_2 = 8$, both of which satisfy the last constraint, so we have a solution.

Our approach to any linear programming problem will be to make it look like Equations 7.5.2 and then read off the solution as we did above. So before we proceed, we urge you to consider the following question: what are the defining properties of Equations 7.5.2? That is, what characteristics of the objective function and the constraints enabled us to see the answer with no real calculations? Jot down your answers. A little later in this explanation we'll give you our answers and you can see how yours compare.

You may also want to consider how these properties would change if we were minimizing instead of maximizing. From here on we will only consider maximization problems, but the same technique applies with minor modifications to minimization. See Problem 7.119.

Normal Form and Restricted Normal Form

The first step in the simplex method is to write the problem in "normal form."[6]

[6]Many sources define "normal form" without the constant c_{n+1} in the objective function. We include it because in the course of the simplex solution such a constant will get added to f even it wasn't there to start with.

Definition: Normal Form of a Linear Programming Problem

A linear programming problem written in "normal form" asks you to maximize an objective function of n variables, $f = c_1x_1 + c_2x_2 + c_3x_3 + \ldots + c_nx_n + c_{n+1}$, subject to $x_i \geq 0$ and the following m linear constraint equations.

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n &= b_2 \\
&\ldots \\
a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \ldots + a_{mn}x_n &= b_m
\end{aligned}$$

All the constants b_i must be non-negative.

Not every linear programming problem looks like the normal form described above, but every linear programming can be *put* into normal form with three tricks.

- Rewrite inequality constraints as equations by introducing extra variables. For example, $6F + 4C \leq 300$ becomes $6F + 4C + S = 300$, $S \geq 0$. S is called a "slack variable" because it takes up the slack between the left and right sides of the original inequality. If $6F + 4C \geq 300$ write $6F + 4C - S = 300$, $S \geq 0$; this S is called a "surplus variable."
- If any of the b_i are negative, multiply that constraint equation by -1.
- Finally, you might have a variable x that doesn't have the constraint $x \geq 0$. If it has a different limit you can fix this easily by replacing it with a new variable. If $x \geq -7$ let $y = x + 7$ or if $x \leq 0$ let $y = -x$. Rewrite your problem in terms of y instead of x and you have the constraint $y \geq 0$. If x isn't constrained replace it with two variables: $x = y - z$. Now $y, z \geq 0$ covers all possible values of x. See Problem 7.120.

Those rules leave an ambiguity if you have a constraint such as $x \geq 2$. Because x is already constrained to be positive you could write $x - S = 2$. Our recommendation, however, is to define a new variable $y = x - 2$ which gives you only one variable (y) and only one constraint ($y \geq 0$) to work with.

For the machine example (Equations 7.5.1) we only need the first of these tricks to get to normal form.

$$\begin{aligned}
\text{objective function: } & f(F, C) = 4F + 2C \\
\text{constraints: } & F + S_1 = 40; \quad C + S_2 = 40; \quad 6F + 4C + S_3 = 300; \quad F, C \geq 0
\end{aligned} \tag{7.5.3}$$

You probably won't be shocked to hear that it is conventional to write these equations without the variables. The augmented matrix for a linear programming problem in normal form is called a "simplex tableau." Notice that we pull the variables in the equation for f to the left side to make them line up with the other equations, so they change sign.

$$\begin{array}{c}
\begin{array}{ccccccc} f & F & C & S_1 & S_2 & S_3 & \end{array} \\
\begin{pmatrix}
1 & -4 & -2 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 40 \\
0 & 0 & 1 & 0 & 1 & 0 & 40 \\
0 & 6 & 4 & 0 & 0 & 1 & 300
\end{pmatrix}
\end{array} \tag{7.5.4}$$

Switching from a list of equations to a matrix is not a change in form—it's just a different way of writing the same form. Given our list of four forms above you might expect our next step to be converting the machine example from "normal form" to "restricted normal form." In this case, however, we are already in restricted normal form. Let's see what that means.

Definition: Restricted Normal Form of a Linear Programming Problem

A "basic variable" is one that has a positive coefficient in one of the constraint equations and zero coefficient in the objective function and all the other constraints. A linear programming problem is in "restricted normal form" if it is in normal form (as defined above) and every constraint has at least one basic variable in it.

Restricted normal form categorizes the variables into two types. In Equation 7.5.4 the basic variables are S_1, S_2, and S_3. (You can easily identify a basic variable by looking for a column with one positive entry and the rest zero.) The other variables, which generally make up the objective function, are "non-basic." In Equation 7.5.4 these are F and C.

When the tableau is in restricted normal form you can immediately read off one solution by setting all the non-basic variables to zero. This particular tableau suggests $F = C = 0$, $S_1 = S_2 = 40$, and $S_3 = 300$. The corresponding value of the objective function comes from the first row, and in this case is $f = 0$.

It is a true but non-obvious result that the solution represented by a tableau in restricted normal form is a basic feasible solution, meaning it corresponds to a vertex of the feasible region.

But we do not yet have an optimal solution. The difference between Equation 7.5.4 ("restricted normal form") and Equations 7.5.2 ("what we actually want") is that the coefficients of the non-basic variables in the first row are negative. That means setting them to zero does not maximize the objective function. (Remember that we moved those coefficients to the left side of the equation. Equation 7.5.2 was optimal because the coefficients in the objective function were all negative, but that means they would appear as positive numbers in the first row of the simplex tableau.)

We can now fully answer our own question "What are the defining properties of Equations 7.5.2?" The answer is all the requirements for normal form, plus the added requirements for restricted normal form, plus one more: all the coefficients in the first row of the tableau must be zero or positive. When you have your equations in that form, you have your solution.

You may recall that as soon as we converted this particular problem into normal form it was in restricted normal form. In this Explanation we will only consider problems of that type. In Section 7.5.3 we will take up the question of how to convert normal form to restricted normal form in cases where you have to do so manually. But for now, we will continue with the machine problem.

Finding the Optimal Solution

Equation 7.5.4 represents one vertex of the feasible region, but it is not the right vertex. The simplex method gives us a way of stepping from this vertex to a better one. Each such step brings us closer to the goal until we reach the maximum value of the objective function.

Here is another way of expressing the same goal. Equation 7.5.4 is in restricted normal form, but it has negative variables in the top row. The simplex method gives us a way of replacing one such variable with a zero, while a different variable in the top row goes from zero to a positive number. Each such step improves our tableau until it reaches the form that we want.

Either way you look at it, each step follows the rules of row reduction.

Look at the second column in Equation 7.5.4, the F. You see a -4 at the top (which is why we want to change this), a 1, a 0, and a 6. We are going to turn F into a basic variable, meaning it will have a non-zero coefficient in only one constraint. That constraint (called the "pivot") will be the second row. We want to make all other entries in the F column go to zero. So we add four times the second row to the first row, and we subtract six times the second row from the fourth row.

$$\begin{array}{ccccccc} f & F & C & S_1 & S_2 & S_3 & \\ \end{array}\qquad\qquad \begin{array}{ccccccc} f & F & C & S_1 & S_2 & S_3 & \\ \end{array}$$

$$\begin{pmatrix} 1 & -4 & -2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 40 \\ 0 & 0 & 1 & 0 & 1 & 0 & 40 \\ 0 & 6 & 4 & 0 & 0 & 1 & 300 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -2 & 4 & 0 & 0 & 160 \\ 0 & 1 & 0 & 1 & 0 & 0 & 40 \\ 0 & 0 & 1 & 0 & 1 & 0 & 40 \\ 0 & 0 & 4 & -6 & 0 & 1 & 60 \end{pmatrix} \qquad (7.5.5)$$

Make sure you followed the two row operations we used! Then look at the results. F has become a basic variable, while S_1 has become non-basic. We are still in restricted normal form so the entire tableau still represents a solution, which is now $f = 160$. That's an improvement over our previous solution, but the -2 at the top of the C column means it's still not optimal.

But before we fix that last problem, let's look at two decisions we made in turning Equation 7.5.4 into Equation 7.5.5.

- *Why did we get rid of F first instead of C?* It was mostly arbitrary. We could have chosen C instead, and we would have eventually reached the same solution we're going to reach this way. It is possible in principle, however, for the simplex method to enter an infinite loop, and you can avoid this with "Bland's rule": always choose the leftmost column with a negative coefficient in the first row.[7] The cycling problem almost never arises in practice and there are algorithms for choosing a column that will move you to the optimal solution as quickly as possible, but Bland's rule provides a good simple approach.
- *Why did we choose the second row as the pivot?* That is not arbitrary! We could have used the fourth row to eliminate F coefficients from all other rows, but we would have ended up with a negative number on the right side of the second row; we would no longer be in normal form. (Try it!) Avoid this problem with the "minimum ratio rule." Having chosen F as the pivot column, look at all the constraints that have positive entries in the F column (ignoring those with zero or negative entries). For each such constraint calculate the ratio of the rightmost coefficient (the constant) to the F coefficient. The pivot is the row with the smallest ratio. (If two rows are tied for smallest Bland's rule says to choose the topmost one.) In Equation 7.5.4 the second row had a ratio of $40/1 = 40$ and the fourth row had a ratio of $300/6 = 50$, so we had to use the second.

To finish the problem, we pick up from Equation 7.5.5. Since C still has a negative coefficient on top we do a pivot somewhere on that column. The ratio for the third row is $40/1 = 40$ and for the fourth row $60/4 = 15$, so we pivot about the fourth row.

$$\begin{array}{ccccccc} f & F & C & S_1 & S_2 & S_3 & \\ \end{array}\qquad\qquad \begin{array}{ccccccc} f & F & C & S_1 & S_2 & S_3 & \\ \end{array}$$

$$\begin{pmatrix} 1 & 0 & -2 & 4 & 0 & 0 & 160 \\ 0 & 1 & 0 & 1 & 0 & 0 & 40 \\ 0 & 0 & 1 & 0 & 1 & 0 & 40 \\ 0 & 0 & 4 & -6 & 0 & 1 & 60 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1/2 & 190 \\ 0 & 1 & 0 & 1 & 0 & 0 & 40 \\ 0 & 0 & 0 & 3/2 & 1 & -1/4 & 25 \\ 0 & 0 & 4 & -6 & 0 & 1 & 60 \end{pmatrix} \qquad (7.5.6)$$

The basic variables are now F, C, and S_2. Since there are no negative numbers in the first row, we've found the optimal solution. Setting the non-basic variables to zero gives the solution $F = 40$, $C = 15$, and $f = 190$. In other words, you should run the fancy machine for 40 hours and use the rest of your budget running the cheap machine. (We expected that, remember?)

Suppose there is a variable whose coefficient in the top row is negative and it doesn't have *any* positive coefficients in the constraints? You'll show in Problem 7.121 that this implies the problem is unbounded.

[7]Bland, Robert G. (May 1977). "New finite pivoting rules for the simplex method". Mathematics of Operations Research 2 (2): 103–107.

EXAMPLE The Simplex Method

Problem:
Maximize $f = 2x_1 + 3x_2 - x_3$ subject to $x_1 + 3x_2 - x_3 \leq 6$, $2x_1 + x_2 + 2x_3 \leq 4$, $x_1, x_2, x_3 \geq 0$.

Answer:
First convert the inequalities to equations using slack variables: $x_1 + 3x_2 - x_3 + S_1 = 6$, $2x_1 + x_2 + 2x_3 + S_2 = 4$. Next write the initial simplex tableau. Remember that the coefficients in f are being pulled to the left of the equation, so they all switch sign.

$$\begin{array}{c} \begin{array}{ccccccc} f & x_1 & x_2 & x_3 & S_1 & S_2 & \end{array} \\ \begin{pmatrix} 1 & -2 & -3 & 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & -1 & 1 & 0 & 6 \\ 0 & 2 & 1 & 2 & 0 & 1 & 4 \end{pmatrix} \end{array}$$

This is in restricted normal form with S_1 and S_2 as basic variables. The x_3 coefficient in the top row is already positive, so we need to do something about the x_1 and x_2 entries; following Bland's rule we choose x_1. (If you want the practice you could redo the problem starting with x_2 and verify that you get to the same final answer.) The ratios of the constant to the x_1 coefficients are $6/1 = 6$ and $4/2 = 2$, so we use row 3 as the pivot. Add the third row to the first row and subtract 1/2 of the third row from the second row.

$$\begin{array}{c} \begin{array}{ccccccc} f & x_1 & x_2 & x_3 & S_1 & S_2 & \end{array} \\ \begin{pmatrix} 1 & 0 & -2 & 3 & 0 & 1 & 4 \\ 0 & 0 & 5/2 & -2 & 1 & -1/2 & 4 \\ 0 & 2 & 1 & 2 & 0 & 1 & 4 \end{pmatrix} \end{array}$$

The next pivot column is x_2. The ratios are $4/(5/2) = 8/5$ and $4/1 = 4$, so the second row has the smallest one this time. Add 4/5 of the second row to the first one and subtract 2/5 of the second row from the third one.

$$\begin{array}{c} \begin{array}{ccccccc} f & x_1 & x_2 & x_3 & S_1 & S_2 & \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 7/5 & 4/5 & 3/5 & 36/5 \\ 0 & 0 & 5/2 & -2 & 1 & -1/2 & 4 \\ 0 & 2 & 0 & 14/5 & -2/5 & 6/5 & 12/5 \end{pmatrix} \end{array}$$

Since all of the coefficients in the first row are positive we've reached the maximum value: $f = 36/5$. This value is obtained by setting all of the non-basic variables to zero, which turns the bottom two rows into $(5/2)x_2 = 4$ and $2x_1 = 12/5$. So the maximum value occurs at $x_1 = 6/5$, $x_2 = 8/5$, $x_3 = 0$.

Stepping Back

The simplex method begins with a linear optimization problem and consists of the following steps to solve it.

1. Use the tricks described above to ensure that every independent variable has a constraint of the form $x_i \geq 0$.
2. Add slack and/or surplus variables to any inequalities to turn them into equations.

3. If any constraint equations have negative constant terms, multiply them by −1. Your problem is now in normal form.
4. Move the variables in the objective function to the left side and write the objective function and all constraints in an augmented matrix, a.k.a. a "simplex tableau."
5. *If your problem isn't in restricted normal form use the "two-phase" method described below to fix that. If the problem is infeasible then it will not be in restricted normal form and the two-phase method will fail in a way we describe below.*
6. Find the solution.
   (a) If none of the variables have negative coefficients in the first row you've found the solution. Skip to Step 7. If there is a variable with a negative coefficient in the first row and no positive coefficients below it the problem is unbounded, and you should stop.
   (b) Choose the leftmost negative coefficient in the first row to change into a basic variable.
   (c) Consider all the positive coefficients in the column for that variable and use the minimum ratio rule to select which one to pivot about. In case of a tie use the topmost of the possible ones as your pivot.
   (d) Use the techniques of row reduction to set all of the coefficients in your pivot column except the pivot itself to zero. This will make that variable basic, and it should make one of the other variables non-basic. (Occasionally this step will also make another variable basic as well. You can simply proceed as usual when this happens.)
   (e) Return to Step 6a.
7. Once the problem is in restricted normal form with no negative coefficients in the first row, you're done. Set all the non-basic variables to zero and read off the values of the basic variables and the objective function.

There are a couple of mistakes to watch out for in this process.

- Remember that you have to bring all the non-constant terms in the objective function to the left side of the equation before you write the tableau, so their signs will flip.
- A tableau is not in restricted normal form just because each constraint has a variable that doesn't appear in any other constraint. You also have to get the coefficient of that variable in the objective function equal to zero.

To minimize a function with the simplex method you have two options. First, you can take the negative of your objective function and maximize that. Alternatively, you can use the simplex method exactly as described here except that you pivot around columns with *positive* entries in the first row, and you're done when all the non-basic variables have negative entries there. See Problem 7.119.

At this point you know how to do basic simplex problems. We encourage you to practice with a few problems, starting with Problem 7.108, and get comfortable with the process. Then come back and read Section 7.5.3 dealing with the "two-phase" simplex method.

7.5.3 Explanation: The Two-Phase Simplex Method

One thing you may not have noticed about our examples so far is that all the constraints (other than the ubiquitous "no negative variables") were of the form $a_1x_1 + a_2x_2 + \ldots \leq b$ where $b > 0$. It's not hard to convince yourself in such a situation that the origin must be one of the vertices. The simplex method begins its vertex-hopping from there.

But replace one of those constraints with $a_1x_1 + a_2x_2 + \ldots \geq b$ or $a_1x_1 + a_2x_2 + \ldots = b$ (still with $b > 0$) and the origin is no longer in the feasible region. That's a problem because the simplex method is designed to move from one vertex to another, so where does it start? Here we discuss a technique you can use to find one vertex—that is, one basic feasible solution. From there you can begin the simplex method.

Here is an alternative way of expressing the same idea. In our previous examples we put our problems into normal form and found that they were also in restricted normal form. Here we discuss the extra step that is required when that doesn't happen.

Remember that in restricted normal form every constraint contains at least one basic variable. That variable's coefficient is positive in that constraint and zero in all other equations. For instance, if your problem has four constraints then you need to rewrite it with at least four basic variables.

One tool for accomplishing that is the standard row reduction operations. Choose any four variables and attempt to make them basic by zeroing out all but one of their coefficients. If you end up with positive numbers in all the right places then your problem is in the right form. If not, choose four different variables and try again. Eventually you will find a feasible solution (or exhaust all possibilities and thus prove that none exists). But this needle-in-a-haystack approach is not practical with large numbers of variables.

We therefore introduce a new trick. This trick is designed to find one basic feasible solution: that is, to put a tableau into restricted normal form. That is the first phase of the "two-phase simplex method" and the second phase then proceeds as we described above.

Consider the following example.

$$\begin{aligned} \text{objective function: } & f(x_1, x_2) = 2x_1 - 3x_2 \\ \text{constraints: } & x_1 + 2x_2 \le 2; \quad x_1 + x_2 \ge 1; \quad x_1, x_2 \ge 0 \end{aligned} \tag{7.5.7}$$

We can put this in normal form by introducing a slack variable in the first constraint and a surplus variable in the second one: $x_1 + 2x_2 + S_1 = 2$, $x_1 + x_2 - S_2 = 1$. We can then write a simplex tableau.

$$\begin{array}{cccccc} f & x_1 & x_2 & S_1 & S_2 & \\ \end{array}$$
$$\begin{pmatrix} 1 & -2 & 3 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 2 \\ 0 & 1 & 1 & 0 & -1 & 1 \end{pmatrix}$$

This looks almost like restricted normal form, but it's not because the S_2 entry is negative. This tableau therefore does not represent a feasible solution: if we set $x_1 = x_2 = 0$ we get $S_2 = -1$, which is not allowed. We need to rewrite this tableau to have two non-negative basic variables.

We begin by defining an entirely new problem, complete with new variables and a new objective function.

1. Add one new variable to each constraint. This example has two constraints so we add z_1 to the first and z_2 to the second. Visually this means adding a new z_1 column (with a 1 in the first constraint and 0 in all other rows), and a new z_2 column (1 in the second constraint and 0 in all other rows).
2. Define a new objective function $f_2 = -z_1 - z_2$. Visually this means replacing the top row with a new row with 1s in the new columns.

So here is the new problem we will solve.

$$\begin{aligned} \text{objective function: } & f_2 = -z_1 - z_2 \\ \text{constraints: } & x_1 + 2x_2 + S_1 + z_1 = 2; \quad x_1 + x_2 - S_2 + z_2 = 1; \quad x_1, x_2, S_1, S_2, z_1, z_2 \ge 0 \end{aligned}$$

We are going to use the simplex method, just as we presented it before, to solve this new problem. But what will that buy us? Optimizing the f_2 that we just made up doesn't optimize the f_1 in the problem. But we'll see below that the solution to this new problem is a basic feasible solution to the original problem, which we can use as a starting point for solving that problem.

In this example our simplex tableau starts here.

$$\begin{array}{c}\begin{array}{ccccccc} f_2 & x_1 & x_2 & S_1 & S_2 & z_1 & z_2 \end{array}\\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 1 & 0 & 2 \\ 0 & 1 & 1 & 0 & -1 & 0 & 1 & 1 \end{pmatrix}\end{array}$$

We begin by making both z variables basic, so we subtract the second row from the first and then subtract the third row from the first.

$$\begin{array}{c}\begin{array}{ccccccc} f_2 & x_1 & x_2 & S_1 & S_2 & z_1 & z_2 \end{array}\\ \begin{pmatrix} 1 & -2 & -3 & -1 & 1 & 0 & 0 & -3 \\ 0 & 1 & 2 & 1 & 0 & 1 & 0 & 2 \\ 0 & 1 & 1 & 0 & -1 & 0 & 1 & 1 \end{pmatrix}\end{array}$$

We are now in restricted normal form (for this problem, not the original problem). To get rid of the $-2x_1$ we pivot about the third row, because it has the lowest ratio.

$$\begin{array}{c}\begin{array}{ccccccc} f_2 & x_1 & x_2 & S_1 & S_2 & z_1 & z_2 \end{array}\\ \begin{pmatrix} 1 & 0 & -1 & -1 & -1 & 0 & 2 & -1 \\ 0 & 0 & 1 & 1 & 1 & 1 & -1 & 1 \\ 0 & 1 & 1 & 0 & -1 & 0 & 1 & 1 \end{pmatrix}\end{array}$$

Now get rid of the $-x_2$, pivoting about the second row. (Both ratios are the same so we chose the topmost of them.)

$$\begin{array}{c}\begin{array}{ccccccc} f_2 & x_1 & x_2 & S_1 & S_2 & z_1 & z_2 \end{array}\\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & -1 & 1 \\ 0 & 1 & 0 & -1 & -2 & -1 & 2 & 0 \end{pmatrix}\end{array} \qquad (7.5.8)$$

The first phase is now over. We find that f_2 reaches a maximum value of zero when $z_1 = z_2 = 0$. (Given that we defined $f_2 = -z_1 - z_2$ for non-negative z-values, this result was predictable from the outset.) But we're now going to return to the original problem, undoing the two steps that we used to go from that problem to this new one.

1. Eliminate both columns that represent the z variables.
2. Replace the top f_2 row with the original f_1 row from the problem.

$$\begin{array}{c}\begin{array}{ccccc} f & x_1 & x_2 & S_1 & S_2 \end{array}\\ \begin{pmatrix} 1 & -2 & 3 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & -1 & -2 & 0 \end{pmatrix}\end{array} \qquad (7.5.9)$$

The point of the first phase was to rewrite the constraints (all rows but the top one). Equation 7.5.8 represented a valid solution to the f_2 constraints with $z_1 = z_2 = 0$, so Equation 7.5.9 must represent a valid solution to the original constraints. And Equation 7.5.8 had two basic variables, so they should still be basic in our new f_1 tableau… right?

Not quite. x_1 and x_2 appear in only one constraint each, but they are no longer basic variables because they also appear in the objective function. So you now have to take an extra step to make them zero in the top row. Both of your humble authors have vivid memories of forgetting this step when we first learned the simplex method, so we particularly caution you to look for it! We will add twice the third row to the first row, and subtract three times the second row from the first row, and then we will really be in restricted normal form.

$$\begin{array}{cccccc} f & x_1 & x_2 & S_1 & S_2 & \\ \end{array}$$
$$\begin{pmatrix} 1 & 0 & 0 & -5 & -7 & -3 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & -1 & -2 & 0 \end{pmatrix}$$

The tableau is now in restricted normal form and the standard simplex method can begin. Pivot about the second row in the S_1 column, then pivot about the second row in the S_2 column.

$$\begin{array}{cccccc} f & x_1 & x_2 & S_1 & S_2 & \\ \end{array} \qquad \begin{array}{cccccc} f & x_1 & x_2 & S_1 & S_2 & \\ \end{array}$$
$$\begin{pmatrix} 1 & 0 & 5 & 0 & -2 & 2 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & -1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 7 & 2 & 0 & 4 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 & 0 & 2 \end{pmatrix}$$

The optimal value of f is 4, and it occurs at $x_1 = 2$, $x_2 = 0$.

EXAMPLE **The Two-Phase Simplex Method**

Problem:
Maximize $f = 4x_1 - 2x_2$ subject to $x_1 + x_2 \le 5$, $3x_1 + x_2 = 6$, $x_1, x_2 \ge 0$.

Answer:
Before you look at our solution, we encourage you to graph this problem on the $x_1 x_2$-plane and figure out the correct answer.

We add a slack variable to the first constraint to turn it into $x_1 + x_2 + S_1 = 5$, and we get the following simplex tableau.

$$\begin{array}{ccccc} f & x_1 & x_2 & S_1 & \\ \end{array}$$
$$\begin{pmatrix} 1 & -4 & 2 & 0 & 0 \\ 0 & 1 & 1 & 1 & 5 \\ 0 & 3 & 1 & 0 & 6 \end{pmatrix}$$

This is not in restricted normal form; it needs one more basic variable. You could try to get another one with row reduction. In this case if you tried with x_1 it would work and if you tried with x_2 it wouldn't. (Try!) With many more variables, however, it would be prohibitive to find a good set of basic variables by trial and error, so we'll use the two-phase simplex method.

Phase 1: Add a new variable to each constraint and maximize the function $f_2 = -z_1 - z_2$.

$$\begin{array}{ccccccc} f_2 & x_1 & x_2 & S_1 & z_1 & z_2 & \\ \end{array}$$
$$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 5 \\ 0 & 3 & 1 & 0 & 0 & 1 & 6 \end{pmatrix}$$

The z_i are the basic variables, so to get this into reduced normal form we need to get their coefficients in the first row to zero. To do that we subtract the sum of the lower two rows from the first one.

$$\begin{array}{ccccccc} f_2 & x_1 & x_2 & S_1 & z_1 & z_2 & \end{array}$$
$$\left(\begin{array}{ccccccc} 1 & -4 & -2 & -1 & 0 & 0 & -11 \\ 0 & 1 & 1 & 1 & 1 & 0 & 5 \\ 0 & 3 & 1 & 0 & 0 & 1 & 6 \end{array}\right)$$

After that we proceed with the usual simplex method. We pivot about the bottom row in the x_1 column, and then we pivot about the middle row in the x_2 column.

$$\begin{array}{ccccccc} f_2 & x_1 & x_2 & S_1 & z_1 & z_2 & \end{array} \qquad \begin{array}{ccccccc} f_2 & x_1 & x_2 & S_1 & z_1 & z_2 & \end{array}$$
$$\left(\begin{array}{ccccccc} 1 & 0 & -2/3 & -1 & 0 & 4/3 & -3 \\ 0 & 0 & 2/3 & 1 & 1 & -1/3 & 3 \\ 0 & 3 & 1 & 0 & 0 & 1 & 6 \end{array}\right), \left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 2/3 & 1 & 1 & -1/3 & 3 \\ 0 & 3 & 0 & -3/2 & -3/2 & 3/2 & 3/2 \end{array}\right)$$

The first phase is over. The optimum solution $f_2 = 0$ occurs when $z_1 = z_2 = 0$, as we hoped. We could also read off the values of the other variables, but we don't care since this isn't actually the problem we wanted to solve. What we care about is that we've turned two of the original variables into basic variables. So we continue…

Phase 2: Toss out the z_i columns and put the original objective function back in.

$$\begin{array}{ccccc} f & x_1 & x_2 & S_1 & \end{array}$$
$$\left(\begin{array}{ccccc} 1 & -4 & 2 & 0 & 0 \\ 0 & 0 & 2/3 & 1 & 3 \\ 0 & 3 & 0 & -3/2 & 3/2 \end{array}\right)$$

Now comes the step we warned you about; we are not in restricted normal form! The variables x_1 and x_2 are no longer basic until we zero out their coefficients in the top row. That requires subtracting 3 times row 2 and adding 4/3 times row 3.

$$\begin{array}{ccccc} f & x_1 & x_2 & S_1 & \end{array}$$
$$\left(\begin{array}{ccccc} 1 & 0 & 0 & -5 & -7 \\ 0 & 0 & 2/3 & 1 & 3 \\ 0 & 3 & 0 & -3/2 & 3/2 \end{array}\right)$$

The pivot column is the -5 on top, and it only has one positive entry below it so we pivot about that 1.

$$\begin{array}{ccccc} f & x_1 & x_2 & S_1 & \end{array}$$
$$\left(\begin{array}{ccccc} 1 & 0 & 10/3 & 0 & 8 \\ 0 & 0 & 2/3 & 1 & 3 \\ 0 & 3 & 1 & 0 & 6 \end{array}\right)$$

And at last we're done. The optimal solution is $f = 8$ and it occurs at $x_1 = 2$, $x_2 = 0$.

Stepping Back

In the examples we solved in Section 7.5.2, all of the constraints were of the "less than or equal" form. Algebraically that meant that every constraint was written with a slack variable and the initial simplex tableau started in restricted normal form, with those slack variables as the initial set of basic variables. That initial basic feasible solution corresponded to setting all of the x_i equal to zero. In other words the initial vertex was the origin, and the simplex method proceeded from there.

In this section we looked at problems where the constraints included equations or "greater than" inequalities, which often means the origin is not in the feasible region. There's no obvious starting vertex, which shows up algebraically in the fact that the initial simplex tableau isn't in restricted normal form. The two-phase simplex method finds you a vertex to start from.

The outline of this process always looks the same.

1. Add one new variable z_i to each constraint. Define a new objective function f_2 that is -1 times the sum of all these new variables.
2. Create the simplex tableau for your new problem. Add the sum of all the constraint rows to the first row so that the new z variables are all basic.
3. Use the simplex method to maximize f_2.
4. Assuming all the z_i are non-basic, remove the z columns and put the original objective function back into the first row.
5. You should now have one variable in each constraint that doesn't appear in any other constraint. Use row reduction techniques to set the first-row coefficients of those variables to zero. You are now in restricted normal form.
6. Use the simplex method to maximize f.

If the first phase doesn't have an optimal solution where the z_i are all non-basic, that means the original problem was infeasible.

7.5.4 Problems: Linear Programming and the Simplex Method

7.102 Equation 7.5.6 represents the final tableau for a linear optimization problem. Convert this matrix into an objective function and a set of constraints. Then explain using words and equations how you can find the optimal solution for the problem so expressed, and how you know your solution is optimal. *Hint*: we provide a similar explanation for the different problem represented by Equations 7.5.2.

For Problems 7.103–7.105 sketch the feasible region in the x_1x_2-plane. Add contour lines of the objective function to your sketch and use them to predict the vertex where f will be maximized. Then calculate f at each vertex and verify your answer.

7.103 $f = x_1 + x_2,\ 2x_1 + x_2 \le 2$

7.104 $f = 2x_1 + 3x_2,\ x_1 + 2x_2 \le 4,\ x_1 + x_2 \le 3$

7.105 $f = x_1 - 4x_2,\ x_1 + 2x_2 \ge 4,\ x_1 + x_2 \le 4$

7.106 In the Explanation (Section 7.5.2) we claimed that a tableau in restricted normal form always corresponds to a vertex of the feasible region. To see why that's true consider our machine example, with the constraints $C \le 40$, $F \le 40$, $4C + 6F \le 300$. The feasible region for this problem is shown in Figure 7.5. Each boundary of this region corresponds to one of the variables in Equations 7.5.3 being equal to zero. For example, the bottom edge has $F = 0$ and the rightmost edge has $S_1 = 0$.

(a) Copy this sketch.

(b) For each vertex of the feasible region, identify which two variables equal zero.

(c) Equation 7.5.4 represents a basic feasible solution to the machine problem. It tells us that if $F = C = 0$—the lower left-hand corner of Figure 7.5—then $f = 0$. Equation 7.5.5 steps from that tableau to a different tableau. Which vertex does that represent, and what solution? Equation 7.5.6 steps to a third tableau; which vertex does that represent, and what solution?

7.107 Suppose a linear programming problem started with two variables x_1 and x_2 and 7 inequalities, not counting $x_1, x_2 \ge 0$. Including the slack and surplus variables, the problem in normal form would have 9 variables.

(a) At each vertex, how many of those variables would equal zero? *Hint*: consider what a sketch of the feasible region would look like *before* introducing slack and surplus variables.

(b) If you had a simplex tableau in restricted normal form for this problem, how many basic (non-zero) variables would it have? How many non-basic variables would it have?

You should have concluded that a vertex of the original problem corresponds to the same condition as a tableau in restricted normal form. Now consider a problem that has three variables with $x_1, x_2, x_3 \geq 0$, plus an additional m constraints. Once again each boundary would correspond to one variable being zero, but now each boundary is a plane.

(c) How many variables equal zero at each vertex? *Hint*: you can answer this easily if you're good at picturing in 3D, but if you're not remember that each boundary is a linear equation in the original three variables. How many of those equations must be simultaneously satisfied to define a point?

(d) How many basic and non-basic variables would a tableau in restricted normal form for this problem have?

Once again you should have concluded that a tableau in restricted normal form corresponds to a set of variable values that occurs at a vertex of the original problem.

7.108 Walk-Through: The Simplex Method. Maximize the function $f = x_1 + 2x_2$ subject to the constraints $x_1 - x_2 \leq 4$, $x_1 + 3x_2 \leq 6$, and $x_1, x_2 \geq 0$.

(a) Write the constraints in normal form by adding a slack variable to the first two inequalities.

(b) Write the initial simplex tableau. It will have one row for the objective function and two rows for the constraints. (The non-negativity conditions are assumed, and don't appear in the tableau.) Remember that the coefficients in f switch signs when you bring them to the left side of the equation.

(c) You'll use the first negative entry in the first row as your pivot column. According to the minimum ratio rule, which row in that column should you use for your pivot?

(d) Pivot about that spot and write the resulting tableau.

(e) What spot (column and row) should you use as your next pivot? How do you know?

(f) Pivot about that spot and write the resulting tableau.

(g) Your tableau should now indicate that the problem is solved. What feature(s) of the tableau let you know this? (If it doesn't indicate that the problem is solved you've made a mistake. Go back and find it.)

(h) What is the maximum value of f and the values of x_1 and x_2 at which it occurs?

7.109 *[This problem depends on Problem 7.108.]* Sketch the feasible region defined by the constraints in Problem 7.108. Add contours of f to your plot, and use those contours to explain how you could know which vertex the maximum of f is on. Verify that your answer matches your answer to Problem 7.108.

For Problems 7.110–7.116 find the maximum value of the function f subject to the given constraints. Assume in each case that all of the independent variables are constrained to be non-negative. You may find it helpful to first work through Problem 7.108 as a model.

For the problems with two variables you can check your answers by drawing the feasible region and using contours of f to find the optimal vertex. We don't recommend that technique for three-variable problems unless you have extremely good 3D drawing skills, and even then we wouldn't recommend it for 4D problems.[8]

7.110 $f = 3x_1 - 2x_2$, $x_1 + x_2 \leq 4$

7.111 $f = -5x_1 + 2x_2$, $x_1 - x_2 \leq 4$, $-x_1 + 2x_2 \leq 2$

7.112 $f = 2x_1 + x_2$, $x_1 - 3x_2 \leq 4$, $-x_1 + 2x_2 \leq 2$

7.113 $f = -2x_1 + x_2$, $x_1 - 3x_2 \leq 4$, $-x_1 + 2x_2 \leq 2$

7.114 $f = x_1 + 3x_2$, $2x_1 - x_2 \leq 3$, $-x_1 + 2x_2 \leq 1$, $x_1 + 2x_2 \leq 2$

7.115 $f = 2x_1 - x_2 + x_3$, $x_1 + x_2 + x_3 \leq 1$, $x_1 + 2x_2 + 3x_3 \leq 2$

7.116 $f = x_1 + 3x_4$, $x_1 + x_2 + x_3 + x_4 \leq 1$, $x_1 - 2x_2 - x_3 + 2x_4 \leq 1$

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[8] But see http://www.felderbooks.com/papers/4dplots.html if you really want to.

7.117 Maximize $f = x_1 - x_2$ subject to the constraints $x_1 + 3x_2 \le 10$, $x_1 \ge 0$, $x_2 \ge 2$. (The last constraint makes this a bit different from the problems in the block above.)

7.118 If all the variables in a linear programming problem are bounded above and below so the feasible region is finite, the problem is guaranteed to be bounded. The converse is not true; you can have a bounded problem even if some variables are unbounded. Consider $x_1 - x_2 \le 2$, $-x_1 + x_2 \le 1$, $x_1, x_2 \ge 0$.

(a) Sketch the feasible region. Explain how your sketch shows that x_1 and x_2 are not bounded from above.

(b) Add contour lines of $f = x_1 - 2x_2$ to your sketch and use them to predict whether f has a maximum in the feasible region, and if so where.

(c) Use the simplex method to maximize f subject to these constraints and verify that the maximum occurs at the vertex you predicted.

(d) Copy your sketch of the feasible region and add contour lines of $g = x_1 + 2x_2$. Predict whether g has a maximum in the feasible region and if so where, and then use the simplex method to check your prediction.

7.119 To minimize a function you can use the simplex method exactly as we've described it, except that you keep pivoting until there are no *positive* coefficients in the top row. Use this method to minimize the function $f = x_1 - x_2$ subject to $x_1 + 2x_2 \le 5$, $2x_1 - x_2 \le 3$, and of course $x_1, x_2 \ge 0$.

7.120 Maximize the function $f = x_1 - 2x_2$ subject to the constraints $x_1 + x_2 \le 2$, $x_1 - x_2 \le 1$, $x_1 \ge 0$. This looks a lot like the problems we've been solving but there's no explicit constraint on x_2. Because this is such a small problem you could draw the feasible region and figure out the limits on x_2, but for large problems it's not always easy to figure out if a variable is bounded, never mind what its limits are. The systematic way to do this is to define $x_2 = y - z$, subject to $y, z \ge 0$. Rewrite the objective function and constraints in terms of y and z and then use the simplex method to maximize f. *Hint*: At the end you may have to make some arbitrary-looking decisions about y and z. Remember that what really matters is what they imply about x_2.

7.121 Consider the following simplex tableau in restricted normal form.

$$\begin{array}{ccccccc} f & x_1 & x_2 & S_1 & S_2 & S_3 & \\ \end{array}$$
$$\begin{pmatrix} 1 & 2 & -1 & 0 & 0 & 0 & 2 \\ 0 & 3 & -2 & 1 & 0 & 0 & 3 \\ 0 & -2 & -3 & 0 & 1 & 0 & 3 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

This does not represent an optimal solution because the coefficient of x_2 in f is negative. But there is no row in that column you can use as a pivot because none of the coefficients are positive. We claimed in the Explanation (Section 7.5.2) that this means the problem is unbounded, but we didn't explain why.

(a) Instead of setting $x_1 = x_2 = 0$ set $x_1 = 0$, $x_2 = 1$. Find the values of S_1, S_2, and S_3 that satisfy all of the constraints, and find the corresponding value of f.

(b) Repeat Part (a) but set $x_2 = 100$.

(c) What happens to f in the limit where you increase x_2 without bound, adjusting the S variables to keep the constraints satisfied?

(d) Change the last entry in the x_2 column from 0 to 1. Now try setting $x_2 = 100$ as you did before and explain why it doesn't work.

7.122 **Walk-Through: The Two-Phase Simplex Method.** In this problem you will maximize the function $f = 2x_1 - 3x_2$ subject to the constraints $x_1 + x_2 \le 4$, $-x_1 + x_2 \le 1$, $x_1 + 3x_2 = 6$, $x_1 \ge 0$, and $x_2 \ge 0$.

(a) Begin by rewriting the inequality constraints as equations. Because both of them are "less-than" inequalities, both will introduce slack variables.

(b) Put the problem into a simplex tableau. As always this requires pulling the objective function variables to the left side (with the f), thus changing all the signs.

(c) How can you tell that your tableau is not in restricted normal form?

(d) It is therefore time for the first phase. Your tableau represents three constraints so add three new variables: z_1 to the first constraint, z_2 to the second and z_3 to the third. (Visually that adds three columns to the tableau, each with one 1 and two 0s.) You will also temporarily abandon your original objective function and set out to maximize a new function, $f_2 = -z_1 - z_2 - z_3$. (Visually this replaces the first row with a different one that represents $f_2 + z_1 + z_2 + z_3 = 0$.) Write the simplex tableau for starting the first phase.

(e) Make the z variables basic by subtracting every constraint—the second, third, and fourth rows—from the first row.

(f) Use the simplex method to optimize your f_2 function.

(g) Starting from the final tableau of the first phase, toss out the z columns and put back in the original objective function: that is, replace this top row with the original top row.

(h) The variables that were basic at the end of the first phase may not be basic any more, because they appear in the top row. Make these variables truly basic by using the constraint rows to eliminate them from the objective row.

(i) Use the simplex algorithm in the usual way to finish this problem. (This is the "second phase.")

7.123 *[This problem depends on Problem 7.122.]* Sketch the feasible region defined by the constraints in Problem 7.122. Add contours of f to your plot, and use those contours to explain how you could know which vertex the maximum of f is on. Verify that your answer matches your answer to Problem 7.122.

For Problems 7.124–7.130 use the two-phase simplex method to find the maximum value of the function f subject to the given constraints. Assume in each case that all of the independent variables are constrained to be non-negative. You may find it helpful to first work through Problem 7.122 as a model. For the problems with two variables you can check your answers by drawing the feasible region and using contours of f to find the optimal vertex.

7.124 $f = 3x_1 - x_2$, $x_1 + 2x_2 \geq 1$, $3x_1 + 2x_2 \leq 5$

7.125 $f = -x_1 + 2x_2$, $2x_1 + 3x_2 \leq 8$, $x_1 - x_2 \leq 1$, $-x_1 + 4x_2 = 3$

7.126 $f = 3x_1 - x_2$, $x_1 - x_2 \geq 2$, $x_1 + 2x_2 \leq 5$

7.127 $f = 2x_1 + x_2$, $x_1 + x_2 \geq 2$, $3x_1 + 2x_2 \leq 7$, $x_1 \leq 1$

7.128 $f = 2x_1 - x_2 + x_3$, $1 \leq x_1 + x_2 + 2x_3 \leq 2$

7.129 $f = 2x_1 - x_2 + 4x_3$, $x_1 - x_2 + 2x_3 \geq 2$, $x_1 + 3x_2 \leq 4$, $x_3 \leq 1$

7.130 $f = x_1 - 2x_2 + x_4$, $x_1 + x_2 + 2x_3 + 3x_4 \geq 1$, $x_1 + x_4 \leq 10$, $x_1 + 2x_2 - x_3 - x_4 = 6$

7.131 Maximize $f = x_1 + 2x_2$ subject to the constraints $2x_1 + 3x_2 \leq 28$, $2x_1 + x_2 \geq 8$, $2 \leq x_1 \leq 8$, $x_2 \geq -2$. (Note that—unlike in the problems above—we cannot assume here that x_2 is a non-negative number.)

7.132 **The Transportation Problem** Your warehouse in Atlanta has 300,000 nails, the one in New York has 200,000 and the one in Boston has 500,000. The stores in Chicago, St. Louis, and Louisville need 400,000, 300,000, and 300,000 nails respectively. Write, but do not solve, a simplex tableau to answer the question: how many nails should each warehouse send to each store? Assume the transportation cost is proportional to the distance, which is given below.

	Atlanta	New York	Boston
Chicago	700	800	1000
St. Louis	500	1000	1200
Louisville	400	800	1000

7.133 *[This problem depends on Problem 7.132.]* How many nails should each warehouse send to each store?

7.134 **The Assignment Problem**[9] Suppose you have three workers that each need to be assigned to a job. The workers have different levels of skill and experience; the cost of employing each one to do each job is given below. Define a set of variables x_{ij}, equal to one if worker i has job j, and zero otherwise. Write, but do not solve, a simplex tableau to minimize the total cost subject to the constraints that each worker has exactly one job and each job has exactly one worker.

	Job 1	Job 2	Job 3
Worker 1	1	3	2
Worker 2	2	3	5
Worker 3	2	2	4

7.135 *[This problem depends on Problem 7.134.]* Which worker should have each job?

7.136 **Exploration: Moving Sand** In 1781 Gaspard Monge published his work on the "soil transport" problem: how to most efficiently

[9]The simplex method is not the most efficient way to solve the assignment problem for large numbers of variables. See, e.g., H.W. Kuhn, The Hungarian Method for the Assignment Problem, Naval Research Logistics Quarterly 2 (1955) 83–97.

move a pile of dirt from a given starting shape to a given final shape. This problem has applications in many areas, including calculating distance between states of a quantum mechanical system.[10] As an example, suppose you have a square sandbox that goes from the origin to the point $(5, 5)$. Initially the height of the sand is given by $S_0 = e^{-x^2-y^2}$. You want to move it to a different corner: $S_f = e^{-(5-x)^2-y^2}$. The cost of moving a unit volume of sand is equal to the distance you move it. In this form the problem involves integrals over the initial and final distributions, but you can turn it into a linear programming problem by breaking the grid into discrete boxes. If each box is 1×1 then you'll have a total of 25 boxes B_{ij}, where i and j each go from 0 to 4.

(a) Make tables of the average values of S_0 and S_f in each of the 25 boxes. *Hint*: once you calculate this for S_0 you can get it for S_1 from symmetry.

(b) Your independent variables are x_{ijkl}, representing the amount of sand moved from B_{ij} to B_{kl}. Define an objective function representing the total cost of movement as a function of these variables. Recall that cost is equal to distance times amount of sand. Naively you have 5^4 independent variables, but you can significantly reduce that number by tossing out any term that involves moving sand from a location where $S_f \geq S_0$ or moving sand to a location where $S_0 \geq S_f$.

(c) Define a set of constraints that reflects the fact that the amount of sand taken from each box equals $S_0 - S_f$ (provided this is positive).

(d) Define a set of constraints that reflects the fact that the amount of sand moved to each box equals $S_f - S_0$ (provided this is positive).

(e) Either write a simplex program or use an existing one to find the minimum cost to move the sand.

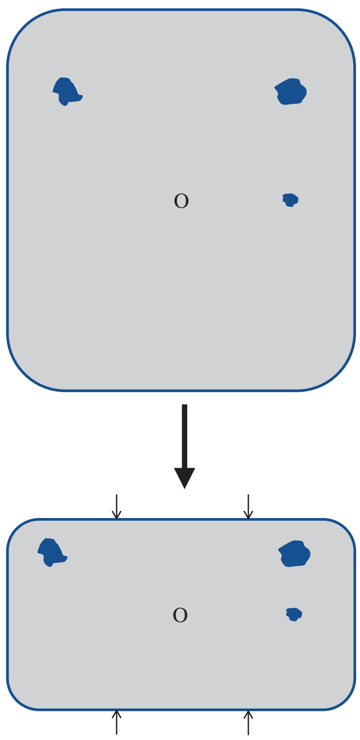
(f) Look at the values of the x_{ijkl} in your final solution and describe in words how the sand was moved. You should find that the answer was predictable.

(g) Find the minimal sand-moving cost for the same initial sand distribution, but with $S_f = (1/2)e^{-(x^2+y^2)/4}$.

[10] Karol Zyczkowskiyx and Wojeciech Slomczynskiz, "The Monge Distance Between Quantal States," J. Phys. A: Math. Gen. 31 (1998) 9095–9104.

7.6 Additional Problems

7.137 The picture below shows a rubber sheet with three small stones on it. Their initial position vectors are $\hat{i}$, $\hat{i}+\hat{j}$, and $-\hat{i}+\hat{j}$, measured from the origin that's marked "O." The rubber sheet is compressed vertically until it is only half as tall as before, as shown below.

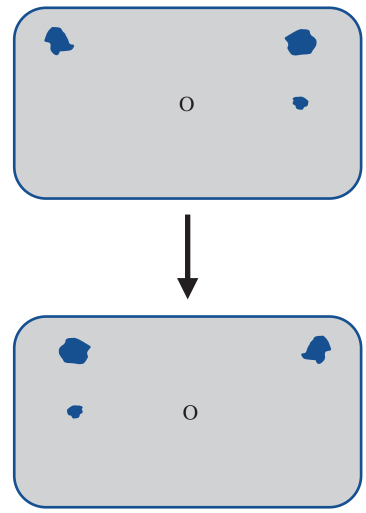


(a) What happens to the x-component of each vector? What happens to the y-component? First answer in words, then write equations that give the new x_1 and y_1 in terms of the old x_0 and y_0 for an arbitrary vector undergoing this transformation.

(b) What are the position vectors for the three stones after the transformation?

(c) Write a matrix that performs the transformation you described in Part (a). In other words, write a matrix that you can multiply by $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ to get $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$. Check your matrix by multiplying it by each of the initial rock positions to make sure it gives the final positions you predicted.

Next you look at a reflection of the rubber sheet in a mirror, so you see all the stones reversed left-to-right.



(d) What happens to the x-component of each vector? What happens to the y-component? First answer in words, then write equations that give the new x_2 and y_2 in terms of the old x_1 and y_1 for an arbitrary vector undergoing this transformation.

(e) What are the position vectors for the three stones after both transformations have been applied?

(f) Write a matrix that performs the transformation you described in Part (d). Check your matrix by multiplying it by each of the pre-reflection rock positions to make sure it gives the final positions you predicted.

Finally, consider the following transformation matrix.

$$\mathbf{M} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

(g) Draw each of the following vectors, and then draw the vector you get after you multiply **M** by the given vector.

i. $\hat{i}$
ii. $\hat{j}$
iii. $-\hat{i}$
iv. $2\hat{i}+2\hat{j}$

(h) Looking at your drawings, what transformation does this matrix perform on vectors? You can't answer that it did such-and-such to this vector and so-and-so to that vector: you need one succinct statement of what effect it has on *all* vectors. (Equivalently you could say what happened to the rubber sheet, analogously to how we described the transformations above.) To test your answer, predict on visual terms where the vector $-\hat{j}$ will end up after this transformation and then multiply **M** by $-\hat{j}$ to check if you got it right.

(i) Multiply **M** by the previous positions of the three stones to find their final positions. Make sure they match what you would expect from your description of this matrix.

(j) Write a single matrix that takes the position vector of a stone, compresses it vertically by a factor of 2, reflects it horizontally, and then does whatever transformation **M** does. Check your matrix by multiplying it by each of the original rock positions to make sure it gives the final positions you calculated in Part (i).

(k) Write a matrix to return the stones from their final positions (after all three transformations) to their initial positions $\hat{\imath}$, $\hat{\imath}+\hat{j}$, and $-\hat{\imath}+\hat{j}$.

7.138 Matrix **A** rotates any point matrix 20° clockwise. Matrix **B** stretches any point matrix by a factor of 3 in the x-direction. *Nothing in this problem should require you to write or multiply any matrices.*

(a) What does matrix $\mathbf{A}^{-1}$ do to a matrix?

(b) What does matrix $\mathbf{B}^{-1}$ do to a matrix?

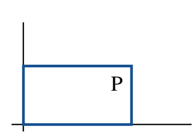
(c) What is the determinant $|\mathbf{A}|$?

(d) What is the determinant $|\mathbf{B}|$?

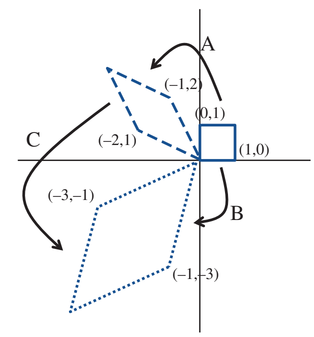
(e) What is the determinant $|\mathbf{B}^{-1}|$?

(f) What is the determinant $|\mathbf{AB}|$?

(g) Matrix **P** draws the rectangle shown below. Draw the shapes represented by point matrices **AP**, $\mathbf{A}^{-1}\mathbf{P}$, **ABP** and **BAP**.



7.139 In the figure below, matrix **A** transforms the solid square into the dashed diamond, and matrix **B** transforms the solid square into the dotted diamond.



(a) Matrix **A** turns the point $(0,0)$ into $(0,0)$ but that's no surprise—any matrix does that. **A** also turns the point $(1,0)$ into $(-1,2)$, and turns the point $(0,1)$ into $(-2,1)$. Finally it turns the point $(1,1)$ into some point in the second quadrant. You can't quite see what that last point is—and that's OK. Using the points that you do know, find matrix **A**.

(b) Matrix **B** turns $(1,0)$ into $(-3,-1)$, while $(0,1)$ becomes $(-1,-3)$. Find matrix **B**.

(c) Matrix **C** transforms the dashed diamond into the dotted diamond. Write a symbolic matrix equation, using only the letters **A**, **B**, and **C** (no numbers!) that expresses the relationship between **C**, **B**, and **A**.

(d) Solve the equation symbolically to find matrix **C**.

(e) Calculate matrix **C**.

(f) The drawing shows two points for which we can clearly see what matrix **C** should do. Test your answer on those two points.

7.140 Write the matrices for rotating 2D shapes by an angle α and rotating them by an angle β. Multiply the two matrices and prove using trig identities that the resulting matrix rotates shapes by the angle $\alpha+\beta$.

7.141 Consider a computer animation package in which every object is represented by a $3\times n$ point matrix: n points, each with x-, y-, and z-coordinates, that the computer will draw in order and connect with line segments. Stretches, reflections, and rotations, as well as composites of these operations, can be represented by matrix multiplications. But one of the simplest operations, a "translation"—retaining the shape and orientation of an object but changing its position in space—is harder in this representation.

(a) Object **A** comprises three points. (You can call the first point (x_1, y_1, z_1) and so on for the others.) Write a matrix operation that will add a constant X to its x-coordinates, Y to its y-coordinates, and Z to its z-coordinates. *Hint*: This operation will not be a matrix multiplication.

(b) Object **B** comprises four points. Write a matrix operation that will perform the same translation to this object.

You should have discovered that you needed a different matrix to perform this same operation on the two objects. This situation is not ideal for programming. Also, the fact that translation is not done by matrix multiplication makes it harder to form compound transformations made of translations and rotations. The solution is to represent every three-dimensional point with a four-dimensional vector, where the fourth coordinate is always a 1.

(c) Write the matrix that represents object **A** in this way. (It won't be $3 \times n$ anymore.)

(d) Multiply the matrix $\mathbf{T} = \begin{pmatrix} 1 & 0 & 0 & X \\ 0 & 1 & 0 & Y \\ 0 & 0 & 1 & Z \\ 0 & 0 & 0 & 1 \end{pmatrix}$ by object **A**. What transformation does it perform?

(e) Does matrix **T** have the same effect on object **B**? How do you know?

7.142 If L is a line 70° counterclockwise from the x-axis then a reflection about L is accomplished by this matrix.

$$\mathbf{M} = \begin{pmatrix} \cos^2 70° - \sin^2 70° & 2\sin(70°)\cos(70°) \\ 2\sin(70°)\cos(70°) & \sin^2 70° - \cos^2 70° \end{pmatrix}$$

(a) Let $\vec{V} = \hat{i}$. Calculate $\mathbf{M}\vec{V}$. (Use decimals instead of carrying around all the sines and cosines.) Plot $\vec{V}$ and $\mathbf{M}\vec{V}$. If they don't appear reflected about L figure out what you did wrong.

(b) For a more rigorous check, you can verify that **M** has the right eigenvectors and eigenvalues. Without doing any calculations, in what directions should the eigenvectors of **M** point and what should their eigenvalues be?

(c) Find the eigenvectors in component form. This part has nothing to do with **M**. Just look at the line L and do the trig to find the vectors that point in the directions you identified. This time leave your answers in terms of trig functions, not as decimals.

(d) Verify that those vectors are eigenvectors of **M** with the eigenvalues you predicted.

7.143 Consider the set of matrices that stretch vectors by a factor λ_1 along the axis $y = x$ and by λ_2 along $y = -x$, for all real values of λ_1 and λ_2.

(a) Write such a matrix in terms of λ_1 and λ_2.

(b) Assuming addition and multiplication by a scalar are defined in the usual way for matrices, is this set of matrices a vector space?

(c) If you found that it is a vector space, what are its dimensions?

7.144 *[This problem depends on Problem 7.143.]* Assuming "addition" is defined by the usual rule of matrix multiplication and multiplication by a scalar is defined in the usual way for matrices, show that the set of matrices described in Problem 7.143 does *not* define a vector space. Given this addition rule, how would you define multiplication by a scalar so that this is a vector space? *Hint*: Look at what distributivity of scalar multiplication with respect to vector addition implies for the eigenvalues of 2**a** with this new rule.

7.145 The space of all 2D rotation matrices $\mathcal{R}$ is a vector space, but not if you define matrix addition and multiplication by a scalar in the usual ways.

(a) Show that if you define addition and multiplication by a scalar in the way we usually do for matrices, $\mathcal{R}$ is not a vector space.

(b) Instead, you can define the sum of two rotation matrices as the rotation matrix for the sum of their two angles. (This is equivalent to multiplying the two matrices in the usual way.) You can similarly define multiplication by a scalar as multiplying the rotation angle by that scalar. With these definitions $\mathcal{R}$ is a vector space. What is its dimension?

7.146 The set of all rank 3 tensors is a vector space if addition and multiplication by a scalar are defined componentwise, as they are for matrices. Show that this vector space obeys distributivity of scalar multiplication with respect to vector addition.

7.147 **Inertia Tensor** In introductory physics you probably learned that a body has a "moment of inertia" I about any possible axis, which plays a role analogous to mass for linear motion. You may not have been taught that **I** is a tensor with nine components. (In this problem and the next we will use the

For Problem 7.149

	Cost per serving ($)	Calories	Sodium (mg)	Sugar (g)
Peanut butter	0.50	94	73	1.5
Salted Lentils	1	230	240	2
Spam	0.20	174	770	0
Soylent green	0.75	400	100	1

symbol **I** for the inertia tensor—no relation to the identity matrix.) For a continuous object of density ρ, $I_{ij} = \int_V \rho \left[\delta_{ij}\left(\sum_k x_k^2\right) - x_i x_j\right] dV$, where $\int_V dV$ means a triple integral over the volume, and x_1, x_2, and x_3 are the three Cartesian coordinates x, y, and z. The "Kronecker delta" δ_{ij} equals 1 if $i = j$ and 0 otherwise, so the sum over k only appears in the diagonal elements of **I**.

(a) Calculate the inertia tensor of a uniform cube of mass M with corners at the origin and the point (L, L, L).

(b) Angular momentum is given by the equation $\vec{L} = \mathbf{I}\vec{\omega}$. If the cube in Part (a) moves with angular velocity $\omega = a\hat{i} + b\hat{j}$ where a and b are constants, calculate its angular momentum.

7.148 *[This problem depends on Problem 7.147.]*

(a) The angular equivalent of Newton's second law is $\vec{\tau} = \mathbf{I}\vec{\alpha}$. Rewrite this equation to give $\vec{\alpha}$ as a function of $\vec{\tau}$ and then answer the equation: if a torque $\tau = \tau_0\hat{k}$ is applied to the cube in Problem 7.147, find the angular acceleration α of the cube.

(b) Find the inertia tensor of the same cube about a set of axes where the z-axis is unchanged and the x-axis goes diagonally through the bottom face of the cube. *Hint*: this can be done more easily with a rotation matrix than by integrating all over again.

(c) The "principal axes" of a body are the ones for which the inertia tensor is diagonal. Find the principal axes of this cube. (The origin remains unchanged at the corner of the cube. If the origin were in the center the principal axes could be guessed from symmetry.)

(d) If you apply a torque about the first of the principal axes you found, in what direction will the cube rotate?

7.149 **The Diet Problem** The company Sumptuous Land Of Plenty has just been awarded a contract to make stew for school lunches. Each serving must have between 600 and 700 calories, no more than 710 mg of sodium, and no more than 10 g of sugar.[11] The company has hired you to create the recipe that meets these requirements for the least possible cost, using the ingredients in the table above. Write, but do not solve, a simplex tableau to answer the question: how many servings of each ingredient should go in the recipe?

7.150 *[This problem depends on Problem 7.149.]* How many servings of each ingredient should go in the recipe?

7.151 **Exploration: The Schwarz Inequality.** The Schwarz inequality says that for any two vectors **a** and **b** in an inner product space:

$$|(\mathbf{a}, \mathbf{b})| \leq ||\mathbf{a}||\ ||\mathbf{b}|| \tag{7.6.1}$$

You'll prove this important inequality in this problem.

(a) First, let **a** and **b** be spatial vectors. Explain why the Schwarz inequality must hold in this case. What must be true of the two spatial vectors in order for the two sides of Equation 7.6.1 to be equal?

Now let **a** and **b** be generalized vectors. Define a new vector

$$\mathbf{c} = \mathbf{a} - \frac{(\mathbf{a}, \mathbf{b})}{||\mathbf{b}||^2}\mathbf{b}$$

(b) Show that **c** and **b** are orthogonal.

[11] The calorie and sodium requirements are from the "National School Lunch Program" guidelines for 6th–8th grade, and the nutrition information for peanut butter and spam comes from nutrition Web sites. Everything else in the problem we just made up.

(c) You can rewrite the definition of $\mathbf{c}$ as $\mathbf{a} = \mathbf{c} + \left[(\mathbf{a}, \mathbf{b})/||\mathbf{b}||^2\right] \mathbf{b}$. Use that equation for $\mathbf{a}$ to calculate $||\mathbf{a}||^2$. Use your result from Part (b) to simplify your answer as much as possible. *Hint*: Remember that $(\mathbf{a}, k\mathbf{b}) = k^*(\mathbf{a}, \mathbf{b})$.

(d) Rearrange your result to Part (c) to prove Schwarz's inequality.

(e) What must be true of $\mathbf{c}$ in order for Schwarz's inequality to be an equality? (Assume that $\mathbf{a}$ and $\mathbf{b}$ are both non-zero.)

(f) Show that Equation 7.6.1 is an equality only if $\mathbf{a}$ and $\mathbf{b}$ are linearly dependent.

CHAPTER 8

Vector Calculus (Online)

8.12 Additional Problems

In Problems 8.190–8.195 find all the field derivatives (gradient, divergence, curl, Laplacian) that can be calculated for the given fields, some of which are scalars and some of which are vectors.

8.190 $f = x^2 - y^2$

8.191 $\vec{f} = 3x^2\hat{i} - 3x^2\hat{j} + (y-z)^2\hat{k}$

8.192 $f = (\rho/z)\cos\phi$

8.193 $\vec{f} = \rho\sin\phi\,\hat{\rho} + \rho\cos\phi\,\hat{}\phi + \left(\rho^2/z\right)\,\hat{k}$

8.194 $f = r^2\sin\theta\cos\phi$

8.195 $\vec{f} = r\sin\theta\,\hat{r} + r\cos\phi\,\hat{\phi}$

In Problems 8.196–8.202 find a function F such that $\vec{\nabla}F = \vec{v}$ or prove that none exists. (This is one way of determining if $\vec{v}$ is conservative. If it is, your F is *negative* the potential function for $\vec{v}$.)

8.196 $\vec{v} = 3\hat{i} + 2\hat{k}$

8.197 $\vec{v} = xy\,\hat{i} + xy\,\hat{j}$

8.198 $\vec{v} = 3y\hat{i} + 3x\hat{j} + 3z\hat{k}$

8.199 $\vec{v} = e^x\left[\sin(x+2y) + \cos(x+2y)\right]\hat{i} + 2e^x\sin(x+2y)\hat{j}$

8.200 $x(\sin y)e^z\hat{i} + \left(y^2 - x(\cos y)e^z\right)\hat{j} + \left(2z + x(\sin\, y)e^z\right)\hat{k}$

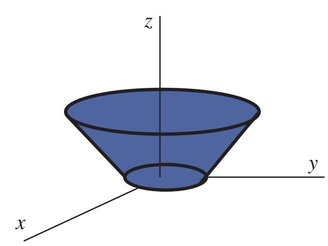
8.201 $\vec{v} = ze^x\hat{i} + \hat{j} + e^x\hat{k}$

8.202 $\vec{v} = \left(ye^{x+z} - 2(x+y+z)\right)\hat{i} + \left(e^{x+z} - 2(x+y+z)\right)\hat{j} + \left(e^{x+z} - (x+y+z)\right)\hat{k}$

Problems 8.203–8.207 ask questions that make no mention of the divergence theorem or Stokes' theorem. But you can use these theorems to make many of these questions easier.

8.203 The electric field in a region of space is $\vec{E} = kx\hat{i}$, where k is a constant. Find the flux of this field through the spherical surface $x^2 + y^2 + z^2 = R^2$. (In electromagnetism "flux" means the surface integral of the electric or magnetic field.)

8.204 Surface S consists of three surfaces: a disk of radius 1 on the xy-plane, a disk of radius 2 at $z = 3$, and a part of a cone that connects the two.



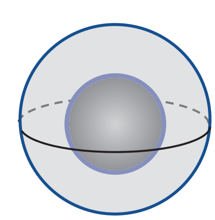
Vector $\vec{f}$ is $y^2\hat{i} + xe^z\hat{j} + \cos(xy)\hat{k}$. Find $\oiint_S \vec{f}\cdot d\vec{A}$.

8.205 Surface S consists of the cone and upper disk from Problem 8.204, but without the bottom disk.

(a) Vector $\vec{f}$ is $2x\hat{i} + 2y\hat{j} + 2z\hat{k}$. Evaluate $\iint_S \left(\vec{\nabla}\times\vec{f}\right)\cdot d\vec{A}$.

(b) Vector $\vec{g}$ is $\left(x^2 + e^y + \sqrt{z}\right)\hat{k}$. Evaluate $\iint_S \left(\vec{\nabla}\times\vec{g}\right)\cdot d\vec{A}$.

8.206 Region V is bounded between the spheres $x^2 + y^2 + z^2 = 9$ and $x^2 + y^2 + z^2 = 25$. The surface S (shown below) is the entire bounding surface of the region.



(a) If a vector field has positive flux through the inner sphere, which way would it have to point? Which way would it point to have positive flux through the outer sphere?

(b) Vector $\vec{f} = x^2\hat{j}$. Evaluate $\oiint_S \vec{f}\cdot d\vec{A}$.

(c) Vector $\vec{g} = (\ln r)\hat{r}$ in spherical coordinates. Evaluate $\iiint_V \left(\vec{\nabla} \cdot \vec{g}\right) dV$.

8.207 The surface S_1 is the cube with corners at $(0, 0, 0)$ and $(3, 3, 3)$, but it is missing the bottom (the side that would lie on the xy-plane). Your job is to add up the curl of $\vec{f} = e^z\hat{i} - 2xz\hat{k}$ along this most-of-a-cube.

(a) Describe and/or draw the curve C that bounds this surface.

(b) Describe and/or draw a simpler surface S_2 that is bounded by the same curve.

(c) Find $\iint_{S_1} \left(\vec{\nabla} \times \vec{f}\right) \cdot d\vec{A}$. You can do this any way you like, but it is easiest to use the results of your previous two answers!

In Problems 8.208–8.213 you'll be given a momentum density $\vec{V}$ for a fluid and something to calculate about it. Spend some time thinking about the easiest method for each one before jumping into calculations. Any letter other than the usual coordinate symbols is a constant.

8.208 $\vec{V} = ke^{(r-R)^2}\hat{r}$. Find the flow rate of the fluid out of a sphere of radius R centered on the origin.

8.209 $\vec{V} = (k/\rho)\hat{\rho}$. Find the flow rate of the fluid out of the portion of the sphere $x^2 + y^2 + z^2 \le R^2$ that lies in the first octant (meaning $x \ge 0$, $y \ge 0$, and $z \ge 0$).

8.210 $\vec{V} = \ln\left(x^2 - \sin x\right)\hat{i} + e^{-2\sqrt{y}}\hat{j} - 2y\hat{k}$. Find the circulation of the fluid around the ellipse $x^2 + 3y^2 = 1$, $z = 0$.

8.211 $\vec{V} = y\left(x^2 + y^2 - 1\right)\sin(x-3)\hat{i} - (x^2 + y^2 - 1)\hat{j} - 2\left(x^2 - y^2 + e^{xy}\right)\hat{k}$. Find the circulation of the fluid around a path that goes on the x-axis from the origin to $(1, 0, 0)$, from $(1, 0, 0)$ to $(0, 1, 0)$ along the curve $x^2 + y^2 = 1$ in the xy-plane, and finally back to the origin along the y-axis.

8.212 Find the line integral of $\left(x^2 + y^3\right)\hat{i} + 3xy^2\hat{j}$ from the origin to the point $x = -3\pi$, $y = 0$ along the path $\rho = \phi$ (where ρ and ϕ are the usual polar coordinates).

8.213 Find the flow rate of $\vec{V} = x\hat{i} + \left(\sin(\ln x) + y\right)\hat{j}$ through the cone $z^2 = 1 - x^2 - y^2$ in the region $z > 0$. *Hint*: You can start by finding the integral out of the surface made of the cone plus an added circle that closes off the bottom.

8.214 An important result in electrostatics is that inside a conductor, the electric field is everywhere zero. What does this say about the potential inside a conductor?

8.215 **The Hubble Flow** In 1929 Edwin Hubble discovered that the universe is expanding. From our point of view this means that all other galaxies are moving away from ours at a rate proportional to their distance from us: $\vec{v} = Hr\hat{r}$. We can to a reasonable approximation consider the density of galaxies to be the same value ρ throughout the observable universe, so the momentum density is $\vec{V} = \rho Hr\hat{r}$. (The variable ρ is commonly used to represent cylindrical coordinates, but it's also commonly used for density. Here we are using ρ for density, while r is the spherical coordinate.)

(a) Use the equation of continuity to derive an expression for the rate of change of the universe's density. Your answer should look like an ordinary differential equation for $\rho(t)$.

(b) If you assume the Hubble parameter H to be a constant, solve that differential equation to estimate how long ago the universe was twice its current density. Use the value $H = 2.4 \times 10^{-18}\ \text{s}^{-1}$.

(c) In reality H is not constant. Einstein's general theory of relativity predicts that for most of the history of the universe H has been approximately $2/(3t)$, where t is time since the big bang. Solve your differential equation for $\rho(t)$ again, using this function for H.

(d) Given that the current age of the universe is about 14 billion years, use your answer to Part (c) to give a more accurate answer for how long ago the universe was at twice its current density.

CHAPTER 9

Fourier Series and Transforms (Online)

9.7 Discrete Fourier Transforms

A Fourier *series* models a function as a discrete set of numbers (coefficients of a sum); a Fourier *integral* models a function as a continuous function (the transform). In both cases the function you start with is continuous. In this section we will use Fourier analysis on a data set that is discrete to begin with: a set of points, or data, rather than an algebraically defined function.

9.7.1 Explanation: Discrete Fourier Transforms

Throughout this chapter we have used Fourier analysis (series or transforms) to see what oscillations make up a particular function. In practice, though, we rarely measure a function such as $f(x) = e^{-x^2}$. We measure data points.

Consider, for example, the Motivating Exercise (Section 9.1). Observers repeatedly measure the velocity of a star over many years. A star with no planets around it should have a constant velocity, a star with one planet should oscillate like a sine wave, a star with two planets should follow a superposition of two sine waves, and so on. A "discrete Fourier transform" starts with individual velocity measurements and finds the oscillations represented by those points, enabling us to determine the presence of planets.

But Figure 9.1 doesn't look like a superposition of sine waves, does it? Real oscillations are buried under random noise caused by limitations of the measuring instruments, atmospheric distortion, and other factors. These random fluctuations are often many times greater than the oscillations themselves. Amazingly, the Fourier transform can generally cut through such noise to find the underlying order.

Below we present the formula for a discrete Fourier transform, but first let's make sure the letters are clear. Take a look at Figure 9.19. We begin with a function $f(x)$ representing N individual points spaced Δx apart. For instance, if we have $f(0)$, $f(20)$, $f(40)$, and $f(60)$, then $\Delta x = 20$ and $N = 4$. (The formula assumes Δx is constant: that is, the points are sampled at evenly spaced intervals.)

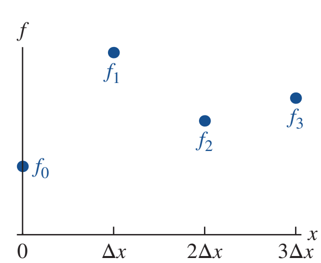


FIGURE 9.19 With four data points $r = 0, 1, 2, 3$.

We refer to the data points as f_0, f_1, f_2, f_3, and so on. We use the letter r as an index for this list. In our example above, f_3 would mean $f(60)$; more generally, f_r means $f(r\Delta x)$.

Our goal is to express the data as *<some coefficient>* times an oscillation of a particular frequency, plus *<a different coefficient>*

times an oscillation of a different frequency, and so on. We refer to these coefficients as $\hat{f}_1, \hat{f}_2$ and so on, using n as an index into this list. Below, then, we present the formula for turning the f_r list of data points into the $\hat{f}_n$ list of frequency coefficients.

The Formula for a Discrete Fourier Transform

Suppose a function $f(x)$ is known at N discrete points: $f_r = f(r\Delta x)$ for $r = 0, 1, \dots, N-1$. The "discrete Fourier transform" (DFT)[a] of f is a set of points $\hat{f}_n$ with $-N/2 \le n \le N/2$ given by the formula:

$$\hat{f}_n = \sum_{r=0}^{N-1} f_r e^{-2\pi i n r/N} \tag{9.7.1}$$

The numbers $\hat{f}_n$ are proportional to the amplitudes of oscillations with frequencies given by:

$$p = \frac{2\pi n}{N\Delta x} \tag{9.7.2}$$

It's easy to glide over those equations too quickly. To slow down, consider the example of $\hat{f}_3$ generated from a set with $N = 100$ data points.

- Equation 9.7.1 tells us how to find $\hat{f}_3$: it is a sum over all the values in the original data set.

$$\begin{aligned}\hat{f}_3 = f_0 + f_1 e^{-6\pi i/100} + f_2 e^{-12\pi i/100}\\ + f_3 e^{-18\pi i/100} + \cdots + f_{99} e^{-6(99)\pi i/100}\end{aligned}$$

  Then $\hat{f}_4$ is a different sum over the entire original data set, and so on. (Such linear combinations can be concisely represented with matrices; see Problem 9.114.)
- Equation 9.7.2 tells us what $\hat{f}_3$ represents: it tells us how much of the original data set oscillates with frequency $6\pi/(100\Delta x)$. Note that the denominator $N\Delta x$ is the entire span of the original data set which means that $2\pi/(N\Delta x)$, $4\pi/(N\Delta x)$, $6\pi/(N\Delta x)$, etc. are all the frequencies that fit perfectly into that domain.

As you can see, calculating N-values for your N data points in this way requires summing N^2 terms. But discrete Fourier transforms are so important that decades of research have gone into algorithms that can quickly calculate the DFT of a list with billions of input points. The need to do so arises more often than you might think, in applications ranging from image manipulation to electronic signal processing.

Interpreting a Discrete Fourier Transform

As with so many of these formulas, Equation 9.7.1 hides a number of subtleties.

- The DFT formula assumes that the function you are modeling is periodic, and that your data represent a full period. For instance (as we mentioned above) if you start with 100 data points with spacing 1, the terms in the DFT represent a period of 100, a period of 100/2, a period of 100/3, and so on; all of them start over at precisely 100. If you *inverse* transform your DFT you will recreate the original 100 data points and then reproduce them again and again.

[a] The acronym "DFT" is widely used for Discrete Fourier Transform, but unfortunately it has a number of other uses. The most problematic for our readers may be "Density Functional Theory," a model of many-body quantum systems used by physical chemists. There is probably less danger of confusion with the UK's "Department For Transport."

- Equation 9.7.1 makes no mention of Δx so the DFT depends on the values of your data points but not on how often they were sampled. If you sample 100 points over the course of a year, $\hat{f}_3$ tells you how much your data oscillate with a period of a third of a year. If you sample the same 100 points in a minute you will get the same $\hat{f}_3$, this time representing a period of 20 seconds.
- Strictly speaking $\hat{f}_n$ is not the amplitude of an oscillation; $\hat{f}_n/N$ is the amplitude. That matters for an inverse discrete Fourier transform, but for most purposes we only care about the *relative* amplitudes of the oscillations so we ignore that detail.
- We said above that Equation 9.7.1 assumes its input (the f_r numbers) are periodic, repeating the same N numbers over and over. The output (the $\hat{f}_n$ numbers) will *also* be periodic with period N. For instance if you put in 100 data points you will find that $\hat{f}_{101} = \hat{f}_1$ and $\hat{f}_{102} = \hat{f}_2$ and so on. In most cases the frequencies of interest are given by $-N/2 \le n \le N/2$.[4] A negative n just means a negative exponent in the complex exponential, but n and $-n$ both represent oscillations with the same period, just as they do in regular Fourier series with complex exponentials.
- Just as with a regular Fourier transform, if the input data are real then $\hat{f}_{-n}$ will always be the complex conjugate of $\hat{f}_n$. Since DFTs are generally used to model measured data, which is by definition real, you can generally get away with only calculating about half of the amplitudes. (In our example above you could calculate $\hat{f}_0$ through $\hat{f}_{50}$ the hard way and then you would know $\hat{f}_{-50}$ through $\hat{f}_{-1}$.)

EXAMPLE Discrete Fourier Transform

Problem:
Find a discrete Fourier transform of the list $f(0) = 1, f(3) = 4, f(6) = 2, f(9) = 3$.

Solution:
We plug $f_0 = 1, f_1 = 4, f_2 = 2, f_3 = 3$, and $N = 4$ into Equation 9.7.1.

$$\begin{array}{lllll}
\hat{f}_0 = 1 & + 4 & + 2 & + 3 & = 10 \\
\hat{f}_1 = 1 & + 4e^{-2\pi i/4} & + 2e^{-2\pi i(2)/4} & + 3e^{-2\pi i(3)/4} & = 1 + 4(-i) + 2(-1) + 3i = -1 - i \\
\hat{f}_2 = 1 & + 4e^{-2\pi i(2)/4} & + 2e^{-2\pi i(2)(2)/4} & + 3e^{-2\pi i(3)(2)/4} & = 1 + 4(-1) + 2 + 3(-1) = -4
\end{array}$$

It's worth going through that exercise to see the patterns in the formula. You can see the r-values counting 0, 1, 2, 3 as you move across the terms in any line, and you can see the n-values counting 0, 1, 2 as you count down the lines.

We could similarly calculate $\hat{f}_{-1}$, but the fact that the inputs were all real guarantees that $\hat{f}_{-1} = \hat{f}_1^* = -1 + i$. (You can do the calculation directly and verify this.) We know for the same reason that $\hat{f}_{-2} = \hat{f}_2^*$, and we know from periodicity that $\hat{f}_{-2} = \hat{f}_2$. The only way those can both be true is if $\hat{f}_2$ is real, which we see it is. With N real inputs, $\hat{f}_{N/2}$ will always be real (assuming N is even). Can you see why the same argument guaranteed that $\hat{f}_0$ would be real as well?

So the DFT, from $n = -N/2$ to $N/2$, is $(-4, -1 + i, 10, -1 - i, -4)$.

Once you have gone through the process once or twice, there's no great value in repeating it. DFTs are meant for extremely large data sets and are performed by computers.

[4] Technically $\hat{f}_{-N/2} = \hat{f}_{N/2}$ so you only need to calculate values from $n = -N/2 + 1$ to $N/2$.

Question: What does $\hat{f}_2 = -4$ represent?

Answer:
From Equation 9.7.2 it represents the amplitude associated with the frequency $2\pi(2)/(4 \times 3) = \pi/3$.

Inverse Discrete Fourier Transforms

When we take the Fourier series of a function we get a list of coefficients c_n. If we multiply each of those coefficients by $e^{\pi inx/L}$ and add them, we recover the original function. In a similar way, a discrete Fourier transform takes as input the value of a function at certain discrete points and gives as output the coefficients $\hat{f}_n$, and the "inverse discrete Fourier transform" calculates the original values f_r from those coefficients.

Inverse Discrete Fourier Transform

The inverse discrete Fourier transform of a set of points $\hat{f}_n$ is

$$f_r = \frac{1}{N} \sum_{n=-N/2+1}^{N/2} \hat{f}_n e^{2\pi inr/N} \tag{9.7.3}$$

(The sum can go over any N consecutive values of n, and it is often written from 0 to $N-1$, but we prefer this way because it focuses on the most physically meaningful values. Clearly for odd N you would have to adjust the limits of the sum.)

Recall that the x-value of each f_r is $r\Delta x$. If we plug $r = x/\Delta x$ into Equation 9.7.3 the exponential becomes $e^{2\pi in/(N\Delta x)x}$, which is why we said above that each mode $\hat{f}_n$ has frequency $2\pi n/(N\Delta x)$. We can also see from this formula that $\hat{f}_n$ is not technically the coefficient of the oscillation; $\hat{f}_n/N$ is.

Remember that the DFT coefficients $\hat{f}_n$ are periodic; they are N numbers that repeat forever. You can generate an inverse DFT from any N such numbers in a row and get a function that perfectly replicates the original points you used to create your DFT. Between those points, however, the behavior may surprise you unless you use the range specified in Equation 9.7.3. See Problem 9.107.

With the DFT and inverse DFT formulas you can translate a data set into "Fourier space" and then recover the original data. That may not seem very useful, but for many applications we can transform a set of data into Fourier space, manipulate it in some way, and then do an inverse transform to get back to regular space. This is illustrated in the example below.

EXAMPLE **Smoothing the Rough Edges**

Problem:
Copy an image from the Web and eliminate all high frequency oscillations from it.

Solution:
We chose an image of Rodin's sculpture "The Thinker." We imported this photo into Mathematica, which stored it as a 3072×2048 grid of numbers. We then asked

Mathematica to take a discrete Fourier transform of that grid of numbers. (We haven't given you the formula for a multivariate discrete Fourier transform, but Mathematica's built-in DFT routines knew just what to do.) The result was a 3072×2048 grid of complex numbers representing the DFT. An inverse Fourier transform of this grid reproduced the original picture exactly.

We then set about 99% of the numbers in the DFT to zero, leaving only the low frequency terms. An inverse Fourier transform of the resulting frequencies produced a softened version of the original. Finally, we set over 99.99% of the numbers in the DFT to zero. An inverse transform of the resulting set was still very recognizable, but blurry.

Rafael Ramirez Lee/Shutterstock.

Image reproduced from 100% of the data in Fourier space, image reproduced from 1% of the data, and image reproduced from 0.006% of the data

This technique can be used to deliberately soften images, but it also provides powerful data compression. You can store pictures in a fraction of the space required for an uncompressed image.

The implications for downloading pictures (in a Web browser for instance) are also important. If you download a picture as pixels the user sees the top row of pixels appear at first, filling in toward the bottom over time. But if you download the same information as frequencies the user sees the entire image almost instantly, sharpening to full resolution over time.

Where Did Those Formulas Come From?

Consider a regular Fourier series for a function with period $2L$: $f(x) = \sum c_n e^{\pi i n x/L}$ where the coefficients are given by

$$c_n = \frac{1}{2L}\int_0^{2L} f(x)e^{-\pi i n x/L}dx$$

If we know the function $f(x)$ we can find the coefficients c_n by integrating, analytically or numerically. If we only know $f(x)$ at N evenly spaced intervals $x = r\Delta x$ then we must approximate this integral with a sum.

$$c_n \approx \frac{1}{2L}\sum_{r=0}^{N-1} f(r\Delta x)e^{-\pi i n r\Delta x/L}\Delta x$$

The spacing Δx is related to the total period $2L$ and the number of sample points N by $\Delta x = 2L/N$. Substituting this in gives

$$c_n \approx \frac{1}{N}\sum_{r=0}^{N-1} f(r\Delta x)e^{-2\pi inr/N}$$

We could have left the formula like this, but historically the terms in a DFT are not defined to make $\hat{f}_n \approx c_n$, but rather $\hat{f}_n \approx Nc_n$, which thus eliminates the $1/N$ from the formula, giving us Equation 9.7.1. A similar calculation can be used to derive Equation 9.7.3 for the inverse discrete Fourier transform.

A Few Practical Notes

While discrete Fourier transforms have been around for centuries, their use didn't become common until Cooley and Tukey published a DFT algorithm that is in many cases millions of times faster than straightforwardly applying Equation 9.7.1.[5] That algorithm is now known as a "Fast Fourier Transform." One of us (GF) runs simulations of the early universe using hundreds of processors to take FFTs of datasets with over 60 billion points, and such large transforms are common in many areas of science and engineering.

As a practical matter, the FFT is most efficient when the number of data points N is a power of 2. If you are working with any other size data set it is usually worth padding the end with extra zeroes to make N a power of 2.

It's also important to remember that a DFT gives us oscillations at the frequencies $p = 2\pi/(N\Delta x), 4\pi/(N\Delta x) \ldots \pi/\Delta x$. That highest frequency $p = \pi/\Delta x$ is called the "Nyquist frequency," and it sets a limit on how high a frequency we can find from our data. The fact that the Nyquist frequency is inversely proportional to Δx makes sense: if we sample once per second, we won't find an oscillation that happens ten times per second! If a function oscillates much faster than our sampling rate, our discrete Fourier transform will be subject to "aliasing." The power that is actually in modes above the Nyquist frequency will appear incorrectly in the lower frequency modes, causing them to come out higher than they should. As a practical matter you should either know ahead of time what range of oscillation frequencies your data is likely to have or figure it out by trial and error, and make sure you sample frequently enough that your Nyquist frequency is higher than the highest significant frequency of your physical system.

9.7.2 Problems: Discrete Fourier Transforms

9.99 Walk-Through: Discrete Fourier Transforms. You've measured the following data points for a function $f(x)$: $f(0) = 2$, $f(2) = 3$, $f(4) = -6$, $f(6) = 0$.

(a) Use Equation 9.7.1 to calculate $\hat{f}_0$, $\hat{f}_1$, and $\hat{f}_2$.

(b) Find $\hat{f}_{-1}$ without using Equation 9.7.1. *This should take no more than 20 seconds.*

(c) What are $\hat{f}_{-2}$ and $\hat{f}_3$? *Again, more than 20 seconds means you're doing it wrong.*

(d) What frequencies p are represented by the terms $\hat{f}_{-1}$, $\hat{f}_0$, $\hat{f}_1$, and $\hat{f}_2$?

For Problems 9.100–9.105 find the discrete Fourier transform of the given list of numbers, going from $n = -N/2$ to $N/2$. Give the smallest magnitude non-zero frequency and the largest frequency (the Nyquist frequency) represented by the DFT.

9.100 $f(0) = 1$, $f(5) = 2$

[5] Cooley, James W.; Tukey, John W. (1965). "An Algorithm for the machine calculation of complex Fourier series". Math. Comput. 19: 297–301. The algorithm had been independently discovered numerous times before Cooley and Tukey, beginning with Gauss in 1805. For a discussion of the FFT algorithm itself see "Numerical Recipes" by Press, et. al.

9.101 $f(0) = 10, f(.6) = -1, f(1.2) = 0, f(1.8) = -7$

9.102 $f(0) = 1, f(.2) = 2, f(.4) = -2, f(.6) = -1$

9.103 $f(0) = 5, f(1) = 4, f(2) = 3, f(3) = 2$

9.104 $f(0) = 5 + i, f(2) = i, f(4) = 3, f(6) = 1 - i$

9.105 $f(0) = 2, f(4) = 4, f(8) = 0, f(12) = -6, f(16) = 0, f(20) = -4$

9.106 Calculate the DFT of $f(0) = 5, f(5) = -5, f(10) = 5, f(15) = -5$. Explain why your results make sense. In other words, how could you have predicted most of the values you found without having to calculate anything?

9.107 Consider the points $f(0) = 2, f(3) = 7, f(6) = -2, f(9) = 1, f(12) = 4, f(15) = -3, f(18) = 2, f(21) = 2$.

(a) Calculate the DFT coefficients $\hat{f}_n$ for $0 \le n \le 7$.

(b) On one plot show the real and imaginary parts of $(1/8)\sum_{n=0}^{7} \hat{f}_n e^{2\pi inx/12}$. Show the original 8 points f_r on the same plot.

(c) Repeat Part (b) for $-3 \le n \le 4$. *Hint*: you can easily get all the $\hat{f}_n$-values you need from the ones you already calculated.

(d) On one plot show the original points f_r and the real and imaginary parts of $(1/8)[(1/2)(\hat{f}_{-4}e^{-8\pi inx/12} + \hat{f}_4 e^{8\pi inx/12} + \sum_{n=-3}^{3} \hat{f}_n e^{2\pi inx/12}]$.

(e) You should have found that all three functions you just defined go through all of the original data points f_r, but behave differently in between them. Looking at your plots, which of these functions is likely to be the most useful for modeling your original function, and why?

9.108 Have a computer generate a table of 256 points $f(x)$ where x goes from 0 to 30. At each point let $f(x) = 2\sin(2x) + 2\sin(5x) + R$, where R is a random number (different at each point) from -4 to 4. Plot the data points on a graph of $f(x)$ vs. x. Then take a discrete Fourier transform of the data and plot the results on a graph of $\hat{f}(p)$ vs. p. Explain why the results appear the way they do.

9.109 Use $x = r\Delta x$, $p = n\Delta p = 2\pi n/(N\Delta x)$, and $\hat{f}_n = Nc_n$ to rewrite Equation 9.7.3 so it looks like the usual expression for a Fourier series. A normal Fourier series has infinitely many terms, but this one will only have N. All the others can be taken to be zero.

9.110 Plug Equation 9.7.1 into Equation 9.7.3 and verify that it does simplify to f_r. At some point in the calculation you should encounter a term of the form $\sum_{j=-N/2+1}^{N/2} e^{2\pi ki/N}$, where k is an integer. This sum equals zero unless $k = 0$. (You can easily figure out what it equals when $k = 0$.)

9.111 Use Equation 9.7.1 to show that $\hat{f}_{n+N} = \hat{f}_n$. Formally a DFT is periodic in frequency space, but in practice that really means all of the frequencies outside the range $|p| \le \pi/\Delta x$ are meaningless.

9.112 Consider the function $f(x) = e^{\cos x}$.

(a) Have a computer generate a table of 200 values of $f(x)$ ranging from $x = 0$ to $x = 99.5$. Plot these values on a graph of f vs. x. (Just plot the values in your table, not the curve for the function.)

(b) Take a discrete Fourier transform of your list of values. What frequency is associated with the n^{th} term in that list? Plot the moduli of the values of the discrete Fourier transform from $n = 1$ to $n = 100$, with the horizontal axis showing the associated frequencies. (Leave out $n = 0$, which just represents the constant term.)

(c) What frequency shows the highest peak in the DFT? What does that tell you about the original function? (*Hint*: you could have predicted this peak before doing the calculations.)

(d) What frequency shows the second highest peak in the DFT? What does that tell you about the original function? (You probably couldn't have predicted this one.)

9.113 Consider the list $f_r = (1, 2, 3, 4, 3, 2, 1, 2, 3, 4, 3, 2)$.

(a) Plot these data points, using the x-values $1, 2, 3, 4, 5, 6, \ldots$.

(b) Take a discrete Fourier transform of the list, then take an inverse discrete Fourier transform, and then plot the resulting points. This plot should look identical to the previous one.

(c) Take a discrete Fourier transform of f_r, set $\hat{f}_0 = 0$ in the resulting list, then take its inverse discrete Fourier transform and plot the results. How does this look different from your original plot?

(d) Take a discrete Fourier transform of f_r, set $\hat{f}_2 = \hat{f}_{-2} = 0$, take an inverse discrete Fourier transform, and plot the results. How does this plot look different from your original plot? *Hint*: some computers store the results in the order $\hat{f}_0$–$\hat{f}_{N-1}$, so the component $\hat{f}_{-2}$ may be stored as $\hat{f}_{N-2}$.

9.114 (*This problem requires linear algebra.*)

(a) Starting from Equation 9.7.1 write a 2×2 matrix for converting a pair of data points f_r into a DFT $\hat{f}_n$. (*Remember that n and r go from 0 to 1.*)

(b) Starting from Equation 9.7.3 write a 2×2 matrix for converting $\hat{f}_n$ into f_r.

(c) Prove that the two matrices you just wrote are inverses. What does this tell you about the two transformations?

(d) Write a matrix to find $\hat{f}_0$, $\hat{f}_1$ and $\hat{f}_2$ from three data points f_0, f_1, f_2.

9.115 Look through the Example "Smoothing the Rough Edges" on Page 4. Repeat this process for another image. Choose an image, import it into a computer algebra program, take a discrete Fourier transform of the image data, set the high frequency components to zero, and inverse transform it to get a new image. You may need to do some trial and error to figure out exactly which components to eliminate. Be aware that when you eliminate Fourier coefficients the inverse discrete Fourier transform will usually give you complex numbers. The easiest way to deal with this is to just take the real part after you take the IDFT. *Hint*: If you use a color image then each pixel will most likely be a list of three numbers instead of a single number. You can still do the same thing. Each component in Fourier space will be a list of three numbers and you can set them all equal to zero for the high frequency modes.

9.8 Multivariate Fourier Series

In this section we extend the idea of Fourier analysis to "multivariate" functions: that is, functions of more than one independent variable.

9.8.1 Discovery Exercise: Multivariate Fourier Series

1. Have a computer plot the function $z(x,y) = \sin(3x)\cos(10y)$ on the domain $-2\pi/3 \le x \le 2\pi/3$, $-2\pi/3 \le y \le 2\pi/3$.
2. Have a computer make an animation of the function $y(x,t) = \sin(3x)\cos(10t)$ on the domain $-2\pi/3 \le x \le 2\pi/3$, $0 \le t \le 2\pi/3$. In other words make a video that starts as a plot of $y(x,0)$ and changes into a plot of $y(x,\Delta t)$, $y(x,2\Delta t)$, and so on up to $y(x,2\pi/3)$, where Δt is a number much smaller than $2\pi/3$.
3. Redo Part 1 changing $\sin(3x)$ into $\sin(5x)$. Then redo it changing $\cos(10y)$ into $\cos(20y)$ (using $\sin(3x)$ again for this one). How did each of those changes affect the plot?
4. Similarly, redo Part 2 first changing $\sin(3x)$ into $\sin(5x)$ and then changing $\cos(10t)$ into $\cos(20t)$. How did each of those changes affect the animation?

9.8.2 Explanation: Multivariate Fourier Series

A multivariate version of our guitar string problem is a rectangular drumhead: that is, a piece of rubber stretched across a rectangular wire frame with length L and width W. See Figure 9.20. The rubber is free to vibrate in the middle, but is tacked down along the border, so the height z must be zero along $x = 0$, $x = L$, $y = 0$, and $y = W$.

Now imagine pulling the center of the drumhead straight up, just as we did with our guitar string in Section 9.4. The resulting shape might reasonably be modeled by the function below.

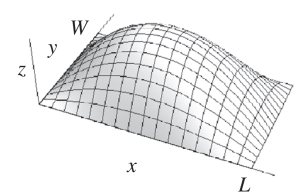


FIGURE 9.20 A rectangular drumhead stretched into the shape $z = k(Lx - x^2)(Wy - y^2)$.

$$z = k(Lx - x^2)(Wy - y^2)$$

The basic idea here is the same as it was for the guitar string. If the drumhead happens to start in a perfect sine wave, it will vibrate at one particular frequency. When the starting condition is not a sine wave we represent it as a sum of sine waves, whose frequencies and amplitudes will result in particular sounds. But before we go further with this particular example, let's step back to discuss what multivariate Fourier series look like mathematically.

One of the Many Formulas for a Multivariate Fourier Series

We have seen four different types of Fourier series: sines-and-cosines, sines-only, cosines-only, and complex exponentials. This leads to sixteen different types of multivariate Fourier series, because you can choose independently for x and y! For instance, consider a function that is even in x with a period of 6π, and neither even nor odd in y with a period of 2π. Each term in the Fourier expansion of this function could look like $\cos(mx/3)\left[A_{nm}\cos(ny) + B_{nm}\sin(ny)\right]$. Then again, you might choose in such a case to use cosines for x and complex exponentials for y, creating $\cos(mx/3)\left[c_{nm}e^{iny} + c_{-nm}e^{-iny}\right]$.

In Appendix G we give the general sine-and-cosine form, and the complex exponential form, for a Fourier series in two variables. (You can easily generalize these formulas to more variables if you need to.) The sine-and-cosine form is messy, with four different terms and a variety of integrals, which is one reason most people stay with the complex exponential form. But the sines and cosines are useful, especially when working with a function defined on a finite domain. Below we give the specific formula that applies to a function that is odd in both x and y, so all the terms are sines. (This particular formula gives as good an example as any, and it happens to be the one we need for our drumhead. Do you see why?)

Multivariate Fourier Series, Sines Only (See Appendix G for More General Formulas)

The Fourier series for a function that is odd in both x and y, with a period of $2L$ in x and a period of $2W$ in y, is:

$$f(x,y)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}D_{mn}\sin\left(\frac{m\pi}{L}x\right)\sin\left(\frac{n\pi}{W}y\right) \tag{9.8.1}$$

The coefficients are given by the equation:

$$D_{mn}=\frac{1}{LW}\int_{-W}^{W}\int_{-L}^{L}f(x,y)\sin\left(\frac{m\pi}{L}x\right)\sin\left(\frac{n\pi}{W}y\right)\,dx\,dy \tag{9.8.2}$$

You will derive that formula in Problem 9.125, using as always the orthogonality of the sine functions.

Solving the Vibrating Drumhead

Because our drumhead is defined on a finite domain, we need to create a periodic extension of it. That means we extend out in the x-direction, and *also* extend out in the y-direction, to cover the whole plane. We have four options for how to do this; for instance, we could do an even extension in x and an odd extension in y. But remember that our drumhead is constrained to $z = 0$ on all four edges, so its vibrational modes are all sines in both x and y. For this reason we will make odd extensions in both directions, as shown in Figure 9.21.

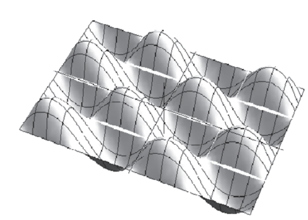


FIGURE 9.21 The rectangular drumhead's initial shape extended periodically as an odd function in both x and y.

We could express this odd extension as a piecewise function, and then calculate coefficients with four separate integrals in the four separate quadrants. However, the symmetry of the situation saves us a lot of the trouble. Our function is odd in both x and y (because we made it so!). That tells us that if we looked for cosine coefficients, they would all come out zero. It also tells us that we can calculate the sine coefficients by integrating in the first quadrant only, and then multiplying the result by four.

$$D_{mn}=\frac{4k}{WL}\int_0^W\int_0^L(Lx-x^2)(Wy-y^2)\sin\left(\frac{m\pi}{L}x\right)\sin\left(\frac{n\pi}{W}y\right)\,dx\,dy$$

The integral is separable, and you can then work out the individual integrals by parts, or hand them to a computer. After all due simplification, we arrive here.

$$D_{mn}=\frac{64kW^2L^2}{\pi^6m^3n^3}\qquad \textit{odd } m \textit{ and } n \textit{ only}$$

So we conclude that our original function can be written in this form.

$$\frac{64kW^2L^2}{\pi^6}\left[\sin\left(\frac{\pi}{L}x\right)\sin\left(\frac{\pi}{W}y\right)+\frac{1}{27}\sin\left(\frac{\pi}{L}x\right)\sin\left(\frac{3\pi}{W}y\right)\right.$$
$$\left.+\frac{1}{27}\sin\left(\frac{3\pi}{L}x\right)\sin\left(\frac{\pi}{W}y\right)+\frac{1}{27^2}\sin\left(\frac{3\pi}{L}x\right)\sin\left(\frac{3\pi}{W}y\right)+\cdots\right]$$

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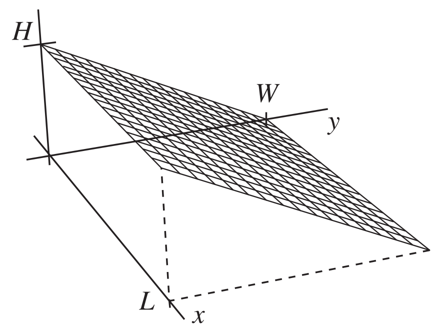
EXAMPLE Multivariate Fourier Series

Problem:

The drawing shows a surface going directly up from 0 at $y = W$ to a height of H when $y = 0$. The function $f(x, y)$ is this figure repeated evenly in the y-direction and oddly in the x-direction. Find the Fourier series for $f(x, y)$.



Solution:

In the x-direction this function is odd with a period of $2L$. In the y-direction this function is even with a period of $2W$. We could use the full sines-and-cosines form in Appendix G, but the sines in y and cosines in x must drop out due to the symmetry, so our function will look like:

$$f(x,y) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} C_{mn} \sin\left(\frac{m\pi}{L}x\right)\cos\left(\frac{n\pi}{W}y\right) \qquad \text{where}$$

$$C_{m0} = \frac{1}{2WL}\int_{-W}^{W}\int_{-L}^{L} f(x,y)\sin\left(\frac{m\pi}{L}x\right)\,dx\,dy$$

$$C_{mn} = \frac{1}{WL}\int_{-W}^{W}\int_{-L}^{L} f(x,y)\sin\left(\frac{m\pi}{L}x\right)\cos\left(\frac{n\pi}{W}y\right)\,dx\,dy \qquad n > 0$$

As the appendix instructed us to, we used $Q = 2$ when $n = 0$ and $Q = 1$ when neither m nor n was zero. (In the drumhead example we were only dealing with sines so the $m = 0$, $n = 0$ terms dropped out. Here the $m = 0$ terms vanish, but the $n = 0$ terms do not.)

We could write the function $f(x, y)$ four times—one in each quadrant—and perform four separate integrals, but the symmetry of the situation allows us to integrate only in the first quadrant and quadruple the answer. In the first quadrant, our function looks like $f(x, y) = H(1 - y/W)$. The formula for the coefficients is therefore:

$$C_{m0} = \frac{2H}{WL}\int_0^W\int_0^L \left(1 - \frac{y}{W}\right)\sin\left(\frac{m\pi}{L}x\right)\,dx\,dy$$

$$C_{mn} = \frac{4H}{WL}\int_0^W\int_0^L \left(1 - \frac{y}{W}\right)\sin\left(\frac{m\pi}{L}x\right)\cos\left(\frac{n\pi}{W}y\right)\,dx\,dy \qquad n > 0$$

The x integrals are trivial. The y integrals can be done with integration by parts, or using the table in Appendix G. The result, after simplifying, is:

$$C_{m0} = \frac{2H}{\pi m} \quad \text{(odd } m \text{ only)} \qquad C_{mn} = \frac{16H}{\pi^3 mn^2} \quad (n > 0, \text{ odd } m \text{ and } n \text{ only})$$

So the original function can be expressed as:

$$f(x,y) = \frac{2H}{\pi}\sum_{m=1}^{\infty}\frac{1}{m}\sin\left(\frac{m\pi}{L}x\right) + \frac{16H}{\pi^3}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{mn^2}\sin\left(\frac{m\pi}{L}x\right)\cos\left(\frac{n\pi}{W}y\right)$$

odd m and n only

What do Those Things Look Like?

When you write a multivariate Fourier series, you write your function as a series of sine waves. If the independent variable is x then one term in your expansion might be $10\cos(2x)$, a sine curve with a wavelength of π meters. If the independent variable is t then a term might be $10\cos(2t)$, an oscillation with a period of π seconds. If you don't understand what those terms represent, finding a series of them won't do you much good.

So what do the terms in our multivariate series represent? If x and y are both spatial then a function like $\sin(3x)\cos(7y)$ can be graphed as a surface, as shown in Figure 9.22. If you start at any point and move in the x-direction, you trace out a sinusoidal oscillation with wavelength $2\pi/3$. If you move in the y-direction you see oscillation with a wavelength of $2\pi/7$. The function is odd with respect to x and even with respect to y. You can verify all that information from the function, but the graph gives you a way to see what it all means visually.

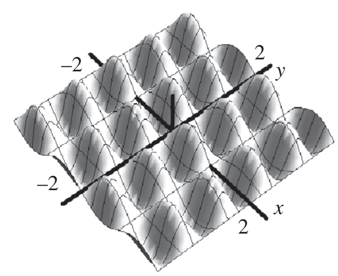


FIGURE 9.22 $f(x,y) = \sin(3x)\cos(7y)$.

We can visualize the same function in a very different way by thinking of one of the independent variables as time. Return to the image of a guitar string moving up and down in the plane of the page. The motion of such a string can be fully described by a function $y(x,t)$, which can in turn be decomposed into a multivariate Fourier series. One term in that series might be $\sin(3x)\cos(7t)$. What does that look like?

Well, at $t = 0$ it just looks like $y = \sin(3x)$. The function $\cos(7t)$ represents the changing amplitude of that function, and the wavelength does not change. So over time, the state evolves as shown in Figure 9.23.

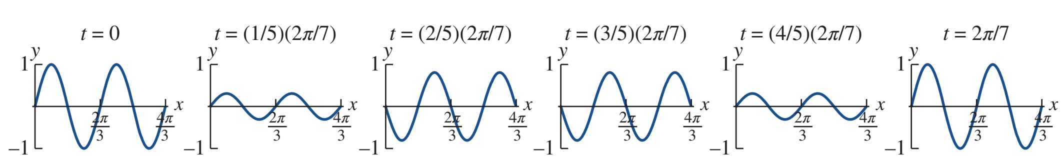


FIGURE 9.23 The function $f(x,t) = \sin(3x)\cos(7t)$ at several moments in time.

Motion like this where all the points on the string oscillate together is sometimes referred to as a "normal mode" of the system. The wavelength (2π divided by the x-frequency) is the distance from one crest to another; the period (2π divided by the t-frequency) is the time between oscillations. In an actual string the two frequencies are related by the "wave equation" which will be discussed more fully in Chapter 11.

Remember that we are not saying that a string always vibrates in a normal mode like $\sin(px)\cos(\omega t)$. We are saying, rather, that more complicated motion can be decomposed into a combination of such normal modes—that's what a Fourier series does—and that representation helps you model the behavior of the string and predict the resulting notes.

9.8.3 Problems: Multivariate Fourier Series

The formulas in Appendix G for multivariate Fourier series will be needed for many of these problems.

9.116 Walk-Through: Multivariate Fourier Series. Imagine an infinite chess board on which all the black squares are raised to the height $z = 1$ and all the white squares are lowered to the height $z = -1$. (Such boards are sometimes used by blind players so they can easily feel where each square begins and ends. Those boards, of course, are finite.) Such a function might be represented as:

$$f(x,y) = \begin{cases} 1 & x \in [0,1), \quad y \in [0,1) \\ -1 & x \in [-1,0), \quad y \in [0,1) \\ 1 & x \in [-1,0), \quad y \in [-1,0) \\ -1 & x \in [0,1), \quad y \in [-1,0) \end{cases}$$

… and so on forever in both directions.

(a) Is this function odd, even, or neither in x? What is its period in x?

(b) Is this function odd, even, or neither in y? What is its period in y?

(c) Your answers so far will tell you what types of functions will appear in the Fourier series: sines or cosines or both, and with what frequencies. Based on that information, write the form of the Fourier series you will need.

(d) One complete period of this function is a square extending from −1 to 1 in both x and y. Explain why, in this problem, you can integrate over a smaller part of that region and still find the complete answer.

(e) Find the coefficients of the Fourier series.

(f) Write the Fourier series for $f(x,y)$ including all terms with m and n less than 4.

In Problems 9.117–9.121 find the Fourier series for the specified function. If *both* x and y are symmetric (even or odd) then use sines and cosines, using only the appropriate one for each variable. Otherwise use the complex exponential form for both variables. The integrals in Appendix G will be helpful for some of these.

9.117

$$f(x,y) = \begin{cases} 1 & x \in [0,1), \quad y \in [0,1) \\ 2 & x \in [-1,0), \quad y \in [0,1) \\ 3 & x \in [-1,0), \quad y \in [-1,0) \\ 4 & x \in [0,1), \quad y \in [-1,0) \end{cases}$$

… repeated periodically in both directions.

9.118 The function $x^2 y$ on $-1 \le x < 1$, $-2 \le y < 2$ and repeated thereafter.

9.119 The function xy^2 on $0 \le x < 1$, $0 < y < 2$ and repeated thereafter.

9.120 The function $4x\sin(3y)$ on $-\pi \le x < \pi$ and repeated thereafter.

9.121 The function e^{2x-y} on $-2\pi \le x < 2\pi$, $-2\pi \le y < 2\pi$.

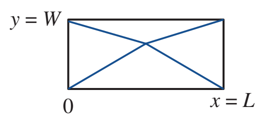
9.122 The Explanation (Section 9.8.2) showed the time evolution of a guitar string that follows the equation $z = \sin(3x)\cos(7t)$. In this problem you are going to do some similar sketches. In each case you will draw a sequence of pictures to show the time evolution of a different string. You can do these quickly by hand; the point is to see how they differ from each other, not to get every detail right.

(a) Draw the time evolution of $z = 2\sin(3x)\cos(5t)$. How does the behavior of the string differ from the version we drew?

(b) Draw the time evolution of $z = \sin(6x)\cos(5t)$. How does the behavior of the string differ from the version we drew?

(c) Draw the time evolution of $z = \sin(3x)\cos(10t)$. How does the behavior of the string differ from the version we drew?

9.123 The Explanation (Section 9.8.2) considered a rectangular drumhead pulled up into a paraboloid. It could instead have been pulled up into four planes.



(a) Assuming the drumhead extends along $0 \le x \le L$ and $0 \le y \le W$, and the middle point is raised to height H above the plane, write the equations for the four planes.

(b) To create a Fourier series for this shape, you need to create a periodic extension. Is it best to use an odd or even extension in x? In y? Explain.

(c) Write the Fourier series for your periodic function and write the integrals for finding the coefficients. You do *not* need to evaluate those integrals for this problem.

9.124 *[This problem depends on Problem 9.123.]*

(a) Evaluate the integrals you found in Problem 9.123 for the Fourier coefficients. (You could do this by hand but it would be very tedious. Whether you do it by hand or on a computer, you will need to consider the case $m = n$ separately from the case $m \ne n$.)

(b) Set $L = 4$, $W = 2$, and $H = 1$ and plot the Fourier series, including all terms up to $m = n = 20$. Verify that it matches the original shape of the drumhead.

(c) Compare this series to the one we found in the Explanation (for the same drumhead with a different starting condition). Do the same combinations of notes appear in both cases? Explain.

9.125 In this problem you will derive the formula for the coefficients of a multivariate Fourier series. We assume that the function $f(x, y)$ is odd in both x and y, and that it is periodic in x with period $2L$ and periodic in y with period $2W$. Under these circumstances the Fourier series looks like Equation 9.8.1.

(a) Multiply both sides of Equation 9.8.1 by $\sin\left(\frac{k\pi}{L}x\right)\sin\left(\frac{p\pi}{W}y\right)$. Then integrate both sides as x goes from $-L$ to L and y goes from $-W$ to W.

(b) The resulting integral is separable. Pull the constant out in front and separate it into one integral in terms of x and one in terms of y.

(c) The right side of your equation now has an infinite number of integrals. Use the orthogonality of the sine functions, and the fact that $\int_{-L}^{L}\sin^2\left(\frac{k\pi}{L}x\right)dx = L$, to evaluate all of them.

(d) Solve the resulting equation for the coefficient D_{kp}.

9.126 *[This problem depends on Problem 9.125.]* Derive the formula for the coefficients of a multivariate Fourier series for a function $f(x, y)$ that is periodic in x with period $2L$, periodic in y with period $2W$, odd in x, and *even* in y. You will not begin with Equation 9.8.1 exactly.

9.127 In the example on Page 11 we found a Fourier series for the function $f(x, y) = H(1 - y/W)$ on the domain $0 \le x \le L$, $0 \le y \le W$. Choose any positive values for H, L, and W, and plot the partial sum of that Fourier series that includes all terms up to $m, n = 11$. Verify that it reproduces the shape of the original function in the appropriate domain.

9.9 Additional Problems

If a problem is not marked with a computer icon then you should be able to do all integrals by hand and/or by using the table in Appendix G.

9.128 In this problem you are going to create a Fourier series for the function $f(x) = 3x$ on the domain $[0, 2]$.

- **(a)** The process begins with a periodic extension of $f(x)$ (neither even nor odd). What is the period of this extension? What is the value of L in the Fourier series formulas?
- **(b)** Write the formulas for a_0, a_n and b_n without yet evaluating the integrals.
- **(c)** Plot the periodic extension from $x = -4$ to $x = 4$.
- **(d)** When you evaluate your integrals you can choose to use any limits of integration that are separated by one full period. Which full period is easiest to use? Explain why by referring to your plot.
- **(e)** Finish finding the Fourier series for $f(x)$.

For Problems 9.129–9.133, find the Fourier series of the indicated function twice, first using sines and cosines and then using complex exponentials. If it is given over a finite domain assume that it extends periodically beyond that domain. (Do not assume an even or odd extension.)

9.129 $f(x) = \sin(2x) + \cos(3x)$

9.130 $f(x) = x + 3$, $-\pi < x < \pi$

9.131 $f(x) = \sin^5 x$. *Hint*: don't just naively accept what the computer gives you if it doesn't make sense.

9.132 $f(x) = x^2 \sin x$, $0 < x < \pi$

9.133 $f(x) = \sin(2x) + \cos(\pi x)$, $0 < x < 1$. *Hint*: the easiest way to evaluate the sine and cosine integrals is to rewrite them as complex exponentials.

9.134 In each part of this problem find a sine/cosine Fourier series for the given function on the given domain. In Part (a) you will do this in the usual way, by evaluating integrals for the coefficients. Using that Fourier series, you should be able to do the remaining parts without evaluating any integrals.

- **(a)** $f(x) = x$ from $x = 0$ to $x = 1$
- **(b)** $g(x) = x + 1$ from $x = 0$ to $x = 1$
- **(c)** $h(x) = -x$ from $x = 0$ to $x = 1$
- **(d)** $i(x) = 2x$ from $x = 0$ to $x = 1/2$

9.135

- **(a)** Write a Fourier series that represents the function $y = x^2$ from $x = -20$ to $x = 20$.
- **(b)** Draw the 1^{st}, 5^{th}, 20^{th}, and 100^{th} partial sums of the resulting series on one plot, along with $y = x^2$.

9.136 The function $f(x)$ equals $2x$ on $-3 \le x \le 3$ and then repeats forever with a period of 6.

- **(a)** Draw $f(x)$ showing at least three full periods.
- **(b)** Create a Fourier series for $f(x)$ based on sines and cosines.
- **(c)** Create a Fourier series for $f(x)$ based on complex exponentials.
- **(d)** Is $f(x)$ odd, even, or neither? How is this reflected in both Fourier series?

9.137 The function $g(x)$ equals $6 - 2x$ on $0 \le x \le 3$ and then repeats forever with a period of 3.

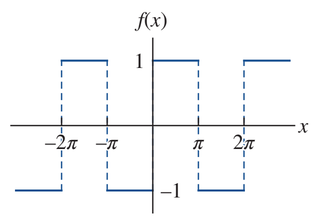
- **(a)** Draw $g(x)$ showing at least three full periods.
- **(b)** Create a Fourier series for $g(x)$ based on sines and cosines.
- **(c)** The function $g(x)$ is neither odd nor even, but $g(x) - 3$ is. Is it even or odd, and how can you see that in the Fourier series you just wrote?
- **(d)** Create a Fourier series for $g(x)$ based on complex exponentials.

9.138

- **(a)** Show that the functions $\sin(n\pi x/L)$ and $\cos(n\pi x/L)$ for all positive integers n form an orthogonal set on the interval $[-L, L]$. (If you're stuck you may find it helpful to look at Section 9.3 Problem 9.32.)
- **(b)** Show that if you add any function $\sin(k\pi x/L)$ with non-integer k the set is no longer orthogonal.

9.139 The drawing below, repeating the same pattern forever to both left and right, represents the function $y(x, t)$ during the time $-2\pi < t < 2\pi$. Then the function suddenly switches: all the

$y = 1$ values move down to $y = -1$ and vice versa, and it stays like that for 4π seconds. Then it switches back…and so on forever.



(a) What is the period of this function in x?

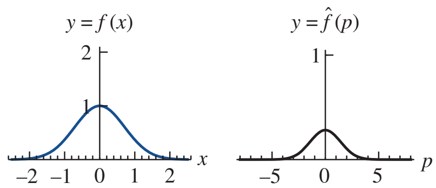
(b) What is the period of this function in t?

(c) Is $f(x, t)$ even in x, odd in x, or neither?

(d) Is $f(x, t)$ even in t, odd in t, or neither?

(e) Create a multivariate Fourier series representation of $y(x, t)$.

9.140 The drawings below represent $y = e^{-x^2}$ and its Fourier transform.



(a) Copy these drawings onto two separate graphs and label them $y = f(x)$ and $y = \hat{f}(p)$.

(b) Add to the first drawing a different graph labeled $g(x) = 2f(x)$, and add to the second drawing a different graph labeled $y = \hat{g}(p)$. These drawings do not need to be exact, but they should show the correct transformations of the original graphs. No computation should be required.

(c) Copy the original drawings again onto separate graphs. Then add new graphs labeled $h(x) = f(2x)$ and $y = \hat{h}(p)$.

9.141  Let $f(x) = x^{-1/3}$ from $-\pi$ to π and repeat thereafter.

(a) This function only satisfies the Dirichlet conditions if $\int_{-\pi}^{\pi} f(x)dx$ is finite. Show that it is. (Because of the vertical asymptote, you have to break it up into $\int_{-\pi}^{0} f(x)dx + \int_{0}^{\pi} f(x)dx$ and show that both parts are finite.)

(b) Use a computer to generate the 20th partial sum of the Fourier series for this function between $x = -\pi$ and $x = \pi$, and then plot that series from $x = -2\pi$ to $x = 2\pi$.

(c) The resulting graph should look a lot like $x^{-1/3}$ between $-\pi$ and π. What does the graph do at $x = 0$ and why? What does the graph do at $x = \pi$ and why?

(d) What are the values of the Fourier series at $x = 0$ and $x = \pi$? Why do these values make sense?

9.142 A simple harmonic oscillator with an external driving force obeys the following differential equation.

$$\frac{d^2x}{dt^2} + 4x = f(t) \qquad (9.9.1)$$

(a) Find the complementary solution that you get if you replace $f(t)$ with 0. Your answer should have two arbitrary constants in it.

(b) Let $f(t) = \sin t$. Find a particular solution that satisfies the differential equation with no arbitrary constants and add it to the complementary solution to get the full solution. *Hint*: guess a solution of the form $x(t) = k \sin t$ and plug it in to find out what k has to be.

(c) Find the general solution if $f(t) = \sin(\omega t)$ where ω is an unspecified constant.

(d) Find the general solution if $f(t)$ is a square wave, equal to 1 from $t = 0$ to $t = 1$, -1 from $t = 1$ to $t = 2$, and repeated periodically thereafter. *Hint*: At the risk of stating the obvious, the solution will involve taking a Fourier series.

9.143 *[This problem depends on Problem 9.142.]* Find the general solution to Equation 9.9.1 if $f(t)$ is a square wave equal to 2 from $t = 0$ to $t = 1$, 0 from $t = 1$ to $t = 2$, and repeated periodically thereafter. *Hint*: you can use your work from Problem 9.142 but you will need to add an additional particular solution.

9.144 The temperature distribution on a disk is best described in polar coordinates, where ρ goes from 0 (the center) to the radius R (the rim), and ϕ goes from 0 to 2π. If the disk has no heat sources or sinks then the temperature distribution throughout the disk is determined by the "boundary condition": the temperature along the rim. In Chapter 11 you will show (here you can take our word for it) that if the outer edge is held at $T(R, \phi) = T_0 \sin(k\phi)$

then the temperature throughout the disk will obey the following equation.

$$T(\rho, \phi) = T_0 \left(\frac{\rho}{R}\right)^k \sin(k\phi)$$

(a) What are the units of the constants k and T_0?

(b) If the temperature on the edge of the disk is continuous, what values of k are allowed?

(c) If you start at a point on the edge where the temperature is T_0 and move directly toward the center, describe how the temperature will change as you move. What if you start at a point on the edge where the temperature is $-T_0$? What about zero?

(d) Write the temperature distribution on the disk if the temperature on the outer ring is $10\sin(4\phi)$.

(e) Write the temperature distribution on the disk if the temperature on the outer ring is $10\sin(5\phi)$.

(f) Describe qualitatively the differences in the two temperature distributions you just wrote down. (You should note two important differences.)

If the temperature on the outer ring is a sum of sines then the temperature throughout the disk will be the sum of the corresponding solutions. For example, if $T(R, \phi) = \sin(\phi) + 15\sin(2\phi)$, then $T(\rho, \phi) = (\rho/R)\sin(\phi) + 15(\rho/R)^2\sin(2\phi)$.

(g) Write the temperature distribution on the disk if the temperature on the outer ring is $2\sin(3\phi) - 6\sin(4\phi)$.

(h) Now consider a disk whose outer ring is held at $T = T_0$ for $0 < \phi < \pi$, and at $T = -T_0$ for $\pi < \phi < 2\pi$. Rewrite the outer ring temperature $T(R, \phi)$ as a Fourier series. Then write the temperature distribution for the entire disk $T(\rho, \phi)$ as a Fourier series.

9.145 **The Gibbs Phenomenon** If a function $f(x)$ satisfies the Dirichlet conditions then its Fourier series converges to its value where it is continuous and converges to the average of its left and right limits at jump discontinuities. However, the convergence at such discontinuities suffers from a problem known as the Gibbs phenomenon. You'll explore this by considering the function $f(x)$ defined as the odd extension of $f(x) = 1$ from $x = 0$ to $x = 1$.

(a) Find the Fourier sine series of $f(x)$.

(b) Plot the partial sum of the Fourier series including the first three non-zero terms. Your plot should go from $x = -1$ to $x = 1$ and should include horizontal lines at $y = -1$ and $y = 1$ for reference.

(c) For negative x the series should be near -1 and for positive x its should be near 1, but in going from negative to positive it "overshoots" by a bit, going higher than 1. Looking at your plot, estimate the amount it overshoots by.

(d) Show all of the partial sums up through the sixth non-zero term together on one plot. As you use more terms, you should find that the amount by which the series overshoots 1 does *not* decrease, but that it moves back down near 1 more quickly.

(e) Plot the 50^{th} partial sum of the Fourier series and show that it still overshoots by the same amount. Zoom in your plot enough to estimate the value of x at which it returns back to 1 after overshooting.

(f) The fact that a Fourier series overshoots a jump discontinuity by an amount that doesn't decrease as you add more terms is the Gibbs phenomenon. Explain how we can still say that the series converges to the function even though this occurs.

CHAPTER 10

Methods of Solving Ordinary Differential Equations (Online)

10.3 Phase Portraits

Just as a slope field (Section 1.4) gives us a way to visualize the solutions to a first-order ODE, a phase portrait is a way of visualizing the solutions to two (or more) coupled first-order ODEs, or to a single second-order ODE.

10.3.1 Explanation: Phase Portraits

In Section 1.7 (see felderbooks.com) we introduced "coupled" differential equations. Such equations occur when two variables depend on each other. For instance, dx/dt may depend on both x and y, and dy/dt may also depend on x and y. A solution would be a pair of equations $x(t)$ and $y(t)$ that solve both equations simultaneously.

Our first example was the math problem of Romeo and Juliet.[2] Romeo's love grows the more Juliet loves him ($dR/dt = J$), while Juliet's love diminishes the more Romeo loves her ($dJ/dt = -R$). The state of the system at any moment is the value of the two functions R and J. If you know those two numbers at any moment in time you can figure out how they will evolve for all future times.

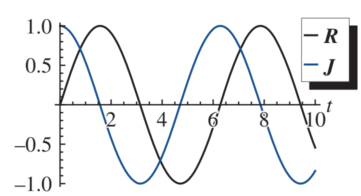


FIGURE 10.1 Romeo and Juliet's feelings for each other oscillate out of phase.

In Chapter 1 we solved these equations and found that R and J oscillate sinusoidally, 90° out of phase, as indicated in Figure 10.1. In this section we will arrive at the same conclusion by a graphical method. Like slope fields, the method of "phase portraits" allows us to visualize the possible behaviors of a system, in this case of two coupled ODEs. Although we're introducing them in the context of this simple problem that we can solve exactly, phase portraits can be used to understand the behavior of systems whose equations can't be solved analytically.

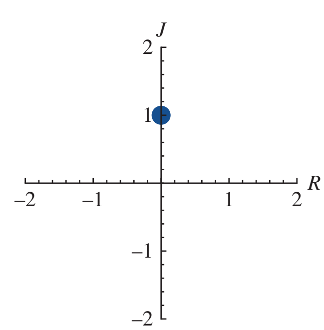


FIGURE 10.2 Juliet loves Romeo.

The phase portrait for the Romeo and Juliet system is a plot with R on one axis and J on another. For instance, Figure 10.2 shows that Juliet loves an indifferent Romeo. Every point in this space represents one possible state of the Romeo-and-Juliet system.

Figure 10.2 is not like most graphs you have worked with. You are accustomed to the variable on the y-axis depending on

[2] adapted with permission from *Nonlinear Dynamics and Chaos* by Steven Strogatz.

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the variable on the x-axis. In this case both the x- and y-axes represent dependent variables. The independent variable, time, does not appear in the diagram at all. So the state $R = 0$, $J = 1$ shown on the plot could be an initial condition ($t = 0$) or it could occur at any other time. We can't say.

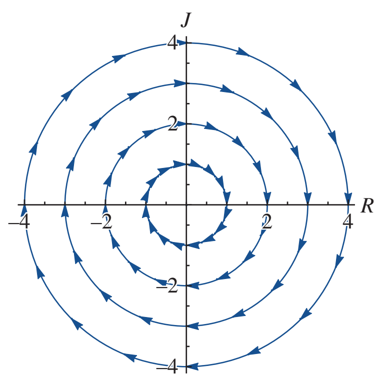


FIGURE 10.3 A phase portrait showing possible solutions to the Romeo and Juliet equations.

What we *can* say, based on the differential equations, is how this state will evolve. $dR/dt = J$ tells us that R will increase; $dJ/dR = -R$ tells us that J will hold steady. So if the system is ever momentarily in the state shown in Figure 10.2, its next shift will be to the right. From there the positive R will start causing J to decrease. If you follow this logical progression you will end up describing a circle back around to the point $(0, 1)$ where we started, and so on forever. This circle describes the same progression, and for the same reasons, that Figure 10.1 described.

But that circle is only the particular solution that starts at the point $(0, 1)$. If the system starts farther from the origin it will trace out a similar circle with a larger radius. We can therefore represent the system by Figure 10.3, a "phase portrait." Just like Figure 10.1, this new representation shows Romeo and Juliet's loves oscillating 90° out of phase with each other in a perpetual cycle of love and hate. Each of these representations has an advantage relative to the other. The advantage of Figure 10.1 is that it includes the time, which Figure 10.3 does not. Looking at the phase portrait we can see that R and J will oscillate, but there's no way to know how long each oscillation takes, or even whether it's going faster at some points of the cycle than at others. The advantage of the phase portrait is that it shows not just a single trajectory, but a whole family of possible trajectories corresponding to different initial conditions. When we look at a phase portrait that includes enough trajectories we can see in one plot all the possible behaviors of the system.

Definition: Phase Portrait

A "phase portrait" is a plot showing the possible solutions to a set of coupled first-order differential equations. Each dependent variable is plotted on one axis. The curves on a phase portrait, usually called "trajectories," show possible behaviors of the system.

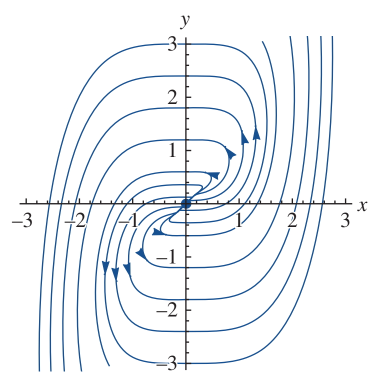
(We'll discuss below how to make phase portraits for second-order differential equations by writing them as sets of coupled first-order equations.)

Each point on a phase portrait represents a possible state of the system. Since the system could presumably start in any possible state, each point can also be said to represent a possible initial condition. For example, we can see from Figure 10.3 that if Romeo and Juliet start with any combination of feelings for each other that has a combined magnitude $R^2 + J^2 = 9$ (in whatever units we might use to measure feelings) then they will oscillate with that same magnitude forever.

If two trajectories crossed each other, one set of initial conditions could lead to two possible outcomes. That is technically possible for non-linear equations, but it takes work to contrive such an example. We will assume that phase portrait trajectories never cross.

EXAMPLE **A Phase Portrait**

Problem:
The figure below is a computer-drawn phase portrait for the equations $\dot{x} = x - y$, $\dot{y} = x^3$. Based on this drawing, describe the possible long-term behaviors of the system.



Solution:
The overall trend is a counterclockwise rotation. Whenever x is positive, y is increasing; in the upper left of the plot x is decreasing and in the lower right x is increasing. Take a glance at the differential equations and convince yourself that this is just what we would expect.

But we also see something else: no matter what initial conditions the system starts in, the amplitude of the oscillation increases over time. The system spirals out toward infinity.

Critical Points and Separatrices

We are going to analyze the behavior of the following system by drawing a phase portrait.

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = x - 1 \tag{10.3.1}$$

The first thing we always look for are the points where $x'(t) = y'(t) = 0$. If the system ever reaches such a "critical point" it will stay there forever. Physically these represent equilibrium states of the system.

Definition: Critical Point

A set of first-order differential equations for the functions $x_1(t)$, $x_2(t)$, … has a critical point at a set of values $(x_1, x_2, \ldots)$ if all of the derivatives $\dot{x}_1$, $\dot{x}_2$, … equal zero there.

A critical point is "stable" (or "attractive") if all the trajectories near that point converge toward it. A critical point is "unstable" (or "repulsive") if all the trajectories near that point move away from it. It is also possible for a critical point to be neither attractive nor repulsive—either because some nearby trajectories approach it and others move away, or because nearby trajectories orbit around the critical point as in Figure 10.3.

Critical points are the only places where a trajectory can begin or end. A trajectory can come in from infinity and end at a critical point, start from a critical point and go off to infinity, start and end at infinity, start and end at critical points, or make a closed loop.

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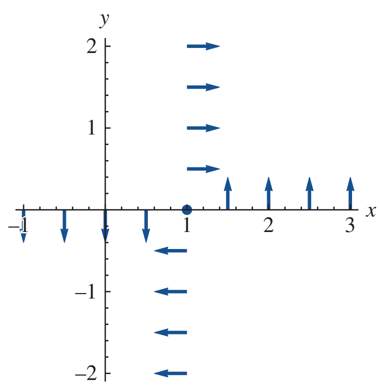
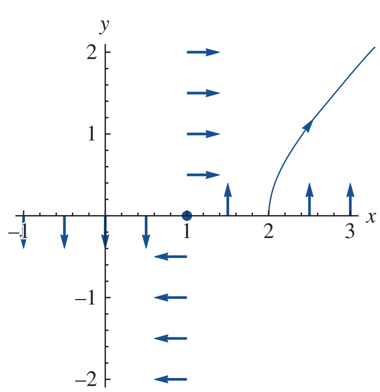
It isn't hard to determine that Equations 10.3.1 have a critical point at $(1,0)$ and nowhere else. So we will start building our phase portrait out from there.

Along the x-axis we have $x'=0$ and $y'=x-1$. So to the right of our critical point the trajectories point straight up, and to the left they point straight down.

Along the line $x=1$ we have $x'=y$ and $y'=0$. So above our critical point the trajectories point directly to the right, and below they point left. Let's draw what we have so far.

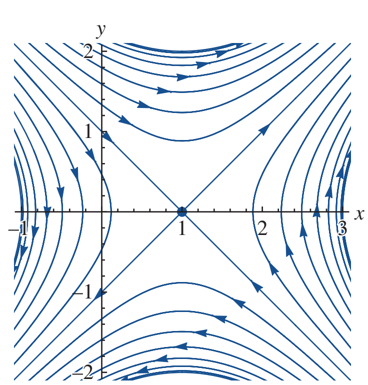


Based on just those arrows and a little bit of thought we can say a surprising amount about this system. For instance, suppose the initial conditions are $x=2$, $y=0$ (directly on one of our arrows, just to make the first step easy). The initial movement will be straight up on the graph: that is, y will increase $(y'=1)$ while x holds steady $(x'=0)$. So a short time later finds us at the point $(2,\Delta y)$. Now y' is still 1, but x' is now a small positive number. Our graph starts to veer slightly to the right. As we move higher the increasing y causes x' to increase, and the increasing x causes y' to increase. So our graph will head toward (∞,∞). In Problem 10.37 you will sketch in a few other curves using a similar process. You may be able to predict much of the behavior just by looking at the drawings we've already done. Trajectories in the upper-right-hand corner will generally head up and right, as our example above did. Trajectories in the lower-left-hand corner will generally head down and left.



Where is the dividing line between these two destinations? In Problem 10.38 you will show that the ultimate destiny of any trajectory in this system depends on what side of $y=1-x$ the initial conditions fall on. Any path above this line will eventually head toward (∞,∞), while any path below it will head toward $(-\infty,-\infty)$. We say that $y=1-x$ is a "separatrix" for this phase portrait, because it separates two regions that exhibit qualitatively different behavior.

If you look closely at the phase portrait you can tell that $y=1-x$ is actually made up of two separatrices, each one a trajectory that comes in from infinitely far away and asymptotically approaches the critical point. Just like a critical point, a separatrix can be attractive, repulsive, or neither. The drawing shows that the separatrices along $y=1-x$ are "repulsors": trajectories near them move away from them over time. The drawing also shows another pair of separatrices along $y=x-1$. They are "attractors": trajectories move toward them over time. All of the separatrices begin or end on the critical point $(1,0)$, which is itself neither attractive nor repulsive.

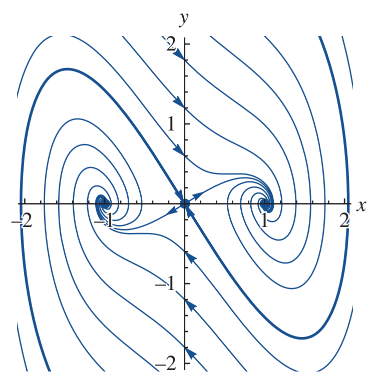


The phase portrait for $\dot{x}=y$, $\dot{y}=x-1$

Drawing phase portraits by hand is not something we do a lot of. It's a worthwhile exercise to go through a few times because it gives you a greater appreciation for the really valuable skill, which is interpreting the behavior of a system from a given phase portrait.

EXAMPLE **More Critical Points**

Problem:
The drawing shows the phase portrait for $dx/dt = y$, $dy/dt = -x^3 + x - y$. Use this drawing to discuss the possible behaviors of the system.



Solution:
It's always helpful to start with the critical points. Setting the first equation equal to zero gives $y = 0$. The second one is zero when $x^3 = x$, with solutions $x = 0, \pm 1$. We can see the three critical points on the drawing.

Around those points the phase portrait shows two qualitatively different behaviors. The system can either spiral in toward the critical point at $x = 1$ or the one at $x = -1$. Dividing those two possibilities are a pair of separatrices, shown in bold. One approaches the origin from the upper left and the other from the lower right. (There's another pair of separatrices connecting the origin to the other critical points, but we're going to focus on the ones shown in bold.) It's far from obvious that the initial conditions $(1, 1)$ lead toward the point $(1, 0)$ and the initial conditions $(3, 1)$ lead toward the point $(-1, 0)$, but it's easy to see those things looking at the phase portrait.

On a technical note, it can often take a lot of trial and error with the computer to plot separatrices since you have to find just the right initial conditions. Later in this section we'll show you a trick that can help with that.

A lot of the point of the previous two examples is that critical points and separatrices reveal a tremendous amount about the possible behaviors of the system.

Second-Order Equations

We said above that phase portraits are for systems of coupled first-order differential equations. They can also be used for second-order equations, however, which include most of the equations used in physics. The trick is to write one second-order equation as two first-order equations. Suppose we have an equation of the form $x''(t) = F(x(t), t)$. We can always define a new function $v(t) = x'(t)$, and now we have the two coupled equations $x'(t) = v(t)$ and $v'(t) = F(x(t), t)$.

EXAMPLE **Damped Simple Harmonic Oscillator**

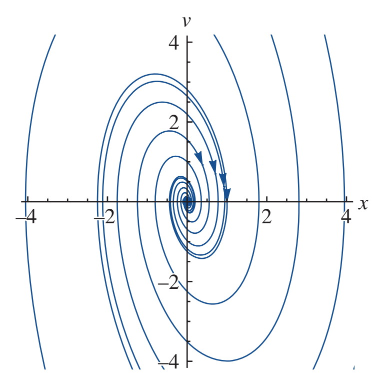
Problem:
A mass on a damped spring obeys the differential equation $\ddot{x} = -4x - \dot{x}$, where x and t are measured in SI units. Find all the critical points for this system, draw a phase portrait for it, and describe the possible behaviors.

Solution:
We begin by writing this as two first-order equations.

$$\dot{x} = v, \quad \dot{v} = -4x - v$$

The only critical point occurs at the origin: $x = v = 0$.

We had a computer graph trajectories for a set of initial conditions laid out along the unit circle in the first quadrant. We can see that all of the trajectories spiral in toward a stable equilibrium at the origin. This point represents $x = v = 0$: the mass is at rest at $x = 0$, as we would expect from a damped oscillator.



Remember that every point on a phase portrait represents one possible state of the system. For a pair of first-order equations for $x(t)$ and $y(t)$, a state is a value for each of those functions. For a second-order equation $\ddot{x} = \ldots$, a state is a value for x and a value for its first derivative—for instance, the position and velocity of the mass at any given moment—so those are the axes of the phase portrait.

Some Tips for Generating Phase Portraits on a Computer

In general, we can get a computer to make a phase portrait by giving it a list of initial conditions and having it numerically solve the differential equations for each one. Each solution will be a pair of functions $x(t)$, $y(t)$, and we can ask the computer to plot the parametrically defined curve (x, y) for each one. Finally we ask it to show all those curves together on one plot. Coming up with a good set of initial conditions can be tricky, and in the end there's no substitute for trial and error, but some guidelines can help.

We begin by finding all of the critical points and make sure that the initial conditions cover the region around those critical points. Sometimes a vertical or horizontal line of points can be useful, sometimes a circle of points around a critical point can help, and sometimes using a mix of the two can help.

Even with all that, the direction of the trajectories can be a problem. If all trajectories spiral in toward the origin and we start at a circle of points around the origin, our phase portrait won't show anything outside that initial circle. One way to deal with that is to solve the equations *backwards*. If our system is $\dot{x} = x + y$, $\dot{y} = x$ then we can generate a valid trajectory by starting at any initial point and solving $\dot{x} = -x - y$, $\dot{y} = -x$. That will trace the trajectory backwards from that point. If we plot the trajectory forwards and backwards from each point we will cover the phase portrait more effectively.

That backwards trick is especially useful for plotting separatrices that asymptotically approach critical points. If you look at the figure in "Example: More Critical Points" above, the separatrices are the only two trajectories that approach the critical point at the origin. Just choosing random initial conditions it could take forever to find one that just happens to be on one of the separatrices. So instead we chose two points near the critical point and evolved the system backwards from those points to plot the separatrices.

Stepping Back: Phase Space

Any physical system has some number of dependent variables. Most commonly these are functions of time. The state of the system at any given time consists of the value of each of those variables and, if they obey second-order differential equations, of their derivatives. Given a set of differential equations for those variables, we can predict the future behavior of the system from the initial conditions.

A phase portrait is a tool for visualizing those behaviors. Each axis is a dependent variable, and taken together those axes define all the possible states of the system. That space of all possible states is called "phase space." In principle, an object moving in 3 dimensions has a 6 dimensional phase space, because to specify its state we need to give all three components of its position and of its velocity. For simpler situations such as an object that can only move in 1 dimension, however, the phase space is 2 dimensional and a phase portrait can provide a useful visualization.

10.3.2 Problems: Phase Portraits

10.34 **Walk-Through: Phase Portraits.** Consider the system described by the equations $dx/dt = x + y$, $dy/dt = -2y$. You're going to draw a phase portrait for this system. By the time the problem is done you're going to have a lot on that drawing, so you might want to start by drawing a big set of axes going from -6 to 6 in both directions. When we ask where a trajectory begins or ends, remember that one possible answer is "at infinity."

(a) Find all critical points of this system. (That is, find all points where both $x'(t)$ and $y'(t)$ are zero.)

(b) Along the x-axis, $y'(t) = 0$.

- **i.** What does that imply about all trajectories that start along that axis? Draw arrows to represent the time evolution of the system along that axis. You will need to think about which way those arrows point.
- **ii.** The positive x-axis is a separatrix for this system. Is it an attractor or a repulsor? Explain how your answer comes from the equations. Where does this trajectory begin and end? (One of the answers will be "at infinity.")
- **iii.** The negative x-axis is another separatrix. Is it an attractor or repulsor? Where does it begin and end?

(c) Analyze the behavior in the first quadrant.

- **i.** At the point $(1, 1)$ the derivative in the x-direction is 2 and in the y-direction it's -2. If the system starts at that point, in what direction will it initially move in the xy-plane? Draw an arrow at $(1, 1)$ pointing in that direction.
- **ii.** What can you say about the direction of the trajectories *everywhere* in the first quadrant? Based on that, describe the behavior of any trajectory that starts in the first quadrant.
- **iii.** Sketch a trajectory beginning at the point $(1, 2)$. You don't need to be exact, but try to have the trajectory at each point going in roughly the correct direction, and be sure to show the right behavior in the limit $t \to \infty$.

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(d) The second quadrant is more complicated.

i. If a trajectory starts slightly to the left of the positive y-axis, what direction will it move in initially? Where will that trajectory go at late times? Sketch one such trajectory, starting at $(-.1, 4)$.

ii. If a trajectory starts at the point $(-2, 2)$, what direction will it move in initially? As soon as it moves away from that point, what will happen to its direction? Where will that trajectory go at late times? Sketch the trajectory starting at $(-2, 2)$.

iii. Explain how your last two answers imply that there must be a separatrix in the second quadrant, passing somewhere in between the points $(-2, 2)$ and $(-.1, 4)$. Is this separatrix attractive or repulsive? Where does it begin and end?

iv. You don't have enough information yet to know the exact shape of this separatrix; you'll work that out in Problem 10.35. For now just sketch in a curve that matches the answers you've given about it. Include arrows showing which direction the separatrix trajectory goes.

v. You drew two trajectories that started in the second quadrant. Now extend them backwards, showing where they would have come from in order to reach the points $(-2, 2)$ and $(-.1, 4)$. As with all of these sketches your goal is to show the correct qualitative behavior, not to plot exact curves.

(e) Describe the behaviors of trajectories in the third and fourth quadrants. For each one you'll have to figure out if you can do it with a simple argument like the first quadrant or if it needs more careful work like the second one. When you're done you should have drawn in one more separatrix and a couple of other sample trajectories showing the possible behaviors of the system.

(f) If there are any regions of your phase portrait where it is not yet clear how the trajectories behave, sketch in enough trajectories to make it clear.

(g) Is the one critical point of the system attractive, repulsive, or neither?

(h) The separatrices divide the graph into four regions. For each region, indicate how trajectories starting in that region will evolve over time. What will the system approach as $t \to \infty$ in each case?

10.35 *[This problem depends on Problem 10.34.]* In Problem 10.34 you found roughly where the separatrices were by looking at the behavior of the system. In some cases that's the best you can do, but in this case you can find the separatrices analytically by guessing (correctly as it turns out) that they are lines. The key is that, for this particular system, the two separatrices you sketched are the only two trajectories that end on the critical point at the origin.

(a) If the system $dx/dt = x + y$, $dy/dt = -2y$ starts out at a point (x, y), what is the initial slope dy/dx of its trajectory? Your answer should be a function of x and y.

(b) Suppose the system starts at a point on the line $y = mx$. In order for the trajectory to stay on that line its initial slope would have to equal m. Using your answer to Part (a), write an equation expressing the statement "Starting at the point (x, mx), the trajectory's initial slope equals m."

(c) Solve that equation for m.

(d) What is the equation for the separatrices other than the positive and negative x-axes? (The separatrices are two halves of the same line, so they have the same equation.)

10.36 *[This problem depends on Problem 10.34.]* Have a computer make a phase portrait for the system $dx/dt = x + y$, $dy/dt = -2y$. Clearly indicate critical points and separatrices. Make sure your phase portrait has enough trajectories to see the behavior in each region, and that it includes arrows showing the directions of the trajectories.

10.37 Consider the system described by Equations 10.3.1 with initial conditions $x = 1$, $y = 2$.

(a) Calculate dx/dt and dy/dt at that point. Based on your answers, in what direction along the phase portrait will this trajectory initially move?

(b) After it moves in that direction for a short time, how will dx/dt and dy/dt change?

(c) By following the curve along in this way, trace the trajectory from that point.

(d) The equations $\dot{x} = -y$, $\dot{y} = -x + 1$ represent the same system evolving backward in time. Starting again at $(1, 2)$

trace in that direction to complete the trajectory.

(e) Draw a similar trajectory starting at the origin.

10.38 The Explanation (Section 10.3.1) claimed that Equations 10.3.1 have a repulsive separatrix along the line $y = 1 - x$. In this problem you will show how the equations lead to the behavior we saw in the computer-generated phase portrait.

(a) Show mathematically that at any point along this line $\dot{y} = -\dot{x}$. What does that imply about the time evolution of the system if the initial condition lies on that line?

Now consider three points. The point $B = (x_0, y_0)$ lies directly on the line $y = 1 - x$. The point $A = (x_0, y_0 + \Delta y)$ lies directly above that point, and $C = (x_0, y_0 - \Delta y)$ directly below.

(b) How does dx/dt at point A compare to dx/dt at point B?

(c) How does dy/dt at point A compare to dy/dt at point B?

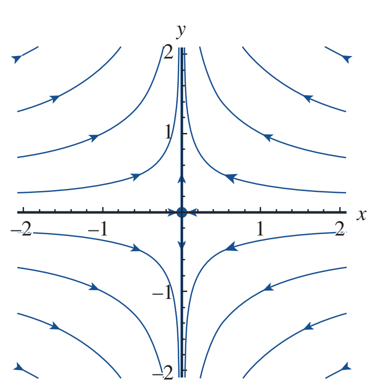
(d) We have seen what will happen to the system if it starts at point B. Based on that and your answers, what will happen to the system if it starts at point A? Will it move closer or farther from the line $y = 1 - x$?

(e) Repeat this analysis for point C.

10.39 *[This problem depends on Problem 10.38.]* Show that Equations 10.3.1 have an attractive separatrix along the line $y = x - 1$.

In Problems 10.40–10.42 you will be given a phase portrait. Classify all of the critical points and separatrices as attractive, repulsive, or neither. Describe how the system will evolve in time. (Your answer will almost always be of the form "if it starts in this region, then…")

10.40



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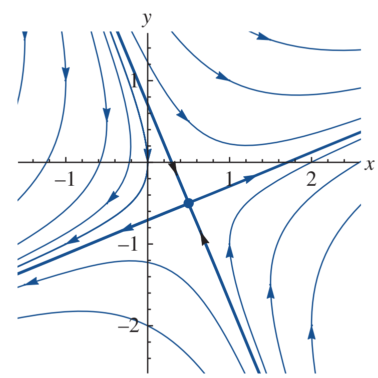

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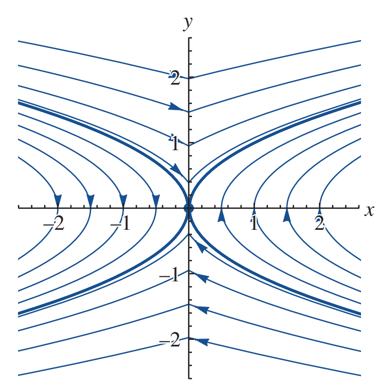


In Problems 10.43–10.50 sketch a phase portrait for the given set of equations. Your phase portrait should show all the critical points and enough trajectories to indicate the possible behaviors of the system.

10.43 $dx/dt = 4y,\ dy/dt = x$

10.44 $dx/dt = y,\ dy/dt = -x - y$

10.45 $dx/dt = y,\ dy/dt = -x + y$

10.46 $dx/dt = y,\ dy/dt = x - y$

10.47 $dx/dt = x + y,\ dy/dt = x - y$

10.48 $dx/dt = y,\ dy/dt = x^2 + y$

10.49 $dx/dt = y^2,\ dy/dt = -x$

10.50 $dx/dt = \rho,\ dy/dt = \phi$.

10.51 Given a pair of equations $x'(t) = f(x, y)$ and $y'(t) = g(x, y)$, write the equations that draw the same trajectories but in the opposite direction.

10.52 Given a pair of equations $x'(t) = f(x, y)$ and $y'(t) = g(x, y)$, write an equation for the slope dy/dx of the trajectory at the point (x, y).

In Problems 10.53–10.58 you will be given a second-order differential equation. Rewrite it as two coupled first-order equations. Have a computer draw the phase portrait for those two equations, and use that phase portrait to predict the possible time evolution of the system. A good answer would look like "If it starts with $x > 5$ moving to the right it will move in a positive direction forever. If it starts at $x > 5$ at rest or moving left slowly enough it will start moving right and continue that way forever. If it starts at $x < 5$ then…"

10.53 $x''(t) + 9x(t) = 0$

10.54 $x''(t) + 5x'(t) + 6x(t) = 0$

10.55 $x''(t) + 5x'(t) + 6x(t) = 2$

10.56 $x''(t) - 5x'(t) + 6x(t) = 2$

10.57 $x''(t) + x^3 = 0$

10.58 $x''(t) + \tan(4\pi x) = 0$

10.59 If you have a second-order differential equation for $x(t)$ you can draw a phase portrait for it where the axes are x and $\dot{x}$. Without knowing anything else about the system, what can you conclude about what the trajectories look like? In other words, what must be true about all phase portraits for second-order equations that is not necessarily true about phase portraits for coupled first-order equations?

10.60 Explain why a trajectory cannot begin or end at any point other than a critical point.

10.61 **Inflationary Cosmology** According to the theory of "inflation," the early universe went through a period during which virtually all of the energy was in the form of a "scalar field." You don't need to know what a scalar field is to solve this problem. All you need to know is that in the simplest model of inflation the field ϕ obeys the differential equation $\ddot{\phi} + m^2\phi + \sqrt{12\pi G(\dot{\phi}^2 + m^2\phi^2)} = 0$.

(a) Define $v = \dot{\phi}$ and rewrite this second-order equation as two coupled first-order equations for ϕ and v.

(b) What is the one critical point for this system?

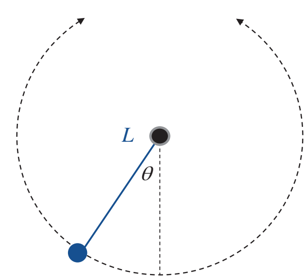
(c) Have a computer draw a phase portrait for the system. You can set the constants G and m equal to 1. There are two separatrices. Where do they begin and end? (In each case one of the answers is "at infinity.") Are they attractive or repulsive?

(d) For a typical trajectory, describe the evolution of the system. What happens at early times, middle times, and late times?

10.62 **Exploration: The pendulum.** A rigid undamped pendulum of length l obeys the differential equation

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0$$

where θ is the pendulum's angle off the vertical and g is the gravitational constant. The usual approach is to say that $\sin\theta \approx \theta$ for small θ, which reduces the equation to the simple harmonic oscillator equation. This approximation only works for small θ, but the equation is difficult to solve more generally. (Go ahead and try. We dare you.)



(a) Rewrite this equation as two coupled first-order equations. What variables should go on the axes of the phase portrait for this system? Draw a set of axes with those labels. You'll fill in the phase portrait as you go through the problem.

(b) What are the critical points for the system? What physical states do these points represent? *Hint: Mathematically there are infinitely many critical points, but there are only two states they can represent.* Draw a few of these points on your plot.

(c) If the pendulum starts at the bottom ($\theta = 0$) and you give it a little push it will swing back and forth. Draw a set of trajectories on your plot representing this motion. Include arrows showing the direction of the trajectories.

(d) If instead you give the pendulum a large push it will swing in circles. Draw a set of trajectories on your plot showing motion in clockwise circles, and another set showing counterclockwise circles. (We said the pendulum is rigid, a rod instead of a string, so the bob always stays a distance L from the center, even when $\theta > \pi/2$.) Again, include arrows on the trajectories.

(e) You've drawn trajectories showing two types of motion, oscillations and circles. Draw separatrices on your plot in between those types of motion. Where do those separatrices begin and end? What do they represent physically?

10.4 Linear First-Order Differential Equations

The simplest type of differential equations are linear and first order, and there is a general formula for solving all such equations (if you can integrate them). Such equations are important in their own right, but they are also important because we often approach more difficult equations by turning them into linear first-order equations—either exactly or approximately.

10.4.1 Discovery Exercise: Linear First-Order Differential Equations

A linear first-order differential equation can always be written in the form:

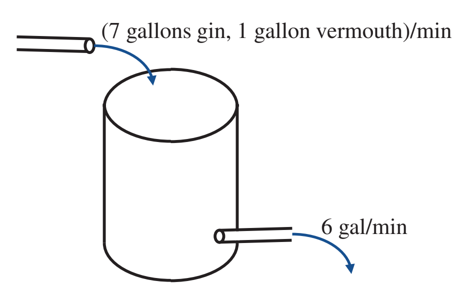
$$\frac{dy}{dx} + a_0(x)y = f(x) \tag{10.4.1}$$

1. It would seem that a more general form would be $a_1(x)y' + a_0(x)y = f(x)$. Why can we claim that Equation 10.4.1 is fully general?
2. Write the complementary homogeneous equation for Equation 10.4.1.
3. Solve your homogeneous equation by separation of variables. Your answer will have an integral with respect to x in it.
4. Confirm that your answer works.

In this section we will find a general formula for solving the *inhomogeneous* Equation 10.4.1. But keep an eye on your homogeneous solution from this exercise; you will see it showing up as part of the general solution, as you might expect.

10.4.2 Explanation: Linear First-Order Differential Equations

The tank in the picture begins with 5 gallons of a mixture of gin and vermouth. Every minute, 7 gallons of gin and 1 gallon of vermouth are added to the tank. The tank is also being drained at 6 gallons/minute. Assuming the mixture in the tank is stirred constantly, find the total amount of gin in the tank as a function of time.



This problem calls on us to create a differential equation for $G(t)$, the total amount of gin in the tank, measured in gallons. There are two effects causing G to change over time.

- The inflow is very simple: every minute, 7 gallons of gin are added to the tank. If this were the only effect, we would write $dG/dt = 7$.
- Assuming the tank is kept evenly mixed, the outflow each minute is the total amount of gin in the tank (G) multiplied by the fraction of the mixture that is drained out. (For instance, if $1/10$ of the mixture flows out, then $G/10$ gallons of gin are drained.) The total amount of the mixture is increasing by 2 gallons every minute, so the total volume after t minutes is $5 + 2t$. That means the fraction of the mixture that is drained every minute is $6/(5 + 2t)$.

We arrive at the following differential equation:

$$\frac{dG}{dt} = 7 + \left(\frac{6}{5+2t}\right)G \tag{10.4.2}$$

Equation 10.4.2 isn't separable, and doesn't suggest any obvious guess—so there go the only two methods we've seen so far. In this section we're going to introduce a new method that is specifically designed for first-order linear equations such as this one: that is, equations that can be expressed as follows.

$$\frac{dy}{dx}+a_0(x)y=f(x) \qquad \textit{the generic first-order linear ODE} \tag{10.4.3}$$

This method can be approached as either a *technique* or a *formula*, and we will solve Equation 10.4.2 both ways to show how the two are comparable.

The Technique

As an analogy to this technique we will quickly review the algebra trick called "completing the square," which is based on the following formula.

$$(x+a)^2=x^2+2ax+a^2 \tag{10.4.4}$$

Every quadratic function involves x^2+2ax for some number a. If you add a^2 then you can use Equation 10.4.4 to turn it into a perfect square, and then just take the square root.

$$\begin{aligned} x^2+10x &= 11 && \textit{the question}\\ x^2+10x+25 &= 36 && \textit{add 25 to make the left side match Equation 10.4.4}\\ (x+5)^2 &= 36 && \textit{because it does match!}\\ x+5 &= \pm 6 && \textit{so } x \textit{ is } -11 \textit{ or } 1 \end{aligned}$$

Now: before you read further, grab a piece of scratch paper and take the derivative, with respect to x, of $ye^{\int a_0(x)dx}$. (You will need a chain rule inside a product rule.) The result provides a template for ODEs, just as Equation 10.4.4 is a template for quadratic functions.

$$\frac{d}{dx}\left(ye^{\int a_0(x)dx}\right)=y'e^{\int a_0(x)dx}+ya_0(x)e^{\int a_0(x)dx}=e^{\int a_0(x)dx}\left[y'+a_0(x)y\right] \tag{10.4.5}$$

Every first-order linear differential equation involves $y'+a_0(x)y$ for some function $a_0(x)$. If you multiply by $e^{\int a_0(x)dx}$ then you can use Equation 10.4.5 to turn it into a perfect derivative, and then just integrate.

As an example let's tackle Equation 10.4.2, starting by putting it in standard form.

$$\frac{dG}{dt}-\left(\frac{6}{5+2t}\right)G=7 \tag{10.4.6}$$

Our a_0 function in this case is $-6/(5+2t)$, so (after a few calculations) $e^{\int a_0(t)dt}=(5+2t)^{-3}$. This function is called the "integrating factor."

Integrating Factor

An integrating factor for a differential equation is a function that you multiply by the equation to allow you to integrate both sides. For a first-order linear ordinary differential equation the integrating factor is

$$I(x)=e^{\int a_0(x)\,dx} \tag{10.4.7}$$

Equation 10.4.7 only works as an integrating factor if your differential equation is in the form of Equation 10.4.3. For instance, if your equation starts with $2y'(x)$ then you have to divide both sides by 2 before finding the integrating factor.

Multiplying both sides of Equation 10.4.6 by its $I(t)$ we get:

$$(5+2t)^{-3}\left(\frac{dG}{dt}\right) - 6(5+2t)^{-4}G = 7(5+2t)^{-3}$$

That may not look like an improvement but Equation 10.4.5 promises us that the left side is the derivative of $G(5+2t)^{-3}$, a fact we urge you to confirm for yourself. (Again, chain rule inside a product rule.) So we can now integrate both sides and solve.

$$G(5+2t)^{-3} = -\frac{7}{4}(5+2t)^{-2} + C$$
$$G = -\frac{7}{4}(5+2t) + C(5+2t)^3$$

We leave it to you to confirm that this is a valid solution to Equation 10.4.2. Note that this technique has something in common with separation of variables: you put in the $+C$ when you integrate and carry it through the math from there, and you end up with the general solution. We never bothered finding separate "particular" and "complementary" solutions. But if you solve the complementary homogeneous equation on your own (by separating variables for instance) you will end up with $C(5+2t)^3$, which comes as no great surprise. We see that $-(7/4)(5+2t)$ is acting as the particular solution in this case.

The Formula

You can solve any specific quadratic equation by completing the square. If you apply the same technique to the generic quadratic equation $ax^2+bx+c=0$ you end up with the quadratic formula. Similarly, you can use the technique discussed above to solve specific differential equations. If you apply the same technique to the generic first-order linear differential equation, you end up with the following formula.

The General Solution to All Linear First-Order Differential Equations

The solution to the equation:

$$\frac{dy}{dx} + a_0(x)y = f(x) \tag{10.4.8}$$

is:

$$y = \frac{1}{I(x)}\int I(x)f(x)\,dx \qquad \textit{where} \qquad I(x) = e^{\int a_0(x)\,dx} \tag{10.4.9}$$

As we discuss below, you do *not* add $+C$ when you integrate a_0 to find I, but you *do* add $+C$ when you integrate If to find y.

This solution involves two integrals, either of which may be impossible. Alas, many differential equations are just impossible to solve, no matter how many techniques you learn. But when you can evaluate those integrals, by hand or by computer, solving a first-order linear ODE is now just a matter of plugging into a formula.

Let's see how that plays out in the example above, which started here.

$$\frac{dG}{dt} - \left(\frac{6}{5+2t}\right)G = 7$$

Note that Equation 10.4.9 prominently features $I(x)$, the integrating factor we saw in the technique above. So we start here with the same calculations we did there:

$$a_0 = -6/(5+2t) \quad \text{so} \quad \int a_0(t)\,dt = -3\ln(5+2t) \quad \text{so} \quad I(t) = (5+2t)^{-3}$$

Now we plug into Equation 10.4.9:

$$G(t) = (5+2t)^3 \int 7(5+2t)^{-3}\,dt = (5+2t)^3\left(-\frac{7}{4}(5+2t)^{-2} + C\right)$$

And we have arrived at the same answer we got the first time.

The definition of I involves an indefinite integral, so why didn't we need an arbitrary constant in I? What would have happened if we had chosen $I(t) = -3\ln(5+2t) + 17$ instead of $I(t) = -3\ln(5+2t)$? You might think we would get a different answer that would also work, but in fact we would have ended up with exactly the same answer. You may want to confirm this right now for this particular example; you will prove it in general in Problem 10.75.

EXAMPLE **First-Order Linear Differential Equation**

Question: Solve the equation $(dy/dx)/x^2 + y/x^3 = 1$.

Solution:
This is a first-order linear equation, but to use the technique of this section we have to put it in the form of Equation 10.4.3, in which nothing is multiplied or divided by dy/dx.

$$\frac{dy}{dx} + \frac{y}{x} = x^2$$

Now we can see that $a_0(x) = 1/x$, so $\int a_0(x)\,dx = \ln x$ and $I(x) = x$. We therefore multiply both sides of the equation by x:

$x(dy/dx) + y = x^3$

The left side of the equation is now the derivative, with respect to x, of xy. (Confirm that!) So we integrate both sides with respect to x and get:

$xy = (1/4)x^4 + C$

Dividing both sides by x solves the problem. Alternatively, we could approach the problem—and reach the same answer—with Equation 10.4.9. Again we need to use the form of the equation with no factor in front of dy/dx, so $f(x) = x^2$:

$$y = \frac{1}{x}\int x^3\,dx = \frac{1}{x}\left(\frac{1}{4}x^4 + C\right) = \frac{1}{4}x^3 + \frac{C}{x}$$

We leave it to you to confirm that this is a valid solution.

10.4.3 Problems: Linear First-Order Differential Equations

10.63 Walk-Through: Linear First-Order ODE. In this problem you will solve the equation $(\csc x)dy/dx - y = 1$.

(a) Rewrite this equation in the form of Equation 10.4.3.

(b) Identify the function $a_0(x)$.

(c) Find the integrating factor $I(x)$ as defined in Equation 10.4.7.

(d) Multiply both sides of the equation by the integrating factor you found in Part (c).

(e) Show that the left side of the resulting equation is the derivative, with respect to x, of $I(x)y(x)$.

(f) Integrate both sides of your answer to Part (d) with respect to x. (Note that Part (e) does half the work for you.) Don't forget to put a $+C$ on the right side!

(g) Solve the resulting equation. You should end up with a formula for $y(x)$ that includes an arbitrary constant.

(h) Show that your final answer is a valid solution to the original differential equation.

10.64 *[This problem depends on Problem 10.63.]* Redo Problem 10.63 using separation of variables. Make sure you get the same answer!

In Problems 10.65–10.72 solve the given first-order linear differential equation. You may find it helpful to first work through Problem 10.63 as a model.

10.65 $dy/dx + 3y = e^{3x}$

10.66 $dy/dx + y = \sin x$ *Hint*: you may find the integrals on Page 485 useful.

10.67 $dy/dx + y/(2x) = e^x/\sqrt{x}$

10.68 $dy/dx = (\sin x)(\cos x - y/\cos x)$

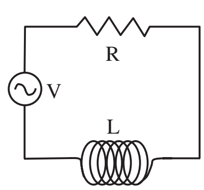
10.69 $\dfrac{dy}{dx} + \left(\dfrac{e^x}{e^x+1}\right) y = \dfrac{1}{e^x}$

10.70 $x^2(dy/dx) = 3xy - x - 1$

10.71 $(y-2)dx + dy/(2x^2+3) = 0$

10.72 $(x^2-4)(dy/dx) = 3x(y+2)$

10.73 The picture shows a circuit with three elements: a voltage source, a resistor, and an inductor.



The general equation governing this circuit is $L(dI/dt) + RI = V(t)$ where L is the inductance of the inductor, I is the current, R is the resistance of the resistor, and $V(t)$ is the voltage output of the source. Assume $R = 10\ \Omega$, $L = 2$ H, and $V(t) = V_0 \sin(\omega t)$.

(a) Write the differential equation for $I(t)$ for this circuit.

(b) Solve it. *Hint*: you may find the integrals on Page 485 useful.

10.74 Consider a special case of Equation 10.4.3 for which $a_0(x) = f(x)$.

(a) Evaluate Equation 10.4.9 using the substitution $u = \int f(x)\,dx$. You should get an answer for $y(x)$ that depends on $f(x)$.

(b) Show that the answer you found in Part (a) works.

(c) Use your result to write down the solution to $dy/dx + x^3 y = x^3$ with little or no work.

10.75 When you calculate $I(x)$ in Equation 10.4.9, there is always an implicit $+C$ in the function. Show that including that constant does not change the final answer you get for the solution y. (Once you have proven this, you can ignore that $+C$ when applying this formula.)

10.76 A 5000 gallon tank is initially filled with pure water. A pipe at the top is pouring 100 gallons per minute of Gluppity-Glupp into the tank, while a drain at the bottom is dumping 200 gallons per minute of mixed water and Glupp into the pond where the Humming-Fish hum. Let $G(t)$ be the number of gallons of Gluppity-Glupp in the tank. You may assume throughout the problem that the tank is well mixed, meaning the ratio of Glupp to water going out the drain equals the ratio of Glupp to water in the entire tank.

(a) What is the total volume in the tank $V(t)$?

(b) What fraction of the tank mixture is Gluppity-Glupp? Your answer should depend on G and t.

(c) How many gallons of Glupp leave the tank each minute through the drain?

(d) Write a differential equation for $G(t)$ that takes into account increases in G through the input pipe and losses through the drain.

(e) Solve the equation to find $G(t)$ using the initial condition given in the problem.

(f) Find the total amount of Gluppity-Glupp dumped into the pond from the start of the process to the time when the tank is empty.

10.77 You invest A dollars per year in a bank account which gains interest at a rate R. That means every year the bank gives you an amount of money equal to R times the amount you have in the bank.

(a) Write a differential equation for the amount of money $M(t)$ you have in the bank. (Assume the investment and interest both occur continuously, not just once per year.)

(b) Assume the amount you invest and the interest rate are both dropping exponentially: $A = 1000e^{-t}$ and $R(t) = .05e^{-t}$ where t is measured in years. Solve for $M(t)$.

(c) If you start with nothing in the bank, how much money do you have after 10 years?

10.78 For as long as anyone can remember the birth and death rates in the city of Foom have been exactly equal, keeping the population constant. At a certain time, however, the birth rate started dropping. To try and compensate the leaders of Foom began allowing immigrants to enter the city each year.

(a) Assuming the birth rate dropped linearly, the death rate remained constant, and the immigration rate grew linearly, write and solve a differential equation for the population of Foom starting at the moment when the birth rate started dropping and immigrants began arriving. (Note that birth rate has units of year^{-1}—it is the number of people born each year divided by the total population—and similarly for death rate. Immigration, on the other hand, has units of people/year.)

(b) Describe the behavior indicated by your solution. Will the population grow without bound? shrink toward zero? Approach a constant value? Does the answer depend on the values of the constants?

10.79 **Make Your Own.** Write a first-order linear differential equation that isn't in this section (including the problems) and solve it using the techniques from this section. (The tricky part is making sure you can evaluate both integrals!)

10.5 Exact Differential Equations

In this section we will encounter differential equations written in the unfamiliar looking form $P\ dx + Q\ dy = 0$. In the specific case where $\partial P/\partial y = \partial Q/\partial x$ this is called an "exact differential equation" and has a simple solution.

This section relies on your prior understanding of differentials, as discussed in Chapter 4.

10.5.1 Discovery Exercise: Exact Differential Equations

The buoyancy B of a hot air balloon is a function of the temperature T of the air inside the balloon and the volume V of the balloon.

1. If the air temperature changes while the volume stays constant, the resulting change in buoyancy is given by:
$$dB = \frac{\partial B}{\partial T} dT$$
Explain why. Your explanation should focus on the meaning of that partial derivative.
2. If the volume changes while the air temperature stays constant, what is the resulting change in buoyancy?
3. If the temperature and volume both change, what is the total resulting change in buoyancy?

10.5.2 Explanation: Exact Differential Equations

The following, believe it or not, is a differential equation.

$$3x^2 y\ dx + x^3\ dy = 0 \tag{10.5.1}$$

This section will answer two questions about that equation and others like it.

- What on Earth does that even mean?
- And oh yeah, how do I solve it?

One way to answer both questions is to rearrange the terms into a more familiar-looking form.

$$\frac{dy}{dx} = \frac{-3x^2 y}{x^3} \tag{10.5.2}$$

We all know what *that* means, and how to solve it. Mathematically it's perfectly valid to turn Equation 10.5.1 into Equation 10.5.2. But for reasons we will explain soon, we want instead to take Equation 10.5.1 on its own terms.

How to Read (and Solve) a Funny-Looking Equation Like That One

Equation 10.5.1 presents a relationship between x and y, but we're going to introduce a new variable f that depends on both x and y. What happens to f if you change x by a small amount dx and change y by a small amount dy? Such a "total differential" is found by using partial derivatives. (Partial derivatives, and equations such as the following, are explained in Chapter 4.)

$$df = \left(\frac{\partial f}{\partial x}\right) dx + \left(\frac{\partial f}{\partial y}\right) dy$$

That's starting to look a bit like Equation 10.5.1. And if we happen to choose $f(x, y) = x^3 y$, then it looks a *lot* like Equation 10.5.1. With that insight we are ready to address the two questions that started us off.

- What does Equation 10.5.1 mean? It means that if we change x by a small amount dx, and change y by a small amount dy, then the function $f(x, y) = x^3 y$ will not change at all: its df will be zero.
- And how do we solve it? If $df = 0$ then f must be a constant, and it doesn't matter what constant. So we write $x^3 y = C$ and we have the general solution, arbitrary constant and all.

EXAMPLE Verifying a Solution to an Exact Differential Equation

Problem:
We said above that $x^3 y = C$ is the solution to Equation 10.5.1. Verify this solution.

Solution:
Let's pretend for the moment that x and y are both functions of a third variable—we will call it t and think of it as time, but it could be anything. Now take the derivative of both sides of the solution with respect to time. (Remember that C is *not* a function, but a constant.)

$$x^3 y = C$$
$$3x^2 \frac{dx}{dt} y + x^3 \frac{dy}{dt} = 0$$

Multiply both sides by dt and you have the differential equation we set out to solve.

You can think about this result geometrically. For any given value of C the equation $x^3 y = C$ traces out a curve. (These are called the "level curves" of the function $x^3 y$.) If you start at the point (x, y) and take a small step along the curve, your dy is related to your dx by the equation of the curve. The differential equation defines that family of curves by saying how dy and dx are related at each point (x, y).

Of course we can rewrite that solution as $y = C/x^3$. You can verify that this is a valid solution to Equation 10.5.2 (or you can solve Equation 10.5.2 by separating variables and verify that you end up with C/x^3). But the method we are presenting does not depend on being able to solve for y; it is a more general technique than that, and we present it now in general form.

Exact Differential Equations: First Definition, and Solution

The equation:

$$P(x, y)\, dx + Q(x, y)\, dy = 0 \tag{10.5.3}$$

is an "exact differential equation" if there exists a function $f(x, y)$ such that

$$\partial f/\partial x = P \quad and \quad \partial f/\partial y = Q \tag{10.5.4}$$

In that case, the general solution is:

$$f(x, y) = C$$

You may still be wondering why we prefer this new, abstract formulation to the more familiar approach represented by Equation 10.5.2. One reason is that many differential equations can't easily be solved in the form $dy/dx =$<*something*>. If we write an equation in the form of Equation 10.5.3 it's easy to check if it is exact and solve it if it is, as we'll explain below. Another reason is that exact differential equations are not limited to two variables. You can solve $P(x,y,z)\,dx + Q(x,y,z)\,dy + R(x,y,z)\,dz = 0$ with the same approach, but you cannot write it in terms of a simple derivative of x, y, or z. Such equations come up in applications where differentials play an important role, and it is important to understand and be able to work with them. (We discuss the role of such equations in thermodynamics in Section 4.10: see felderbooks.com.)

How Do You Know If Your Equation is Exact?

Here's a different definition of "exact."

> **Exact Differential Equations: Second (Equivalent) Definition**
>
> Equation 10.5.3 represents an exact differential equation if and only if:
>
> $$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \tag{10.5.5}$$

You can verify this quickly on the example we gave above. The function $P = 3x^2y$ so $\partial P/\partial y = 3x^2$. And $Q = x^3$ so $\partial Q/\partial x = 3x^2$. Because the two come out the same, we know the equation is exact.

Some textbooks use Equation 10.5.4 as a definition of exact (as we have), and others use Equation 10.5.5. We will show below that the two definitions are equivalent, but first let's talk about why it is useful to have both.

You start with a problem in the form of Equation 10.5.3: that is, you are given the functions $P(x,y)$, and $Q(x,y)$. Equation 10.5.4 tells us that they define an exact differential equation if "there exists a function $f(x,y)$ such that…" If you can find such a function, you have the whole problem solved. But how do you know if such a function exists or not?

By contrast, Equation 10.5.5 gives you an easy method based only on the given P and Q to determine if a given equation is exact. But even if it is, it doesn't give you any hint of the solution.

So a common approach is to begin with Equation 10.5.5. If $\partial P/\partial y$ and $\partial Q/\partial x$ are not equal, then the equation is not exact, and you must move on to other methods. If they are equal, then you integrate to find the function f that you *know* must exist, and that will give you the solution. We discuss later in the section how to generalize this rule to equations with more than two variables.

How Do You Solve an Exact Differential Equation?

The example below illustrates the method we just discussed. First we use Equation 10.5.5 to check that the equation is exact. Once we know that, our goal is to solve the equations $\partial f/\partial x = P$ and $\partial f/\partial y = Q$—a process that is easier to demonstrate (below) than to explain (here).

EXAMPLE **Exact Differential Equation**

Question: Solve this differential equation.

$$2x \ln y \, dx + \left(\frac{x^2}{y} + 6e^{2y} \right) \, dy = 0$$

Solution:
We begin by determining if this is an exact equation, based on Equation 10.5.5.

$$P(x,y) = 2x \ln y, \quad \text{so} \quad \frac{\partial P}{\partial y} = \frac{2x}{y}$$
$$Q(x,y) = \frac{x^2}{y} + 6e^{2y}, \quad \text{so} \quad \frac{\partial Q}{\partial x} = \frac{2x}{y}$$

Because the two answers are equal, this is an exact differential equation. We now set out to find the function f that Equation 10.5.4 promises.

We begin with $\partial f/\partial x = 2x \ln y$. This means that f could be $x^2 \ln y$, but it doesn't have to be exactly that. If we add any constant to that function, or any pure function of y, then $\partial f/\partial x$ will remain unchanged. So we write

$$f(x,y) = x^2 \ln y + g(y)$$

Based on that equation, $\partial f/\partial y = x^2/y + g'(y)$. But $\partial f/\partial y$ must be Q, which in this case is $x^2/y + 6e^{2y}$. So we see that $g'(y) = 6e^{2y}$, meaning $g(y) = 3e^{2y}$.

We have now found that $f(x,y) = x^2 \ln y + 3e^{2y}$ has the appropriate partial derivatives. The solution to our differential equation is therefore:

$$x^2 \ln y + 3e^{2y} = C$$

(When we integrated $g'(y)$ to get $g(y)$ we could have included $+C$, but it would have gotten absorbed in the arbitrary constant we set f equal to anyway.)

If the middle step in that example made you suspect that our original differential equation was carefully contrived, you're absolutely right. What if the coefficient of dy had not looked like x^2/y plus a pure function of y? That would mean the original equation was not exact. It would have failed the $\partial P/\partial y = \partial Q/\partial x$ test, and no $f(x,y)$ function could be found.

We've given two completely different definitions of "exact." Why are they the same?

Equation 10.5.4 tells us there exists a function $f(x,y)$ such that $\partial f/\partial x = P$. Take the derivative of both sides of that equation with respect to y and you get this.

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial P}{\partial y}$$

Similarly, you can start with $\partial f/\partial y = Q$ and take the derivative of both sides of that equation with respect to x.

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial Q}{\partial x}$$

Since partial derivatives commute, those two quantities must be equal, so Equation 10.5.4 leads us to Equation 10.5.5.

If you're familiar with vector calculus, we can reframe everything we have said more concisely. Equation 10.5.3 is exact if $P(x,y)\hat{i} + Q(x,y)\hat{j}$ represents a *conservative vector field.* (See Section 8.11.) Recall that a vector field $\vec{V}$ is conservative if there exists a function f such that $\vec{V} = \vec{\nabla} f$, which is what Equation 10.5.4 says. We also know that $\vec{V}$ is conservative if and only if $\vec{\nabla} \times \vec{V} = \vec{0}$, which is what Equation 10.5.5 says.

This interpretation also suggests the way to generalize Equation 10.5.5 to three-variable equations.

Exact Differential Equations: Three-Variable Definition

The equation $P(x,y,z)dx + Q(x,y,z)dy + R(x,y,z)dz = 0$ is exact if and only if $\vec{V} = P\hat{i} + Q\hat{j} + R\hat{k}$ represents a conservative vector field: in other words, if $\vec{\nabla} \times \vec{V} = \vec{0}$.

The curl operator is not defined in more than three dimensions, however, so for four or more variables just use the brute force method. See Problems 10.92–10.93.

Integrating Factors

Recall from Section 10.4 that we sometimes multiply both sides of a differential equation by the same thing—an "integrating factor." One use of this technique is to *make* an exact equation where there was none.

EXAMPLE **Integrating Factor**

Problem:
Solve the equation $3y\ dx + x(2y+1)\ dy = 0$

Solution:
As given, the equation is not exact. ($\partial P/\partial y \neq \partial Q/\partial x$.) You can try multiplying both sides of the equation by x, and it's still not exact. Try a few other things. (Go ahead, we'll wait.)

Now we'll multiply both sides of the equation by $x^2 e^{2y}$.

$3x^2 y e^{2y}\ dx + x^3 e^{2y}(2y+1)\ dy = 0$

Ta-da! $\partial P/\partial y$ is now the same as $\partial Q/\partial x$. With a bit more work (as in the previous example) you can solve it.

$$x^3 y e^{2y} = C$$

We know that is the solution because if $f(x,y) = x^3 y e^{2y}$ then $\partial f/\partial x = 3x^2 y e^{2y}$ and $\partial f/\partial y = x^3 e^{2y}(2y+1)$. You should confirm, however, that this also solves the original differential equation.

We hope you can easily confirm for yourself that the original equation in that example was not exact, that multiplying by x would not have made it exact, and that multiplying by $x^2 e^{2y}$ did. But none of that suggests how you can come up with integrating factors on your own. There is no general answer to that question, but there is a formula you can use in some cases.

The Integrating Factor to Make an Exact Differential Equation

Given an equation that is in the form of Equation 10.5.3 but is not exact, you want to find an integrating factor I that will make it exact. If such a factor exists that is a function of x and not of y, then it is:

$$I(x) = e^{\int \left[\frac{1}{Q} \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} \right) \right] dx} \tag{10.5.6}$$

If a factor exists that is a function of y and not of x, then it is:

$$I(y) = e^{\int \left[\frac{1}{P} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \right] dy} \tag{10.5.7}$$

In many cases—such as the example we worked above—the integrating factor is a function of both x and y, in which case those formulas won't help. But a formula that works sometimes is better than no formula at all.

These formulas follow from Equation 10.5.5. In general, an integrating factor $I(x, y)$ makes a differential equation exact if:

$$\frac{\partial}{\partial y}(IP) = \frac{\partial}{\partial x}(IQ)$$

Applying the product rule turns this into:

$$I\frac{\partial P}{\partial y} + P\frac{\partial I}{\partial y} = I\frac{\partial Q}{\partial x} + Q\frac{\partial I}{\partial x}$$

If it happens that I is a function of x only, then $\partial I/\partial y = 0$. Dropping that term, dividing both sides by QI, and rearranging leads to:

$$\frac{1}{I}\left(\frac{dI}{dx}\right) = \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$$

The left side of this equation is the derivative with respect to x of $(\ln I)$, so integrating both sides and then exponentiating leads to Equation 10.5.6. A similar argument leads to Equation 10.5.7 for integrating factors that only depend on y. (See Problem 10.101.)

EXAMPLE **Finding an Integrating Factor**

Problem:
Solve the equation $2xy \sin y \, dx + x^2 y \cos y \, dy = 0$

Solution:
The equation as given is not exact. Applying Equation 10.5.6 we begin with:

$$\frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = \frac{1}{x^2 y \cos y}\left(2x \sin y + 2xy \cos y - 2xy \cos y\right)$$

It looks promising, with most of the terms in the parentheses canceling, but there's a problem. When you simplify this, you're going to end up with a function of both x and y. Integrating with respect to x, and exponentiating, is not going to make those

"y"s go away. Equation 10.5.6 only gives us a useful integrating factor if it gives us a function of x only, so this won't help. (You can finish integrating and exponentiating this, but you'll find that it doesn't make the differential equation exact.)

Let's see if Equation 10.5.7 works out any better.

$$\frac{1}{P}\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)=\frac{1}{2xy\sin y}\left(2xy\cos y-2x\sin y-2xy\cos y\right)$$

That's more like it! That simplifies to $-1/y$; $\int(-1/y)dy=-\ln y$, so $e^{-\ln y}=1/y$ is the integrating factor. This turns our original equation into:

$$2x\ \sin y\ dx+x^2\ \cos y\ dy=0$$

which is exact, as promised. The solution is $x^2\sin y=C$.

10.5.3 Problems: Exact Differential Equations

10.80 **Walk-Through: Exact ODE.** $2\cos(2x+y)\ dx+\left(\cos(2x+y)+3\sin y\right)\ dy=0$

- **(a)** This problem is in the form of Equation 10.5.3. What are the functions $P(x,y)$ and $Q(x,y)$?
- **(b)** Use Equation 10.5.5 to show that this is an exact differential equation.
- **(c)** Because this is an exact differential equation, there must exist a function $f(x,y)$ such that $\partial f/\partial x=P$ and $\partial f/\partial y=Q$. To begin finding it, write down the general solution to the equation $\partial f/\partial x=P$ for the $P(x,y)$ that you wrote above. Note that your solution at this stage will involve an arbitrary function $g(y)$.
- **(d)** Take the partial derivative with respect to y of the $f(x,y)$ function you wrote in Part (c). Set the result equal to your $Q(x,y)$ function and solve to find $g(y)$.
- **(e)** Write the function $f(x,y)$ and confirm that it fulfills Equation 10.5.4.
- **(f)** Write the solution to the differential equation.
- **(g)** Assuming that x and y are both functions of t, verify that your answer solves the original differential equation.

10.81 *[This problem depends on Problem 10.80.]* Rewrite the differential equation in Problem 10.80 in the form $dy/dx=$ <*some function of x and y*>. Show that your final solution to Problem 10.80 is a valid solution to this differential equation.

In Problems 10.82–10.88 determine if the given differential equation is exact or not. If it is, solve it. You may find it helpful to first work through Problem 10.80 as a model.

10.82 $x\ dx+y\ dy=0$

10.83 $y\ dx+x\ dy=0$

10.84 $y\ dx-x\ dy=0$

10.85 $(6x+6xy+10y)\ dx+(3x^2+10x+14y)\ dy=0$

10.86 $(10x+3y+8)\ dx+(4x+4y)\ dy=0$

10.87 $\dfrac{y}{(x+y)^2}dx-\dfrac{x}{(x+y)^2}dy=0$

10.88 $\left(\dfrac{x}{\sqrt{x^2-y}}+2x\right)dx-\left(\dfrac{1}{2\sqrt{x^2-y}}+2y\right)dy=0$

10.89 In special relativity the length of an object is given by the formula $L=L_0\sqrt{1-v^2/c^2}$ where L_0 is the "rest length" of the object, v is its speed, and c, the speed of light, is a constant.

- **(a)** If the rest length of the object increases by a small dL_0, calculate the resulting change dL in the length.
- **(b)** If the object increases its speed by a small dv, calculate the resulting change dL in the length.
- **(c)** Write a differential equation that says "Both L_0 and v increased by small amounts in such a way that there was no net change in the length." Your

differential equation should be in the form of Equation 10.5.3.

(d) Solve your differential equation.

10.90 A light source of strength S is shining on an object x meters away. You have measured that when you increase the strength of the source the illumination of the object increases according to $\partial I/\partial S = k/x^2$, where k is a positive constant. You've also measured that when you move the object farther from the source the illumination deceases: $\partial I/\partial x = -2kS/x^3$. Write and solve a differential equation of the form *<something>dS+<something>dx* = 0 that says "Both S and x increased by small amounts in such a way that there was no net change in the illumination."

10.91 Measurements of the electric field in a region give $\vec{E} = (-2xe^{-x^2-y^2}+1)\hat{\imath} - 2ye^{-x^2-y^2}\hat{\jmath}$. The electric potential V is related to the electric field by $\partial V/\partial x = -E_x$ and $\partial V/\partial y = -E_y$. Find $V(x,y)$, up to an arbitrary additive constant.

10.92 In this problem you'll solve the equation $y\,dx + (x + 2y\sin z)dy + y^2\cos(z)dz = 0$.

(a) To check that this equation is exact, define a vector $\vec{V} = y\hat{\imath} + (x+2y\sin z)\hat{\jmath} + y^2\cos(z)\hat{k}$ and show that $\vec{\nabla}\times\vec{V} = \vec{0}$.

(b) Since the equation is exact there must be a function $f(x,y,z)$ such that $\vec{\nabla}f = \vec{V}$. To find that function, first solve $\partial f/\partial x = y$. Your answer should contain an arbitrary function $g(y,z)$.

(c) Using your answer to Part (b), calculate $\partial f/\partial y$ and set it equal to $x + 2y\sin z$. By solving the resulting equation you should be able to find f up to an arbitrary function $h(z)$.

(d) Finally, set $\partial f/\partial z = y^2\cos z$ and solve to find f up to an arbitrary constant.

(e) Write the solution to the differential equation in the form $f = C$, where C is the (only) arbitrary constant in the solution.

10.93 With four or more variables, you can't use the curl to test if a differential equation is exact, so you simply have to start trying to solve it and see if it works. Consider two differential equations that we will call D_1 and D_2.

$$D_1: \ (yzt + e^z)dx + xzt\,dy + (xyt + xe^z)dz + (xyz + 2t)\,dt = 0$$

$$D_2: \ (yzt + e^z)dx + xzt\,dy + (xyt + e^z)dz + (xyz + 2t)\,dt = 0$$

Show that one of them is not exact. Show that the other one is exact, and solve it.

10.94 In this problem you will solve the equation $y\,dx + 2\tan x\,dy = 0$ by using an integrating factor.

(a) Show that the equation as given is *not* exact.

(b) Multiply both sides of the equation by $y\cos x$.

(c) Show that the resulting equation is exact, and solve it.

10.95 In this problem you will solve the following equation by using an integrating factor.

$$\frac{3y^2+2y}{x}\,dx + (xy^2 - 3y)\,dy = 0$$

(a) Show that the equation as given is *not* exact.

(b) Multiply both sides of the equation by $1/(xy)$.

(c) Show that the resulting equation is exact, and solve it.

In Problems 10.96–10.99 use Equations 10.5.6 and 10.5.7 to find an appropriate integrating factor and solve the equation.

10.96 $dx + 2x\cos y\,dy = 0$

10.97 $e^{x+2y}dx + \left(2 + 2/y\right)e^{x+2y}dy = 0$

10.98 $2xy\,dx + 3x^2\,dy = 0$

10.99 $\dfrac{2\ln(x^2+y)}{x}dx + \dfrac{\ln(x^2+y)}{x^2}dy = 0$

10.100 $P(x,y)dx + Q(x,y)dy = 0$ is an exact differential equation with solution $f(x,y) = C$. Is $P(x,y)dx + (Q(x,y)+7)dy = 0$ also an exact differential equation? If not, why not? If so, what is the solution?

10.101 Show that if an integrating factor exists that is a function of y only, it must be given by Equation 10.5.7.

10.102 The "thermodynamic identity" for a gas in a sealed container (constant number of molecules) relates the change in internal energy U of the gas to changes in its entropy S and volume V: $dU = T\,dS - P\,dV$ where T and P are the temperature and pressure of the gas. For a monatomic ideal gas these are given by

$$T = \frac{C}{V^{2/3}}e^{(2/3)S/(Nk_B)}, \quad P = \frac{CNk_B}{V^{5/3}}e^{(2/3)S/(Nk_B)}$$

The constants N and k_B are the number of molecules and Boltzmann's constant, and C is a another constant that depends on N. Starting from the thermodynamic identity, derive a relationship between S and V that must hold for an ideal gas undergoing a process at constant internal energy. (If you know enough thermodynamics there are easier ways to derive this. You could do that to check yourself, but for this problem you should derive it by setting $dU = 0$ and solving the resulting equation.)

10.103 The "thermodynamic identity" for a gas relates the change in internal energy U of the gas to changes in its entropy S, volume V, and number of molecules N: $dU = TdS - PdV + \mu dN$. T, P, and μ are the temperature, pressure, and "chemical potential" of the gas. For a monatomic ideal gas these are given by

$$T = \frac{h^2 N^{2/3}}{2\pi e^{5/3} m k_B V^{2/3}} e^{(2/3)S/(Nk_B)}$$

$$P = \frac{h^2 N^{5/3}}{2\pi e^{5/3} m V^{5/3}} e^{(2/3)S/(Nk_B)}$$

$$\mu = -\left(\frac{S}{Nk_B} - \frac{5}{2}\right) \frac{h^2 N^{2/3}}{2\pi e^{5/3} m V^{2/3}} e^{(2/3)S/(Nk_B)}$$

Here m is the mass of a molecule, h is Planck's constant, and k_B is Boltzmann's constant. Starting from the thermodynamic identity, derive a relationship between S, V, and N that must hold for an ideal gas undergoing a process at constant internal energy. (If you know enough thermodynamics there are easier ways to derive this. You could do that to check yourself, but for this problem you should derive it by setting $dU = 0$ and solving the resulting equation.) *Hint*: rather than taking the curl, it's easier in this case to just look for a function that has the right partial derivatives.

10.104 You are in charge of the production line at Spacely Sprockets. Mr. Spacely has told you to increase production, but he refuses to increase your budget. You decide to accomplish this by putting less metal in each sprocket. Let C be your total cost, S be the number of sprockets you produce, and M be the grams of metal in each sprocket. Taking into account the grams of metal per sprocket M and the discounts you get for bulk buying, $\partial C/\partial S = M(S+10)/\sqrt{(S+10)^2 - 100}$ and $\partial C/\partial M = \sqrt{(S+10)^2 - 100}$. Find an equation relating S and M that will keep your total costs fixed.

10.105 You're conducting experiments on a flat table. The experiment produces varying amounts of heat in different places, and a series of measurements tells you that $\partial T/\partial x = e^y$ and $\partial T/\partial y = xe^y$. Sketch the isotherms (curves of constant temperature) on the surface.

10.106 You're conducting experiments on a flat table. The experiment produces varying amounts of heat in different places, and a series of measurements tells you that $\partial T/\partial x = \sin(y^2 + x) + x\cos(y^2 + x)$ and $\partial T/\partial y = 2xy\cos(y^2 + x)$. Sketch the isotherms (curves of constant temperature) on the surface.

10.107 **Make Your Own.**

(a) Write an exact differential equation that isn't in this section (including the problems) and solve it using the techniques from this section.

(b) Write a differential equation that is not exact, but that can be made exact by multiplying both sides by $y/(\ln x)$. *Hint*: This is very easy once you have done Part (a).

(c) Find a function $Q(x, y)$ to make $\sin(x + 2y)\, dx + Q(x, y)\, dy = 0$ an exact differential equation.

10.6 Linearly Independent Solutions and the Wronskian

The “Wronskian” is a tool that can help you determine if you have found linearly independent solutions to a differential equation. That, in turn, helps you know when you have found the general solution.

10.6.1 Discovery Exercise: Linearly Independent Solutions and the Wronskian

1. Each of the following functions is a valid solution to the differential equation $y''(x) + k^2 y(x) = 0$. But four of these functions represent (in different forms) the *general solution,* and the other three do not. Which ones?
(a) $y(x) = \sin(kx)$
(b) $y(x) = A\sin(kx) + B\cos(kx)$
(c) $y(x) = A\sin(kx + \phi)$
(d) $y(x) = e^{ikx}$
(e) $y(x) = Ae^{ikx} + B\sin(kx)$
(f) $y(x) = Ae^{ikx} + Be^{-ikx}$
(g) $y(x) = Ae^{ikx+\phi}$

See Check Yourself #66 in Appendix L

2. Each of the following functions is a valid solution to the differential equation $4x^2 y''(x) + y(x) = 0$. But two of these functions represent (in different forms) the *general solution,* and the other three do not. Which ones?
(a) $y(x) = \sqrt{x}$
(b) $y(x) = \sqrt{x}\ln x$
(c) $y(x) = A\sqrt{x} + B\sqrt{x}\ln x$
(d) $y(x) = A\sqrt{x} + B\sqrt{x}(1 + \ln x)$
(e) $y(x) = (A + B)\sqrt{x}\ln x$

10.6.2 Explanation: Linearly Independent Solutions and the Wronskian

The differential equation $y'' + k^2 y = 0$ has two solutions: $y_1 = \sin(kx)$ and $y_2 = \cos(kx)$. Those are both valid individual solutions, so superposition says we can combine them to form another solution.

$$y = Ay_1(x) + By_2(x) = A\sin(kx) + B\cos(kx) \tag{10.6.1}$$

Equation 10.6.1 is not just a solution; it is the general solution. All solutions can be written as forms of Equation 10.6.1 with the proper choices of the constants A and B.

Let’s play that discussion back with a slight change. The differential equation $y'' + k^2 y = 0$ has two solutions: $y_1 = 3\sin(kx)$ and $y_2 = -5\sin(kx)$. Those are both valid individual solutions, so superposition says we can combine them to form another solution.

$$y = Ay_1(x) + By_2(x) = 3A\sin(kx) - 5B\sin(kx) \tag{10.6.2}$$

Equation 10.6.2 is a valid solution, but it is *not* the general solution. It’s just an obfuscated way of writing $A\sin(kx)$. Many valid solutions, such as $y = 2\cos(kx)$, cannot fit into this form.

The second solution is not general because its two solutions y_1 and y_2 are *linearly dependent*. They are of course not the same function, but one is a constant multiple of the other.

Definition and Use: Two Linearly Dependent Functions

Two functions $y_1(x)$ and $y_2(x)$ are "linearly dependent" if and only if $y_1 = ky_2$ for some constant k.[3]

If $y_1(x)$ and $y_2(x)$ are solutions to a linear second-order homogeneous differential equation, then $Ay_1(x) + By_2(x)$ is also a solution. But it is the *general* solution if and only if y_1 and y_2 are linearly independent functions.

In our examples above it was obvious which functions were linearly dependent. In other cases it can be harder to tell, but there is a general test you can apply.

Definition and Use: The Wronskian

The "Wronskian" of two functions $y_1(x)$ and $y_2(x)$ is:

$$W(x) = y_1 y_2' - y_1' y_2$$

Given a linear, second-order, homogeneous differential equation $y''(x) + a_1 y'(x) + a_0 y(x) = 0$ where a_1 and a_0 are continuous functions on an open interval I (this interval might be "all real numbers" but it doesn't have to be), and given two functions $y_1(x)$ and $y_2(x)$ that solve this equation on I…

- The Wronskian $W(x)$ of y_1 and y_2 is zero everywhere in I, or it is non-zero everywhere in I. (It cannot be zero for some x-values and non-zero for others.)
- If $W(x) = 0$ then the two solutions are linearly dependent.
- If $W(x) \neq 0$ then the two solutions are linearly independent, and $Ay_1(x) + By_2(x)$ is therefore the general solution.

EXAMPLE **The Wronskian**

In Chapter 12 we will show that the linear second-order homogeneous differential equation $(1 - x^2)y'' - 2xy' + l(l + 1)y = 0$ on the open interval $-1 < x < 1$ has the following two solutions.

$$y_1 = 1 - \frac{l(l+1)}{2!}x^2 + \frac{l(l+1)(l-2)(l+3)}{4!}x^4 - \frac{l(l+1)(l-2)(l+3)(l-4)(l+5)}{6!}x^6 + \ldots$$
$$y_2 = x - \frac{(l-1)(l+2)}{3!}x^3 + \frac{(l-1)(l+2)(l-3)(l+4)}{5!}x^5 - \frac{(l-1)(l+2)(l-3)(l+4)(l-5)(l+6)}{7!}x^7 + \ldots$$

The function $Ay_1(x) + By_2(x)$ represents the general solution if and only if y_1 and y_2 are linearly independent—that is, if their Wronskian is non-zero. We begin by taking derivatives.

$$y_1' = -l(l+1)x + \frac{l(l+1)(l-2)(l+3)}{3!}x^3 - \frac{l(l+1)(l-2)(l+3)(l-4)(l+5)}{5!}x^5 + \ldots$$
$$y_2' = 1 - \frac{(l-1)(l+2)}{2!}x^2 + \frac{(l-1)(l+2)(l-3)(l+4)}{4!}x^4 - \frac{(l-1)(l+2)(l-3)(l+4)(l-5)(l+6)}{6!}x^6 + \ldots$$

[3]Technically the function $f(x) = 0$ is linearly dependent with all other functions. Our definition therefore leaves out the case $y_2(x) = 0$ and $y_1(x) \neq 0$. But we are looking here for non-trivial solutions to homogeneous differential equations, so we're going to ignore the zero function for the rest of this section.

The Wronskian is $W(x) = y_1 y_2' - y_1' y_2$. Multiplying every term in y_1 by every term in y_2' does not look practical, but remember that we don't need to prove that $W(x) = 0$ for all x-values; any x-value in our interval can serve as a representative for the entire interval. And it's easy to check at $x = 0$. The first term of $y_1 y_2'$ is 1. Every other term of $y_1 y_2'$, and *every* term of $y_1' y_2$, will go to zero at $x = 0$. We conclude that $W(0) = 1$ which means y_1 and y_2 are linearly independent, so we have the general solution.

We hope the example above demonstrates that the Wronskian is both easy and powerful. But its use rests on the three claims we made in the box "Definition and Use: The Wronskian" and none of these three claims is obvious. We are going to establish the connection between the Wronskian and linear dependence in two ways. The explanation below presents a coherent way to think about what the Wronskian represents and why it demonstrates linear dependence. That conceptual discussion will then allow us to figure out how to generalize the Wronskian to more than two functions. In Problems 10.116–10.117 you will prove two of these claims in a different way: more direct, but possibly less useful in the long run.

Why are Those Three Assertions True?

In justifying the claims we made about the Wronskian, it's easiest to start with the last one. If two functions y_1 and y_2 are linearly dependent then $y_1 = ky_2$. That in turn means $y_1' = ky_2'$, and you can easily plug these in to the definition of the Wronskian and conclude that $W = 0$. Thus if $W \neq 0$, the two functions are linearly independent.

The second claim is the converse of the third: if $W = 0$ everywhere in the interval I then y_1 and y_2 must be linearly dependent in that interval. First let's rewrite the assertion that $W = 0$ at some point $x = a$.[4]

$$W(a) = 0 \quad \Leftrightarrow \quad y_1(a) y_2'(a) = y_2'(a) y_1(a) \quad \Leftrightarrow \quad \frac{y_1(a)}{y_2(a)} = \frac{y_1'(a)}{y_2'(a)} \tag{10.6.3}$$

At the point $x = a$, $y_1 = ky_2$ for some k. (That's always true unless $y_2(a) = 0$.) Equation 10.6.3 tells us that if the Wronskian is zero, the slope of y_1 is also k times the slope of y_2. And now we arrive at the heart of the argument:

If y_1 starts out k times higher than y_2 and increases k times faster than y_2, then it will stay k times higher than y_2.

Of course that's only true for a moment. But remember the first claim: if $W = 0$ anywhere in interval I, then $W = 0$ everywhere in the interval. So as the function climbs with y_1'/y_2' remaining always the same as y_1/y_2, the functions will retain the relationship $y_1 = ky_2$ (as in Figure 10.4), which is what we're trying to show. You can make this particular argument more rigorous by treating the curve as a succession of line segments, and then taking a limit; you will reach this same conclusion in a slightly different way in Problem 10.116.

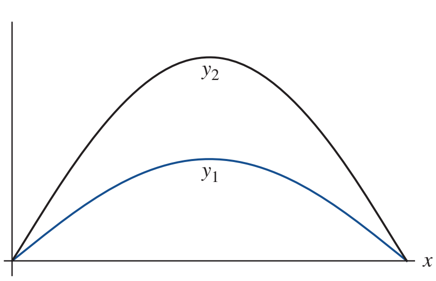


FIGURE 10.4 Two linearly dependent functions. If $y_2 = 2y_1$ then at each point y_2 must be rising or falling twice as fast as y_1.

[4] Our algebra is not valid if $y_2 = 0$ or $y_2' = 0$, but it's not hard to show that the basic conclusions still hold in those cases.

The two arguments we've made so far apply to any two smooth functions[5] y_1 and y_2. If they are linearly dependent then $W=0$, and if $W=0$ throughout some interval I then the functions are linearly dependent in that interval. What remains is the first claim: $W=0$ either everywhere or nowhere in I. That claim is not guaranteed for two arbitrary functions, but it is guaranteed if those functions are both solutions to the same linear second-order differential equation.

Why? We said above y_1 will continue to be proportional to y_2 if y_1' continues to be proportional to y_2'. By the same logic, y_1' and y_2' will continue to be proportional if y_1'' and y_2'' are proportional. (For example, the functions e^x and $x+1$ have the same value and first derivative at $x=0$, but they diverge because their second derivatives are different.)

This is where the differential equation comes in. A second-order linear differential equation can be written $y''=-a_1(x)y'-a_0(x)y$. If you know y and y' at a point, then the differential equation tells you y''. If y_1 and y_2 are both solutions to the same such equation and $y_1/y_2=y_1'/y_2'=k$, then y_1''/y_2'' must also equal k. By differentiating both sides of the differential equation you can get an expression for y''' and similarly conclude that $y_1'''/y_2'''=k$, and so on up to arbitrarily high orders. Since all the higher order derivatives are proportional, y_1 will continue to grow k times faster than y_2, and W will continue to be zero.

Problem 10.117 will present a more algebraic and more rigorous proof of the first claim. In our opinion, however, this hand-waving argument gives more useful intuition for the Wronskian than the algebraic proof, which is why we put this one in the explanation and that one in the problems.

The Wronskian as a Determinant, or, What if There are More than Two Functions?

If you remember how to take the determinant of a 2×2 matrix then it's easy to see that the following determinant is the Wronskian of y_1 and y_2.

$$W=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \tag{10.6.4}$$

Sometimes determinants appear as helpful mnemonics, such as with the cross product or curl—useful, but coincidental—but this is far more than that. Every column in a matrix is a vector. If the determinant of the matrix is zero, then these vectors are linearly dependent. That is one of the primary purposes of the determinant, and look what it means here.

$$W=0 \quad\Longleftrightarrow\quad <y_1,y_1'>=k<y_2,y_2'> \quad\Longleftrightarrow\quad y_1=ky_2,\ y_1'=ky_2'$$

We have already seen that, because y_1 and y_2 are solutions to the same second-order linear ODE, those two relationships are enough to guarantee that the functions are linearly dependent. The payoff for this approach is that it generalizes seamlessly to higher levels. A third-order equation requires three linearly independent functions, a fourth-order requires four, and so on. So we need a general method for determining if n functions are linearly independent—provided, once again, that they are all solutions to the same nth-order linear differential equation.

What does that even mean? If two vectors $\vec{A}$ and $\vec{B}$ are linearly dependent, then $\vec{A}=k\vec{B}$ for some scalar k. For three or more vectors linear dependence is subtler: if $2\vec{A}-3\vec{B}=10\vec{C}$ then the vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$ are linearly dependent even if no two of them are.

The same relationship holds for functions. Consider the functions e^{6x}, $\sin x$, and $2e^{6x}-3\sin x$. Any two of these functions are linearly independent, but the set of three functions is not. If all three of them solved the same third-order linear homogeneous ODE, you could not combine them to form the general solution.

[5]If they aren't differentiable then W isn't defined.

So it can become a subtle business to look at three (or more!) functions and determine if they are linearly dependent. But the Wronskian generalizes to any level because determinants themselves generalize to any level. Consider three functions $y_1(x)$, $y_2(x)$, and $y_3(x)$. The Wronskian of these three functions is given by the following determinant.

$$W(x) = \begin{vmatrix} y_1(x) & y_2(x) & y_3(x) \\ y_1'(x) & y_2'(x) & y_3'(x) \\ y_1''(x) & y_2''(x) & y_3''(x) \end{vmatrix}$$

If the Wronskian is zero then the vectors $<y_1, y_1', y_1''>$, $<y_2, y_2', y_2''>$and $<y_3, y_3', y_3''>$ are linearly dependent. If we also know that these three functions are solutions to the same third-order linear homogeneous differential equation, then their higher order derivatives are all linear functions of these three numbers. So the linear dependence of these three variables is sufficient to conclude that the functions are linearly dependent.

10.6.3 Problems: Linearly Independent Solutions and the Wronskian

In Problems 10.108–10.114 you will be given a linear differential equation and a set of solutions valid on the interval $(-\infty, \infty)$. (You won't actually use the differential equation, but we include it to emphasize that this technique only applies to functions that are solutions to a common ODE.) Use the Wronskian to determine if the functions are linearly independent or dependent.

10.108 $y'' = -k^2 y$, $y_1(x) = \sin(kx)$, $y_2 = \cos(kx)$

10.109 $y'' = -k^2 y$, $y_1(x) = 3\sin(kx)$, $y_2 = -5\sin(kx)$

10.110 $y'' = -k^2 y$, $y_1(x) = \sin(kx)$, $y_2 = e^{ikx}$

10.111 $y'' = -k^2 y$, $y_1(x) = \sin(kx)$, $y_2 = \cos(kx)$, $y_3 = e^{ikx}$

10.112 $y''' - 6y'' + 11y' - 6 = 0$, $y_1(x) = 4e^x - 2e^{2x} + e^{3x}$, $y_2(x) = -e^x + 2e^{2x} - e^{3x}$, $y_3(x) = 5e^x + 2e^{2x} - e^{3x}$

10.113 $y'' - 2xy' + 2ky = 0$,

$$y_1(x) = 1 - \frac{2k}{2!}x^2 + \frac{2^2 k(k-2)}{4!}x^4 - \frac{2^3 k(k-2)(k-4)}{6!}x^6 + \dots,$$

$$y_2(x) = x - \frac{2(k-1)}{3!}x^3 + \frac{2^2(k-1)(k-3)}{5!}x^5 - \frac{2^3(k-1)(k-3)(k-5)}{7!}x^7 + \dots$$

10.114 $y_1(x) = x - 2x^2 - (1/3)x^3 + \dots$, $y_2(x) = 3 + 3x - 33x^2 + 35x^3 + \dots$, $y_3(x) = 3 - 27x^2 + 36x^3 + \dots$ (For this one we are not giving an ODE or a pattern for the rest of the series. Just assume you have three series solutions that start like this.)

10.115 Let $f(x) = 2e^x$ and $g(x) = \sin x + \cos x$ on the interval $[-\pi, \pi]$.

(a) Calculate the Wronskian for these two functions.

(b) Evaluate the Wronskian at $x = 0$.

(c) Evaluate the Wronskian at $x = \pi/2$.

(d) Explain why your answers don't contradict what we've said about the properties of the Wronskian.

(e) Are these two functions linearly dependent on this interval?

10.116 In this problem you will prove the second of our claims about the Wronskian: if the Wronskian of two functions is zero throughout an interval I then the two functions must be linearly dependent.

(a) We defined two linearly dependent functions by the equation $y_1 = ky_2$ for some constant k, which we can also write as:

$$\frac{y_1(x)}{y_2(x)} = k$$

Take the derivative with respect to x of both sides of that equation.

(b) Based on your answer to Part (a), argue that if $W = 0$ then y_1 and y_2 must be linearly dependent.

(c) Why would your argument above not work if $W = 0$ only at a point, and not within a non-zero interval? If you're stuck you may find it helpful to look at our discussion of the functions e^x and $1 + x$ in the Explanation (Section 10.6.2).

10.117 In this problem you will prove the first of our three claims about the Wronskian. If two functions $y_1(x)$ and $y_2(x)$ are both solutions to the same equation $y''(x) + a_1(x)y'(x) + a_0(x)y(x) = 0$ on an open interval I then the Wronskian $W = y_1y_2' - y_1'y_2$ is zero everywhere in I, or it is non-zero everywhere in I. The strategy will be to write and solve an ODE for $W(x)$.

(a) Calculate W'. Your answer should be in terms of y_1, y_2, y_1'', and y_2''. Simplify your answer as much as possible.

(b) Since you know y_1 is a solution to the ODE you can rewrite y_1'' in terms of y_1 and y_1'. Use this substitution and the corresponding one for y_2'' to write W' in terms of y_1, y_2, y_1', and y_2'. Simplify your answer as much as possible.

(c) Replace $y_1y_2' - y_1'y_2$ with W in your expression for W'. The result should only depend on W and a_1.

(d) You just wrote a linear first-order differential equation for $W(x)$. Solve this equation by separating variables to find a formula for $W(x)$ in terms of the unknown function $a_1(x)$. Include an arbitrary constant in front of your solution.

(e) Explain how you can tell by looking at your solution that $W(x)$ is never zero unless $W(x)$ is always zero.

10.118 In this problem you will fill in some of the missing algebra in our discussion of the Wronskian. Consider two functions $y_1(x)$ and $y_2(x)$ that solve the following equation.

$$y''(x) + a_1(x)y'(x) + a_0(x)y(x) = 0 \qquad (10.6.5)$$

Suppose that at $x = c$ we know two facts: $y_1(c) = ky_2(c)$ and $y_1'(c) = ky_2'(c)$. (In the explanation we called the point $x = a$ but we don't want to confuse that a with the coefficient functions in the ODE.)

(a) Show that $y_1''(c) = ky_2''(c)$. *Hint*: This is not guaranteed without the differential equation!

(b) Take the derivative of both sides of Equation 10.6.5.

(c) Show that $y_1'''(c) = ky_2'''(c)$.

Using similar logic you can easily show that $y_1''''(c) = ky_2''''(c)$ and so on for all higher derivatives.

10.119 Consider two functions $f(x)$ and $g(x)$ such that $f(0) = 3g(0)$.

(a) First, suppose both functions are lines. What must be the relationship between their first derivatives (slopes) if $f(x)$ is to continue being three times $g(x)$ as they move to the right?

(b) If $f(x)$ and $g(x)$ are not necessarily lines, then the relationship between $f'(0)$ and $g'(0)$ that you wrote in Part (a) is not enough to show that $f(x) = 3g(x)$. Why not?

(c) If $f(x) = 3g(x)$ for all x-values, then what must be the relationship between $f''(0)$ and $g''(0)$?

10.9 Reduction of Order and Variation of Parameters

"Reduction of Order" and "Variation of Parameters" are two different formulas that can be used to find solutions to a differential equation based on other known solutions.

10.9.1 Discovery Exercise: Reduction of Order

Consider the equation

$$x^2y''(x) - (2x^2 + x)y'(x) + (x^2 + x)y(x) = 0 \tag{10.9.1}$$

1. Confirm that $y_1 = e^x$ is a solution to this equation.
2. The second solution is harder to guess, but you can make it easier by writing it in the form $y_2 = u(x)y_1(x)$, where y_1 is the solution we just gave you and $u(x)$ is an unknown function. Plug this into the differential equation to get a differential equation for $u(x)$. Simplify your answer as much as possible.

   See Check Yourself #67 in Appendix L
3. Your equation for u should have u'' and u' in it, but not u by itself. Make the substitution $v = u'$ to get a first-order differential equation for v.
4. Solve the equation you wrote for $v(x)$ and use that to find $u(x)$.
5. Write the general solution to Equation 10.9.1. Plug it in and verify that it works.

The technique you just used is called "reduction of order." When you have one solution y_1 to a linear, second-order ODE , the guess $y_2 = uy_1$ will give you a *first-order* ODE to solve for u'. This is one of two techniques you will learn about in this section.

10.9.2 Explanation: Reduction of Order and Variation of Parameters

Given a linear, inhomogeneous, second-order[8] differential equation, you can solve it if you can do the following three steps (as discussed in Section 10.2).

1. Find two linearly independent functions that solve the complementary homogeneous equation. Here we will call these functions $y_{c1}(x)$ and $y_{c2}(x)$.
2. Find a particular solution to the original inhomogeneous equation. We will refer to this solution as $y_p(x)$.
3. The general solution you are looking for is the sum of the particular and complementary solutions: $y_p(x) + C_1y_{c1}(x) + C_2y_{c2}(x)$.

"Reduction of order" helps with Step 1: it assumes you have found one solution to the complementary equation, and it gives you a way to find a second. "Variation of parameters" can take care of Step 2: it assumes you have already found both linearly independent solutions to the homogeneous equation, and it gives you a way to get from those to a particular solution of the inhomogeneous equation.

[8]In this section we are focusing on second-order equations but the methods can be generalized to higher orders.

Reduction of Order

A linear second-order homogeneous differential equation should have two linearly independent solutions. Reduction of Order is a formula that starts with one of these solution and finds the second. If you did the Discovery Exercise (Section 10.9.1) you've used this approach on one particular equation. You start with one solution y_{c1} and write the second one in the form $y_{c2} = u(x)y_{c1}(x)$. When you plug this into the differential equation you get a *first-order* differential equation for u', which you can solve by separation of variables.

In Problem 10.182 you'll apply this technique to a generic second-order linear ODE. The result is the following formula.

Reduction of Order

Given a second-order homogeneous differential equation $y''(x) + a_1(x)y'(x) + a_0(x)y(x) = 0$, and given one solution $y_{c1}(x)$, a second solution is given by:

$$y_{c2} = uy_{c1} \tag{10.9.2}$$

where u is a function that satisfies the equation:

$$\ln(u') = -\int \left(2\frac{y'_{c1}}{y_{c1}} + a_1\right) dx \tag{10.9.3}$$

These formulas come with the same warning as several other techniques in this chapter: make sure to write your differential equation in the form given above, which includes having no factor in front of $y''(x)$.

Equation 10.9.3 gives you the function $u'(x)$. There is no guarantee that you can integrate that to find the function you need. (Sorry: no technique is perfect.) But if you can, Equation 10.9.2 then gives you the second solution you need to find the general solution.

In the example below we use reduction of order to derive a result that we pulled out of a hat in Section 10.2: the second solution to a Cauchy-Euler equation with only one power solution.

EXAMPLE **Reduction of Order**

Question: Solve the equation $x^2y'' - 5xy' + 9y = 0$.

Solution:

The decreasing powers of x suggest the guess x^k.

$$y = x^k \quad \rightarrow \quad y' = kx^{k-1} \quad \rightarrow \quad y'' = k(k-1)x^{k-2}$$

Plug this into the differential equation.

$$k(k-1)x^k - 5kx^k + 9x^k = 0 \quad \rightarrow \quad k^2 - 6k + 9 = 0 \quad \rightarrow \quad k = 3$$

So $y = x^3$ is one valid solution. But because the quadratic equation for k had a double root, we don't have a second solution. That's where reduction of order comes in. We begin by writing the differential equation in the proper form—that is, make sure there is no coefficient in front of the y'' term.

$$y'' - \frac{5}{x}y' + \frac{9}{x^2}y = 0$$

Now we plug $y_{c1} = x^3$ and $a_1 = -5/x$ into Equation 10.9.3.

$$\begin{aligned}
\ln(u') &= -\int \left(2\frac{(3x^2)}{x^3} - \frac{5}{x}\right) dx \\
\ln(u') &= -\int \frac{1}{x} dx = -\ln x = \ln\left(x^{-1}\right) \\
u' &= \frac{1}{x} \\
u &= \ln x
\end{aligned}$$

So the second solution to this equation (from Equation 10.9.2) is $y_{c2} = x^3 \ln x$. Note that we did not need arbitrary constants in our integration; we will put them into our general solution.

$$y = Ax^3 + Bx^3 \ln x$$

You can (and should) plug this solution back into the original differential equation and verify that it works.

Variation of Parameters

Variation of Parameters is a formula for finding a particular solution to an inhomogeneous differential equation, based on two (already found) solutions to the complementary homogeneous equation.

Variation of Parameters

Given a second-order differential equation $y''(x) + a_1(x)y'(x) + a_0(x)y(x) = f(x)$, and given two linearly independent solutions y_{c1} and y_{c2} to the complementary homogeneous equation $y''(x) + a_1(x)y'(x) + a_0(x)y(x) = 0$, a particular solution to the inhomogeneous equation is given by:

$$y_p(x) = uy_{c1} + vy_{c2} \tag{10.9.4}$$

where u and v are functions that satisfy the equations:

$$\begin{aligned}
u' &= \frac{y_{c2}}{y'_{c1}y_{c2} - y_{c1}y'_{c2}} f(x) \\
v' &= \frac{-y_{c1}}{y'_{c1}y_{c2} - y_{c1}y'_{c2}} f(x)
\end{aligned} \tag{10.9.5}$$

These formulas come with the same warning as several other techniques in this chapter: make sure to write your differential equation in the form given above, which includes having no factor in front of $y''(x)$.

Equations 10.9.5 give you the functions $u'(x)$ and $v'(x)$. There is no guarantee that you can integrate those to find the functions you need. (Sound familiar?) But if you can, Equation 10.9.4 then gives you a particular solution, which you combine with your already-found complementary solutions to find the general solution.

Below we will give an example demonstrating the process, and then show where the formula comes from.

EXAMPLE Variation of Parameters

Question: Solve the equation $x^2\left(d^2y/dx^2\right) - 2y = \ln x$.

Solution:

We start by rewriting this in the correct form, which means dividing by x^2 so the coefficient of $y''(x)$ is one. That leaves $y''(x) - (2/x^2)y = (\ln x)/x^2$, so $f(x) = (\ln x)/x^2$.

Next we solve the complementary homogeneous equation $y''(x) - (2/x^2)y = 0$, which we will attack by plugging in the guess $y = x^k$.

$$k(k-1)x^k - 2x^k = 0 \quad \rightarrow \quad (k^2 - k - 2)x^k = 0 \quad \rightarrow \quad k = -1 \text{ or } k = 2$$

So our two homogeneous solutions are:

$$y_{c1} = \frac{1}{x} \qquad \textit{and} \qquad y_{c2} = x^2$$

These are the solutions to the complementary homogeneous equation, and we will need them when we write our general solution. But now we are going to plug them into Equations 10.9.5 to find the new functions u and v that we need for our *particular* solution.

$$u' = \frac{x^2}{-(1/x^2)x^2 - (1/x)(2x)}\frac{\ln x}{x^2} = -\frac{1}{3}\ln x$$
$$v' = -\frac{1/x}{-(1/x^2)x^2 - (1/x)(2x)}\frac{\ln x}{x^2} = \frac{1}{3}\frac{\ln x}{x^3}$$

These can be integrated by parts to give

$$u = \frac{1}{3}x - \frac{1}{3}x\ln x$$
$$v = -\frac{1}{12x^2} - \frac{1}{6x^2}\ln x$$

(You do not need to include $+C$ because we are not looking for a general solution; any particular solution will do.) Equation 10.9.4 tells us how to put those together to find a particular solution to our equation: $y_p(x) = uy_{c1} + vy_{c2}$. After a bit of algebra, this gives

$$y_p(x) = \frac{1}{4} - \frac{1}{2}\ln x$$

Remember that the general solution is a sum of the particular and homogeneous solutions!

$$y(x) = \frac{1}{4} - \frac{1}{2}\ln x + \frac{C_1}{x} + C_2x^2$$

We leave it to you to confirm that this solves the original equation. (Go on, it's good for you.)

Where Did That "Variation of Parameters" Formula Come From?

A linear second-order differential equation can be written in the form:

$$\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = f(x) \tag{10.9.6}$$

You might think that we're going to start there and derive Equation 10.9.4, but that isn't the plan: we're going to treat Equation 10.9.4 as a guess, plug it in, and end up at Equation 10.9.5. As always, the guess justifies itself when it works.

So we begin with our guess and take some derivatives. (This involves a few product rules, but also some clever grouping.) Remember as we go that we are starting with y_{c1} and y_{c2} (the already-found solutions to the complementary homogeneous equation) and looking for u and v.

$$\begin{aligned}
y_p(x) &= uy_{c1} + vy_{c2} \\
y_p'(x) &= u'y_{c1} + uy_{c1}' + v'y_{c2} + vy_{c2}' \\
&= uy_{c1}' + vy_{c2}' + (u'y_{c1} + v'y_{c2}) \\
y_p''(x) &= u'y_{c1}' + uy_{c1}'' + v'y_{c2}' + vy_{c2}'' + (u'y_{c1} + v'y_{c2})' \\
&= uy_{c1}'' + vy_{c2}'' + (u'y_{c1}' + v'y_{c2}') + (u'y_{c1} + v'y_{c2})'
\end{aligned}$$

Plugging all that into Equation 10.9.6,

$$\begin{aligned}
uy_{c1}'' + vy_{c2}'' + (u'y_{c1}' + v'y_{c2}') + (u'y_{c1} + v'y_{c2})' + a_1[uy_{c1}' + vy_{c2}' + (u'y_{c1} + v'y_{c2})] \\
+ a_0(uy_{c1} + vy_{c2}) = f(x)
\end{aligned}$$

Rearranging:

$$\begin{aligned}
u(y_{c1}'' + a_1y_{c1}' + a_0y_{c1}) + v(y_{c2}'' + a_1y_{c2}' + a_0y_{c2}) + a_1(u'y_{c1} + v'y_{c2}) + (u'y_{c1}' + v'y_{c2}') + (u'y_{c1} \\
+ v'y_{c2})' = f(x)
\end{aligned}$$

Now comes the good part. y_{c1} is a solution to the complementary homogeneous equation, which means by definition that $y_{c1}'' + a_1y_{c1}' + a_0y_{c1} = 0$. Similarly of course for y_{c2}. So both of the first terms go away, leaving this.

$$a_1(u'y_{c1} + v'y_{c2}) + (u'y_{c1}' + v'y_{c2}') + (u'y_{c1} + v'y_{c2})' = f(x) \tag{10.9.7}$$

You might think we're going to keep manipulating until we end up with u=<*something*> but that is not possible, because there is not just one unique solution. And we don't need one: we only need *one particular* solution, or in other words, anything that works. The easiest way to make Equation 10.9.7 work is to choose u and v such that:

$$\begin{aligned}
u'y_{c1} + v'y_{c2} &= 0 \\
u'y_{c1}' + v'y_{c2}' &= f(x)
\end{aligned}$$

These are now algebra equations, which you can easily solve for u' and v' to arrive at Equation 10.9.5.

10.9.3 Problems: Reduction of Order and Variation of Parameters

10.177 Walk-Through: Reduction of Order. In this problem you will solve the equation $x^4y'' + (x + x^3)y' - (1 + x^2)y = 0$.

(a) Show that $y_{c1} = x$ is one valid solution.

(b) Rewrite this equation in the form $y''(x) + a_1(x)y'(x) + a_0(x)y(x) = 0$. Identify the functions $a_1(x)$ and $a_0(x)$.

(c) Use Equation 10.9.3 to find the function $u(x)$.

(d) Use Equation 10.9.2 to find the second solution $y_{c2}(x)$ to this equation.

(e) Verify that your $y_{c2}(x)$ is a valid solution to this differential equation.

(f) Write the general solution to this differential equation.

10.178 In Section 10.2 we discussed the equation $y'' + 6y' + 9y = 0$. The guess $y = e^{kx}$ leads to one solution, $y = e^{-3x}$. We then suggested trying $y = xe^{-3x}$, but gave no indication of where this second solution came from. Use reduction of order to find this second solution for yourself.

10.179 In Section 10.2 we discussed the equation $x^2y'' + 5xy' + 4y = 0$. The guess $y = x^k$ leads to one solution, $y = 1/x^2$. We then suggested trying $y = (\ln x)/x^2$, but gave no indication of where this second solution came from. Use reduction of order to find this second solution for yourself.

10.180 One solution to the equation $xy'' + (3 - 2x)y' + (x - 3)y = 0$ is $y = e^x$. Find a second solution that is linearly independent of the first.

10.181 Find the general solution to the equation $x^3y'' + (x^2 + x)y' - (x + 1)y = 0$. You will need to begin by playing around until you find one simple solution that works. Reduction of order will then give you the second solution. The last step of finding $u(x)$ involves a tricky integral, but you can evaluate it by parts.

10.182 In this problem you will derive the formula for reduction of order. Remember the scenario: an existing solution $y_{c1}(x)$ has been found, and you are now looking for a second solution by guessing $y_{c2}(x) = u(x)y_{c1}(x)$. The goal is to find the unknown function $u(x)$.

(a) Begin by plugging the function $y_{c2} = uy_{c1}$ into the differential equation $y''(x) + a_1(x)y'(x) + a_0(x)y(x) = 0$. (This will involve a few product rules.) Write your answer in the form

$$u''\text{<something>} + u'\text{<something>} + u\text{<something>} = 0$$

(b) Explain why the u<something> must be zero, and can therefore be ignored. *Hint*: remember what y_{c1} stands for.

(c) The resulting differential equation contains u' and u'' but no u. As we saw in Section 10.8 this suggests the substitution $v = u'$. Write the resulting first-order differential equation for $v(x)$. (This is where "reduction of order" gets its name.)

(d) The resulting differential equation is separable. Solve for $v(x)$. If all goes well, you should end up deriving Equation 10.9.3.

10.183 Consider the differential equation $y'''(x) = y(x)$.

(a) This equation has one relatively obvious solution (beside $y(x) = 0$). Write it down and call it $y_{c1}(x)$.

(b) To find the other two solutions, begin by substituting $y_2(x) = u(x)y_{c1}(x)$ into the differential equation, using the $y_{c1}(x)$ you found in Part (a). Write the resulting third-order differential equation for $u(x)$.

(c) Convert the third-order equation for u into a second-order equation for $v = u'$ and solve it. Then use your solution to find u.

(d) Write the general solution to $y'''(x) = y(x)$.

10.184 Walk-Through: Variation of Parameters. In this problem you will solve the equation $(\sin x)\left[y''(x) + y(x)\right] = 1$.

(a) Rewrite this equation in the form $y''(x) + a_1(x)y'(x) + a_0(x)y(x) = f(x)$. Identify the functions $a_1(x)$, $a_0(x)$, and $f(x)$.

(b) Write the complementary homogeneous equation. Find two linearly independent solutions. (We hope you can see them both just by looking.)

(c) Calling the two functions you wrote in Part (b) $y_{c1}(x)$ and $y_{c2}(x)$, plug them into Equation 10.9.5 so you have two equations for the functions $u'(x)$ and $v'(x)$.

(d) Integrate your answers to find $u(x)$ and $v(x)$. You should be able to do both integrals by hand.

(e) Use Equation 10.9.4 to put your solutions together into a particular solution.

(f) Put your answers together to write the general solution to the original differential equation. Your solution should include two arbitrary constants C_1 and C_2.

(g) Verify that your solution works.

In Problems 10.185–10.187 solve the given equation by variation of parameters. You may find it helpful to first work through Problem 10.184 as a model.

10.185 $y'' - 10y' + 25y = e^{5x}/x^2$

10.186 $y'' - 4y' = 8e^{kx}$ (where k is a constant)

10.187 $2y'' - 5y' + 2y = e^{4x}$

Problems 10.188–10.192 have four parts each.

- Write the complementary homogeneous equation. In some cases we will give you a solution y_{c1} to this equation; if we don't, play around a bit until you find one.
- Use reduction of order to find the other solution y_{c2} to the complementary homogeneous equation.
- Use variation of parameters to find a particular solution y_p to the original equation.
- Write the general solution to the original equation.

10.188 $y''(x) + (3\tan x)y'(x) - 2y(x) = \cos^4 x$. Begin with $y_{c1} = \sin x$.

10.189 $(x-1)y''(x) - xy'(x) + y(x) = (x-1)^2$. Begin with $y_{c1} = e^x$.

10.190 $xy''(x) - (2x+1)y'(x) + (x+1)y(x) = x^2$

10.191 $y''(x) + (\tan x - (2/x))\,y'(x) - \left(\tan(x)/x - (2/x^2)\right) y(x) = x\cos x$. Begin with $y_{c1} = x$.

10.192 $y''(x) - (2\cot x + 1)y'(x) + (1 + \cot x + 2\cot^2 x)y(x) = \sin x$. Begin with $y_{c1} = \sin x$.

10.193 Variation of parameters tells you the derivatives $u'(x)$ and $v'(x)$, but you have to integrate to find the functions u and v. Since those are indefinite integrals, they should normally include arbitrary constants. Recall, however, that the general solution you get is $y(x) = uy_{c1} + vy_{c2} + C_1 y_{c1} + C_2 y_{c2}$, where C_1 and C_2 are arbitrary constants. Explain why, if you add arbitrary constants to u and v, it doesn't change this solution.

10.194 In this section we have focused on second-order differential equations, but variation of parameters can be used for linear equations of any order. In this problem you will derive and use the formula for the first-order equation:

$$\frac{dy}{dx} + a_0 y = f(x) \tag{10.9.8}$$

You will be looking for a solution of the form $y_p = u(x)y_c(x)$ where u is an unknown function and y_c is a solution to the complementary homogeneous equation.

(a) Find y_p' by the product rule. Then plug y_p and y_p' into Equation 10.9.8.

(b) Collect the terms that have u in them, and factor out the u.

(c) The resulting terms in parentheses add up to zero, and can be dropped. Why?

(d) Solve the remaining equation for $u'(x)$. This is the equation you have been looking for.

(e) Use your formula to solve the following equation.

$$\frac{dy}{dx} + \left(\frac{3}{x}\right) y = \frac{1}{x^4 + 5}$$

(You will begin by solving the complementary homogeneous equation by separation of variables.)

10.195 A 1 kg block is attached to a spring with spring constant 12 N/m and damping force $F_d = -bv$, $b = 7$ N·s/m. The block is acted on by an external force F_e.

(a) Write the differential equation for the position of the block, $x(t)$.

(b) Find the complementary solutions to this equation.

(c) Use variation of parameters to find a particular solution. Your answer will include integrals of the unknown function $F_e(t)$.

(d) Using your result from Part (c), find the particular solution $x_p(t)$ for each of the following driving forces.

i. $F_e = 5$ N

ii. $F_e = ae^{t/\tau}$, with $a = 2$ N and $\tau = 2$ s

iii. $F_e = ate^{t/\tau}$ with $a = 3$ N/s and $\tau = 2$ s

10.196 The charge on the capacitor in an RLC circuit obeys the equation $Q''(t) + 6Q'(t) + 5Q(t) = V'(t)$, where $V(t)$ is the voltage at the voltage source.

(a) Find the complementary solutions to this equation.

(b) Use variation of parameters to find a particular solution. Your answer will include integrals involving the unknown function $V'(t)$.

(c) Using your result from Part (c), find the particular solution $Q_p(t)$ for each of the following driving forces. (Be careful to use $V'(t)$ and not $V(t)$ in your formulas.)

i. $V = 5$

ii. $V = e^{2t}$

iii. $V = te^t$

10.197 In Chapter 11 we will solve for the motion of a string of length L held fixed at both ends and subjected to a uniform, time-dependent external driving force. We will reduce that problem to solving the following differential equation in which $b_{yn}(t)$ is the function we are solving for, $a(t)$ represents the time dependence of the external force, and n, L, and v are constants.[16]

$$-\frac{n^2\pi^2}{L^2} b_{yn}(t) - \frac{1}{v^2}\frac{d^2 b_{yn}(t)}{dt^2} = \frac{4}{n\pi}a(t)$$

(a) Find the complementary solutions to this ODE for $b_{yn}(t)$.

(b) Use variation of parameters to find a particular solution. Your answer will include integrals with the unknown function $a(t)$ in them.

(c) Find a particular solution $b_{yn}(t)$ for $a(t) = k$ (a constant external force).

(d) Find a particular solution $b_{yn}(t)$ for $a(t) = t$ (a linearly increasing external force).

10.198 *[This problem depends on Problem 10.197.]* In Problem 10.197 you found the Fourier coefficients for the motion of a vibrating string subject to an external driving force $a(t)$. In this problem, use a computer to find a particular solution for $b_{yn}(t)$ for the external force law: $a(t) = \sin t$.

[16]Some physical background you don't need for this problem: $b_{yn}(t)$ is the nth Fourier coefficient of the shape of the string $y(x)$ at time t, for odd n only. For even n the Fourier coefficients obey the same equation but with 0 on the right-hand side.

10.13 Additional Problems

Throughout these problems, $H(x)$ refers to the Heaviside function and $\delta(x)$ to the Dirac delta function.

In Problems 10.294–10.315 solve the given differential equation.

- If no initial conditions are given, find the general solution. If initial conditions are given, find the specific solution corresponding to those initial conditions.
- Express your answer as a function $f(t)$ if it's easy to do so, but some answers are best expressed as an equation relating f and t.
- If your answer is a Laplace transform $F(s)$, leave it in that form.
- No computers should be required.

Hint: Some of the first-order equations are written with df/dt and some with dt and df separate. In some cases you may find it helpful to rewrite the equation the other way.

10.294 $d^2f/dt^2 + 5df/dt + 6f = 12$

10.295 $df/dt + 4t^3 f = t^3$, $f(0) = 9/4$

10.296 $df/dt = \sec(f + t^2) - 2t$

10.297 $d^2f/dt^2 + 25f = e^{5t}$, $f(0) = f'(0) = 1$

10.298 $d^3f/dt^3 + 4d^2f/dt^2 = 8$, $f(0) = 1$, $f'(0) = 0$, $f''(0) = -1$

10.299 $(f^3 + t^2)dt + (3f^2 t + 2f)df = 0$

10.300 $df/dt + t^2 f = t^2 f^4$

10.301 $(\sin t + f \tan t)dt - df = 0$

10.302 $(e^{f-t} + 1)dt - df = 0$, $f(0) = 0$

10.303 $f\left(\dfrac{df}{dt}\right) = \dfrac{t}{e^{f^2+t} + 2(f^2 + t)} - \dfrac{1}{2}$

10.304 $df/dt = (f^2 + t)/(f - 2tf)$, $f(0) = 1$

10.305 $(f/t - t^2 e^{-t})dt + (1/t + (f^2/t)e^{-t})df = 0$

10.306 $d^2f/dt^2 - df/dt + e^{2t} f = 0$

10.307 $d^2f/dt^2 + 4df/dt + 3f = \delta(t - 1)$, $f(0) = 0$, $f'(0) = 2$

10.308 $df/dt = te^{f^2/t^2}/f + f/t$

10.309 $td^2f/dt^2 + df/dt - (4/t)f = 8/t$

10.310 $(t^2 f^2 + f)dt + t\, df = 0$, $f(1) = 1$. *Notice that the initial condition occurs at $t = 1$.*

10.311 $(f^2 + 2tf)dt - (tf + t^2)df = 0$

10.312 $df/dt + f/t = 2t^3 + 7$

10.313 $d^2f/dt^2 + f = \cos^2 t$

10.314 $d^2f/dt^2 + 2df/dt + f = \sin t$

10.315 $df/dt + 3f + 2\int_0^t f\, dt = e^{-t}$, $f(0) = 1$

In Problems 10.316–10.322, solve the given differential equation by substitution. Start by figuring out if the equation fits into one of the three special cases: Bernoulli, homogeneous, or $y(x)$ only appears inside derivatives. If it does, you immediately know the right substitution to use. If it doesn't, then you'll have to look at the equation and try to find the right substitution. (If you try something that doesn't work, try something else!)

10.316 $dy/dx = (y/x)^2 + (y/x)$

10.317 $e^y\left[(d^2y/dx^2) + (dy/dx)^2\right] + (1 - e^y) = 0$

10.318 $dy/dx = (y/x)^2 - (y/x)$. (Your substitution will lead you to a separable equation that can be integrated using partial fractions.)

10.319 $y''(x) = (y'(x)/x)^2 - (y'(x)/x)$. (Your substitution will lead you to a separable equation that can be integrated using partial fractions.)

10.320 $(d^2y/dx^2) - (dy/dx) - e^{2x}y = 0$

10.321 $xy(dy/dx) = x^2 + y^2$

10.322 $y(dy/dx) = x^2 + y^2$

10.323 (In this problem you will solve the differential equations you set up in the motivating exercise, Section 10.1 , but you do not need to have done that exercise to do this problem.) An object with mass $m = 1$ kg is attached to an ideal spring with spring constant $k = 9$ N/m, subject to a drag force $F_{drag} = -bv$ with $b = 6$ N·s/m, and also subject to an external driving force $F_e(t)$. In each case below the solution you write should be the general solution to the ODE.

(a) Write the differential equation for $x(t)$ assuming no driving force, $F_e = 0$. Solve the equation using guess and check.

(b) Write the differential equation for $x(t)$ assuming a driving force F_e that is a constant 3 N from $t = 0$ to $t = 10$ s. Solve it using Laplace transforms. Your final answer will be in the form of a Laplace transform $X(s)$, and will depend on the unspecified initial conditions x_0 and v_0.

(c) Write and solve the differential equation for $x(t)$ assuming an unspecified driving force $F_e(t)$. Your final answer will have the function $F_e(t)$ in it.

(d) Solve the differential equation for $x(t)$ assuming $F_e = t^5 e^{-3t}$.

10.324 In this problem you'll solve the equation $t^2\ddot{x} + t\dot{x} + x = 0$ twice.

(a) Find the general solution by using guess and check.

(b) Go back to the differential equation and find the general solution again, this time by using the substitution $u = \ln t$.

(c) You probably found two very different looking answers. In fact your guess-and-check answer should have been complex. Using properties of logs and exponents write that solution as a real function and show that it matches the solution you got from variable substitution.

10.325 The equation $x''(t) + 5x'(t) + 4x = 2\sin(2t)$ with initial conditions $x(0) = x'(0) = 0$ represents a damped, driven oscillator.

(a) Find the solution by using guess and check.

(b) Find the solution again by using variation of parameters. *Hint*: you may find the integrals on Page 485 useful.

(c) Find the general solution a third time by using a Laplace transform. Use a computer only to find the inverse transform at the end (and, if you want, to take an integral in the middle).

10.326 For each function below, draw the function and find its Laplace transform (by hand).

(a) $H(t-1) - H(t-2)$

(b) $H(t-4) - H(t-5)$

(c) $H(t-1) - H(t-2) + H(t-4) - H(t-5)$

(d) $H(t-1) + [H(t-2) - H(t-3)]e^{-t}$

10.327 Let $f(t) = (1/a)[H(t-2) - H(t-2-a)]$ where a is a constant.

(a) Draw a quick sketch of $f(t)$, assuming a is a reasonably small positive number.

(b) Find $\mathcal{L}[f(t)]$. Your answer will of course be in the form $F(s)$ but it will still have the constant a in it.

(c) Find $\lim_{a\to 0} F(s)$.

(d) What is $\lim_{a\to 0} f(t)$? Does your Laplace transform make sense for this function?

10.328 Let $f(t) = \sum_{n=0}^{\infty} H(t-n)$.

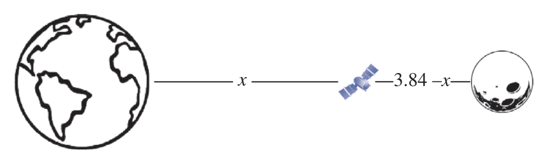
(a) Sketch $f(t)$.

(b) Find the Laplace transform $F(s)$. Your answer should be in closed form, i.e. with no summation.

10.329 The moon is about 384,000 km from Earth. A body in between the Earth and the moon experiences gravitational pulls in opposite directions from the two bodies. If we place the origin at the center of the Earth then the object's position obeys the differential equation

$$\frac{d^2x}{dt^2} = -\frac{2.98}{x^2} + \frac{.0366}{(3.84-x)^2}$$

measuring distance in 100,000s of kilometers and time in days. (You can do the problem in SI units, but the numbers are messier.)



(a) Explain why this equation is only valid for objects in between the Earth and the moon. In other words, how can you tell that it must give incorrect values if the object is at $x < 0$ (behind the Earth) or at $x > 3.84$ (beyond the moon)?

(b) Find the value of x at which the Earth's and moon's pulls exactly balance. Explain physically why you would expect this equilibrium value to be stable or unstable.

(c) Draw a phase portrait for this equation and confirm that there is an equilibrium point where you predicted and that it has the character you predicted.

10.330 **A Brief Voltage.** The charge buildup q on the capacitor in an RLC circuit obeys the following equation.

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = V(t)$$

Consider a circuit with $L = 1$ H, $R = 8\ \Omega$, $C = 1/15$ F, and $V(t) = 9\left[H(t-1/3) - H(t-2/3)\right]\sin(3\pi t)$ Volts.[16] The nature of this problem suggests

[16]These aren't particularly realistic inductance and capacitance values but it's harder to focus on the process when you're juggling numbers like 6×10^{-9} F.

Laplace transforms, but in this problem you will solve for $q(t)$ a different way. Assume $q(0) = 1$ C and $\dot{q}(0) = 0$.

(a) Sketch the function $V(t)$ for $t \geq 0$.

(b) For all $t < 1/3$ the voltage is 0. Use the method of guess and check to find the solution during this time period, subject to the initial conditions.

(c) Use your answer from Part (b) to find q and $\dot{q}$ at the moment when the voltage source turns on. If you punch your answer into a calculator, hold onto at least three digits of the answer.

(d) For the next 1/3 seconds the voltage is $\sin(3\pi t)$. Use the method of guess and check to find the solution during this time period. The initial conditions $q(1/3)$ and $\dot{q}(1/3)$ will come from your answer to Part (c).

(e) Use your answer to Part (d) to find q and $\dot{q}$ at the moment when the voltage source turns off.

(f) Find the solution after the voltage source turns off.

10.331 **Orthogonal Trajectories.** It is sometimes useful to find families of curves perpendicular to each other. For example, in two dimensions a set of charges creates field lines and equipotential curves that are orthogonal (perpendicular). If you know the formula for one set of curves, it's generally possible to find a differential equation that you (hopefully) can solve to find the orthogonal curves. As a first, simple example, consider the field created by a single point charge in 2D. You can measure that the equipotential curves are concentric circles: $x^2 + y^2 = r^2$. Each value of r corresponds to a different equipotential curve.

(a) What curves would you think would be perpendicular to those circles? Answer visually before going through the math.

(b) Use implicit differentiation to find the slope $m_{eq} = dy/dx$ of the equipotentials (circles) as a function of x and y.

(c) Since the field lines are orthogonal to the equipotentials, $m_{fl} = -1/m_{eq}$. Write that statement as a differential equation: dy/dx as a function of x and y for the field lines.

(d) Solve that differential equation to find $y(x)$ for the field lines. Your answer should have one arbitrary constant, so it represents a family of curves. Describe that family of curves. Does it match what you predicted?

(e) Now suppose a less symmetric configuration of charges led to elliptical equipotentials: $(x/2)^2 + y^2 = r^2$. Use the procedure outlined above to find $y(x)$ for the field lines. (Despite the name, "field lines" are not in general linear.)

10.332 *[This problem depends on Problem 10.331.]* Plot a representative sample of the elliptical equipotentials and their corresponding field lines from Part (e) of Problem 10.331.

10.333 *[This problem depends on Problem 10.331.]* Problem 10.331 walked you through the basic process of finding trajectories orthogonal to a family of curves, but it sidestepped a difficulty that can often arise. Consider as an example the curves $y = kx^4$, where k is the parameter that varies from one curve to the next (just as r was in Problem 10.331).

(a) First try finding the orthogonal trajectories exactly as you did in Problem 10.331. What you should find is an equation for $y(x)$ that includes *two* arbitrary constants, the original one k plus the new one that gets introduced by the integration. That doesn't make sense, so we need to make a slight change.

(b) Here's the problem. On each of our original curves k is a constant, and we treated it as such. But each orthogonal trajectory intersects many of the original curves, passing through many values of k, so on these new curves k is not a constant. We therefore have to eliminate k from this process before writing m_{fl}. So solve $y = kx^4$ for k and *then* take the derivative of both sides.

(c) Now finish the process. Find dy/dx for the new trajectories, separate variables, and integrate to find the orthogonal trajectories.

CHAPTER 11

Partial Differential Equations (Online)

11.12 Additional Problems

11.232 The temperature along a rod is described by the function $T(x,t) = ae^{-(bx^2+ct^2)}$.

(a) Sketch temperature as a function of position at several times. Your vertical and horizontal axes should include values that depend on (some of) the constants a, b, and c.

(b) Sketch temperature as a function of time at several positions. Your vertical and horizontal axes should include values that depend on (some of) the constants a, b, and c.

(c) Make a three-dimensional plot of temperature as a function of position and time. *For this part and the next you may assume any positive values for the constants in the temperature function.*

(d) Make an animation of temperature along the rod evolving in time.

(e) Describe the behavior of the temperature of the rod.

(f) Does this temperature function satisfy the heat equation 11.2.3?

11.233 The solution to the PDE $4(\partial z/\partial t) - 9(\partial^2 z/\partial x^2) - 5z = 0$ with boundary conditions $z(0,t) = z(6,t) = 0$ is $z(x,t) = \sum_{n=1}^{\infty} D_n \sin(\pi n x/6) e^{(-\pi^2 n^2+20)t/16}$. In this problem you will explore this solution for different initial conditions. If you approach this problem correctly it requires almost no calculations.

(a) From this general solution we can see that one particular solution is $z(x,t) = 2\sin(\pi x/6)e^{(-\pi^2+20)t/16}$. What is the initial condition that corresponds to this solution?

(b) Describe how the solution from Part (a) evolves in time.

(c) If the initial condition is $z(x,0) = \sin(5\pi x/6)$, what is the full solution $z(x,t)$?

(d) Describe how the solution from Part (c) evolves in time.

(e) If the initial condition is $z(x,0) = \sin(\pi x/6) + \sin(5\pi x/6)$, what is the full solution $z(x,t)$?

(f) Two of the solutions above look nearly identical at late times; the third looks completely different. (If you can see how these solutions behave from the equations, you are welcome to do so. If not, it may help to get a computer to make these plots for you.) Which two plots look similar at late times? Explain why these two become nearly identical and the third looks completely different.

(g) If you had a non-sinusoidal function such a triangle or a square wave as your initial condition, you could write it as a sum of sinusoidal normal modes and solve it that way. Based on your previous answers, which of those normal modes would grow and which would decay? Explain why almost any initial condition would end up with the same shape at late times. What would that shape be?

In Problems 11.234–11.236 assume that the string being described obeys the wave equation (11.2.2) and the boundary conditions $y = 0$ at both ends.

11.234 A string of length π begins with zero velocity in the shape $y(x,0) = 5\sin(2x) - \sin(4x)$.

(a) Guess at the function $y(x,t)$ that will describe the string's motion. The constant v from the wave equation will need to be part of your answer.

(b) Demonstrate that your solution satisfies the wave equation and the initial and boundary conditions. (If it doesn't go back and make a better guess!)

11.235 A string of length π is given an initial blow so that it starts out with $y(x,0) = 0$ and

$$\frac{\partial y}{\partial t}(x,0) = \begin{cases} 0 & x < \pi/3 \\ s & \pi/3 \le x \le 2\pi/3 \\ 0 & 2\pi/3 < x \end{cases}$$

(a) Rewrite the initial velocity as a Fourier sine series.

(b) Write the solution $y(x,t)$. Your answer will be expressed as a series.

(c) Let $v = s = 1$ and have a computer numerically solve the wave equation with these initial conditions and plot the result at several different times. Then make a plot of this numerical solution and of the 10th partial sum of the series solution at $t = 2$ on the same plot. Do they match?

11.236 A string of length L begins with $y(x,0) = 0$ and initial velocity $\frac{\partial y}{\partial t}(x,0) = \begin{cases} x & 0 \le x \le L/2 \\ L - x & L/2 \le x \le L \end{cases}$. Find the solution $y(x,t)$.

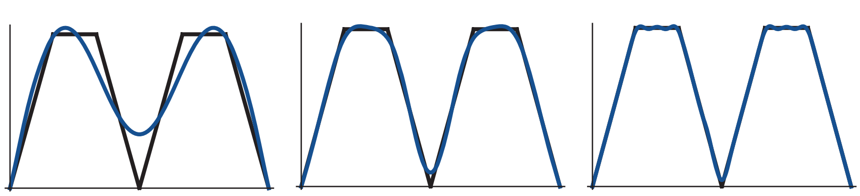
For Problems 11.237–11.239

(a) Solve the given problem using separation of variables. The result will be an infinite series.

(b) Plot the first three non-zero *terms* (not partial sums) of the series at $t = 0$ and at least three other times. For each one describe the shape of the function and how it evolves in time.

(c) Plot successive partial sums at $t = 0$ until the plot looks like the initial condition for the problem. Examples are shown below of what constitutes a good match.

(d) Having determined how many terms you have to include to get a good match at $t = 0$, plot that partial sum at three or more other times and describe the evolution of the function. How is it similar to or different from the evolution of the individual terms in the series?



This is a bad match　　Still not good enough　　This is a good match

11.237
$$\frac{\partial^2 y}{\partial t^2} - \frac{\partial^2 y}{\partial x^2} + y = 0,\ y(0,t) = y(4,t) = 0,$$
$$y(x,0) = \begin{cases} 1 & 1 < x < 2 \\ -1 & 2 \le x < 3 \\ 0 & elsewhere \end{cases},\ \frac{\partial y}{\partial t}(x,0) = 0$$

11.238
$$\frac{\partial y}{\partial t} - t^2 \frac{\partial^2 y}{\partial x^2} = 0,\ y(0,t) = y(3,t) = 0,$$
$$y(x,0) = \begin{cases} x & 0 \le x < 1 \\ 1 & 1 \le x \le 2 \\ 3 - x & 2x \le 3 \end{cases}$$

11.239 $\partial^2 y/\partial t^2 - \partial y/\partial t - \partial^2 y/\partial x^2 + y/4 = 0$, $y(0,t) = y(1,t) = 0$, $y(x,0) = x(1-x)$, $\partial y/\partial t(x,0) = x(x-1)$

Appendix I gives a series of questions designed to guide you to the right solution method for a PDE. For Problems 11.240–11.243 answer the questions in that appendix until you get to a point where it tells you what solution method to try, and then solve the PDE using that method. As always, your answer may end up in the form of a series or integral.

As an example, if you were solving the example on Page 648 you would say

1 The equation is linear, so we can move to step 2 and consider separation of variables.

2(a) The equation is not homogeneous, which brings us to…

2(e) We can't find a particular solution because the domain is infinite and anything simple (e.g. a line) would diverge as x goes to infinity. So we can't use separation of variables.

3 The domain is infinite so we can't use eigenfunction expansion.

4 The equation involves a first derivative of x, so we can't use the Fourier transform method.

5 Time has a semi infinite domain ($0 \le t < \infty$) and t appears only in the derivatives, so we can try a Laplace transform.

Having come to this conclusion, you would then finish the problem by solving the PDE using the method of Laplace transforms.

11.240 $\partial y/\partial t - \alpha^2(\partial^2 y/\partial x^2) + \beta^2 y = 0$, $y(0,t) = y(L,t) = 0$, $y(x,0) = \kappa\left(x^3 - Lx^2\right)$

11.241 $\dfrac{\partial y}{\partial t} - \dfrac{\partial^2 y}{\partial x^2} = 1$, $y(0,t) = y(3,t) = 0$, $y(x,0) = \begin{cases} 1 & 1 \le x \le 2 \\ 0 & \textit{elsewhere} \end{cases}$

11.242 $\partial^2 u/\partial t^2 - c^2(\partial^2 u/\partial x^2) = \cos(\omega t)\sin^2(\pi x/L)$, $u(0,t) = u(L,t) = u(x,0) = \dot{u}(x,0) = 0$

11.243 $\partial^2 y/\partial t^2 - c^2(\partial^2 y/\partial x^2) = \alpha^2\cos(\omega t)e^{-\beta^2(x/x_0)^2}$, $y(x,0) = \dot{y}(x,0) = 0$, $\lim_{x\to-\infty} y(x,t) = 0$, $\lim_{x\to\infty} y(x,t) = 0$

In Problems 11.244–11.257 solve the given PDE with the given boundary and initial conditions. The domain of all the spatial variables is implied by the boundary conditions. You should assume t goes from 0 to ∞.

If your answer is a series see if it can be summed explicitly. If your answer is a transform see if you can evaluate the inverse transform. Most of the time you will not be able to, in which case you should simply leave your answer in series or integral form.

11.244 $\partial y/\partial t - 9(\partial^2 y/\partial x^2) = 0$, $y(0,t) = y(3,t) = 0$, $y(x,0) = 2\sin(2\pi x)$

11.245 $\partial y/\partial t - 9(\partial^2 y/\partial x^2) = 0$, $y(0,t) = y(3,t) = 0$, $y(x,0) = x^2(3-x)$

11.246 $\partial y/\partial t - 9\left(\partial^2 y/\partial x^2\right) = 9$, $y(0,t) = y(3,t) = 0$, $y(x,0) = 2\sin(2\pi x)$. *Hint*: After you find the coefficients, take special note of the case $n = 6$.

11.247 $\partial u/\partial t - \partial^2 u/\partial x^2 + \partial u/\partial x = e^{-t}$, $u(x,0) = u(0,t) = u(1,t) = 0$ *Hint*: depending on how you solve this, you may find the algebra simplifies if you use hyperbolic trig functions.

11.248 $\partial u/\partial t - \alpha^2(\partial^2 u/\partial x^2) - \beta^2(\partial^2 u/\partial y^2) = 0$, $u(0,y,t) = u(L,y,t) = u(x,0,t) = u(x,L,t) = 0$, $u(x,y,0) = \sin(\pi x/L)\sin(4\pi x/L)$

11.249 $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 - \partial^2 u/\partial z^2 = 0$, $u(0,y,z) = u(L,y,z) = u(x,0,z) = u(x,L,z) = u(x,y,0) = 0$, $u(x,y,L) = V$

11.250 $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 - \partial^2 u/\partial z^2 = 0$, $u(0,y,z) = u(L,y,z) = u(x,0,z) = u(x,L,z) = 0$, $u(x,y,0) = V_1$, $u(x,y,L) = V_2$ *Warning: the answer is long and ugly looking.*

11.251 $\partial y/\partial t - \partial^2 y/\partial x^2 = t\sin(3\pi x)$, $y(0,t) = y(1,t) = y(x,0) = 0$

11.252 $\partial y/\partial t = \partial^2 y/\partial x^2 + (1/x)(\partial y/\partial x) - y/x^2$, $y(0,t) = y(1,t) = 0$, $y(x,0) = x(1-x)$. You can leave an unevaluated integral in your answer.

11.253 $\dfrac{\partial y}{\partial t} - \alpha^2\dfrac{\partial^2 y}{\partial x^2} + \beta^2 y = \begin{cases} x & 0 < x < 1 \\ 1 & 1 \le x \le 2 \\ 3-x & 2 < x < 3 \end{cases}$, $y(0,t) = y(3,t) = 0$, $y(x,0) = x(3-x)$

11.254 $\dfrac{\partial y}{\partial t} - \dfrac{\partial^2 y}{\partial x^2} = 0$, $y(0,t) = y(3,t) = 0$, $y(x,0) = \begin{cases} 1 & 1 \le x \le 2 \\ 0 & \textit{elsewhere} \end{cases}$

11.255 $\partial y/\partial t - \partial^2 y/\partial x^2 = 0$, $y(0,t) = 0$, $y(1,t) = 1$, $y(x,0) = x^2$

11.256 $\partial^2 y/\partial t^2 - \alpha^2 t(\partial^2 y/\partial x^2) = 0$, $y(0,t) = y(\pi,t) = 0$, $y(x,0) = 5\sin(3x)$, $\partial y/\partial t(x,0) = 0$

11.257 $\partial^2 y/\partial t^2 - v^2(\partial^2 y/\partial x^2) = e^{-(x+t)^2}$, $y(x,0) = \partial y/\partial t(x,0) = 0$, $\lim_{x\to\infty} y(x,t) = \lim_{x\to-\infty} y(x,t) = 0$

11.258 The electric potential in a region without charges obeys Laplace's equation (11.2.5). Solve for the potential on the domain $0 \le x < \infty$, $0 \le y \le L$, $0 \le z \le L$ with boundary conditions $V(0,y,z) = V_0$, $V(x,0,z) = V(x,y,0) = V(x,L,z) = V(x,y,L) = 0$, $\lim_{x\to\infty} V = 0$.

11.259 You are conducting an experiment where you have a thin disk of radius R (perhaps a large Petri dish) with the outer edge held at zero temperature. The chemical reactions in the dish provide a steady, position-dependent source of heat. The steady-state temperature in the disk is described by Poisson's equation in polar coordinates.

$$\rho^2\frac{\partial^2 V}{\partial \rho^2} + \rho\frac{\partial V}{\partial \rho} + \frac{\partial^2 V}{\partial \phi^2} = \frac{\rho}{R}\sin\phi$$

(a) Begin by applying the variable substitution $\rho = Re^{-r}$ to rewrite Poisson's equation.

(b) What is the domain of the new variable r?

(c) Based on the domain you just described, the method of transforms is appropriate here. You are going to use a Fourier sine transform. Explain why this makes more sense for this problem than a Laplace transform or a Fourier cosine transform.

(d) Transform the equation. The formula is in Appendix G. You can evaluate the integral using a formula from that appendix (or just give it to a computer). Then solve the resulting ODE. Your general solution will have two arbitrary functions of p.

(e) The boundary conditions for ϕ are implicit, namely that $\hat{V}_s(\phi)$ and $\hat{V}_s'(\phi)$ must both have period 2π. You should be able to look at your solution and immediately see what values the arbitrary functions must take to lead to periodic behavior.

(f) Take the inverse transform. (You can get the formula from Appendix G and use a computer to take the integral.) Then substitute back to find the solution to the original problem in terms of ρ.

11.260 Exploration: A Time Dependent Boundary
A string of length 1 obeys the wave equation 11.2.2 with $v = 2$. The string is initially at $y = 1 - x + \sin(\pi x)$ with $\partial y/\partial t(x, 0) = x - 1$. The right side of the string is fixed $(y(1, t) = 0)$, but the left side is gradually lowered: $y(0, t) = e^{-t}$.

(a) First find a particular solution $y_P(x, t)$ that satisfies the boundary conditions, but does not necessarily solve the wave equation or match the initial conditions. To make things as simple as possible your solution should be a linear function of x at each time t.

The complete solution will be $y(x, t) = y_C(x, t) + y_P(x, t)$ where y_P is the solution you found in Part (a) and y_C is a complementary solution, still to be found.

(b) What boundary conditions and initial conditions should y_C satisfy so that y satisfies the boundary and initial conditions given in the problem?

(c) Using the fact that $y = y_C + y_P$ and $y(x, t)$ solves the wave equation, figure out what PDE y_C must solve. The result should be an inhomogeneous differential equation.

(d) Using the method of eigenfunction expansion, solve the PDE for y_C with the boundary and initial conditions you found.

(e) Put your results together to write the total solution $y(x, t)$.

(f) Based on your results, how will the string behave at very late times $(t >> 1)$? Does the particular solution you found represent a steady-state solution? Explain.

CHAPTER 12

Special Functions and ODE Series Solutions (Online)

12.9 Proof of the Orthgonality of Sturm-Liouville Eigenfunctions

In Section 12.8 we claimed that the eigenfunctions of a Sturm-Liouville problem, Equations 12.8.2–12.8.3, are orthogonal and complete. The proof of completeness is beyond the scope of this chapter[11], but we prove orthogonality below.

12.9.1 Explanation: Proof of the Orthgonality of Sturm-Liouville Eigenfunctions

When we claim that the sine functions are orthogonal, we mean the following.

$$\int_{-\pi}^{\pi} \sin(nx)\sin(mx)dx = 0 \text{ if } n \text{ and } m \text{ are distinct integers}$$

(The word "distinct" here means $m \neq n$, so we are dealing with two different functions.) That isn't too hard to prove. And you can imagine that with a bit more work we might be able to prove the same thing about any two distinct Legendre polynomials $P_n(x)$ and $P_m(x)$, or about two distinct Bessel functions $J_p(nx)$ and $J_p(mx)$. Each new function, each new normal mode, would have a different proof.

One of the remarkable accomplishments of Sturm-Liouville theory is to prove orthogonality relationships for all of these functions and many more with one relatively short bit of algebra. The proof is not based on the functions themselves, but on the differential equations that they solve.

Let y_m and y_n be two eigenfunctions of the Sturm-Liouville problem, Equations 12.8.2–12.8.3, with distinct eigenvalues λ_m and λ_n. To say they are solutions of the ODE means

$$\frac{d}{dx}\left(p(x)\frac{dy_m}{dx}\right) + q(x)y_m(x) + \lambda_m w(x)y_m(x) = 0$$
$$\frac{d}{dx}\left(p(x)\frac{dy_n}{dx}\right) + q(x)y_n(x) + \lambda_n w(x)y_n(x) = 0$$

We multiply each of these equations by the other eigenfunction and then subtract them.

$$\left[y_n\frac{d}{dx}\left(p(x)\frac{dy_m}{dx}\right) - y_m\frac{d}{dx}\left(p(x)\frac{dy_n}{dx}\right)\right] + \left(\lambda_m - \lambda_n\right) w(x)y_m(x)y_n(x) = 0 \qquad (12.9.1)$$

[11]See e.g. Birkhoff, Garrett and Rota, Gian-Carlo, "On the Completeness of Sturm-Liouville Expansions," The American Mathematical Monthly, Vol. 67, No. 9 (Nov., 1960), pp. 835–841.

Next we integrate from $x = a$ to $x = b$. You'll show in Problem 12.135 that this gives

$$p(x)\left[\left(y_n(x)y_m'(x) - y_m(x)y_n'(x)\right]_a^b + \left(\lambda_m - \lambda_n\right)\int_a^b w(x)y_m(x)y_n(x)dx = 0 \qquad (12.9.2)$$

We're trying to prove that the integral on the right equals zero, so we need to show that the term in square brackets on the left is zero. Since that term is evaluated at $x = a$ and $x = b$ we need to show that $p(a)[y_n(a)y_m'(a) - y_m(a)y_n'(a)]$ is zero, and likewise at $x = b$. The tool we can use here is the boundary condition.[12]

$$\begin{aligned} c_1 y_m(a) + c_2 y_m'(a) &= 0 \\ c_1 y_n(a) + c_2 y_n'(a) &= 0 \end{aligned} \qquad (12.9.3)$$

We multiply the first boundary condition by y_n and the second one by y_m and subtract, and we conclude that the term in square brackets is zero at $x = a$. A similar argument holds at $x = b$. (We have to divide by c_2 in the process, so this argument isn't valid if $c_2 = 0$. You'll show in Problem 12.136 that the conclusion still holds in that case.)

In sum, the fact that y_m and y_n both satisfy the original ODE led us to Equation 12.9.2. The fact that they both satisfy the boundary conditions at a and b led us to conclude that the first term in that equation is zero, so we were left with the orthogonality condition we were trying to prove.

12.9.2 Problems: Proof of the Orthgonality of Sturm-Liouville Eigenfunctions

12.133 In this problem you will prove that any distinct solutions y_m and y_n of the simple harmonic oscillator equation $y''(x) + \lambda y(x) = 0$ with boundary conditions $x(0) = x(L) = 0$ are orthogonal on the interval $0 \le x \le L$. You can do this by showing that the solutions are sines and then showing that sines are orthogonal, and you can also just say "this is an example of a Sturm-Liouville problem," but you're not going to do either of those. Instead you're going to follow the steps of the general Sturm-Liouville proof in the Explanation (Section 12.9.1).

(a) Write an equation that asserts "If you write the SHO differential equation with eigenvalue λ_m then the solution is eigenfunction y_m."

(b) Write an equation that asserts "If you write this differential equation with eigenvalue λ_n then the solution is eigenfunction y_n."

(c) Multiple your equation from Part (a) by $y_n(x)$ and your equation from Part (b) by $y_m(x)$. Then subtract the two equations.

(d) Integrate both sides of the resulting equation from 0 to L. Then use integration by parts to reduce second derivatives to first derivatives.

(e) Use the boundary conditions to show that part of the resulting expression must be zero, and complete the proof of the orthogonality of the solutions to this equation. *Hint*: the first boundary condition for the generic Sturm-Liouville problem involves two constants called c_1 and c_2. It is possible for neither constant to be zero, and it is possible for one constant to be zero, but it is not possible for both constants to be zero because then you would be missing a boundary condition. This fact will be useful towards the end of the proof.

12.134 Why is there no $q(x)$ term in Equation 12.9.1?

12.135 Fill in the steps to get from Equation 12.9.1 to Equation 12.9.2. Start by integrating both sides of the equation from a to b

[12] You may recall from Section 12.8 that these boundary conditions don't apply at $x = a$ if $p(a) = 0$, but in that case the term on the left of Equation 12.9.2 is trivially zero at $x = a$.

and then use integration by parts to take the d/dx off of the py' terms.

12.136 Write out the steps in going from Equations 12.9.3 to showing that $p(a)[y_n(a)y'_m(a) - y_m(a)y'_n(a)] = 0$. You will have to handle $c_2 = 0$ as a special case.

12.137 We proved that $\int_a^b w(x)y_m(x)y_n(x)dx = 0$ for any two eigenfunctions y_m and y_n of the Sturm-Liouville problem where $m \neq n$. What step in our proof is invalid if $m = n$?

12.138 The functions $y_k = e^{kx}$ are solutions of the equation $y''(x) + \lambda y(x) = 0$ for $\lambda = -k^2$, but they are not orthogonal to each other. Explain why this does not violate what we've said about Sturm-Liouville theory.

12.10 Special Application: The Quantum Harmonic Oscillator and Ladder Operators

In this section we're going to derive the possible states of a particle in a potential field $V = (1/2)kx^2$, an important problem in quantum mechanics. To do that we have to introduce the technique of "ladder operators." To do *that* we have to spend some time on the notation of operators, and especially on the idea of a "commutator." As usual, the math you pick up along the way will apply to a variety of different physical situations.

If you haven't worked much with differential operators we recommend starting with the Discovery Exercise, which will give you some practice with them.

What we are not going to do in this section is explain the basics of quantum mechanics: Schrödinger's equation, eigenstates and energy levels, normalization, and so on. If you have never seen any quantum mechanics this section may assume too much for you. If you are particularly interested we have two papers on the subject. (We highly recommend reading the first one before the second one.)

- `http://www.felderbooks.com/papers/quantum.html` is a non-mathematical introduction to some fundamental ideas of quantum mechanics, how those ideas radically depart from classical physics, and why this radical departure is necessitated by experimental results.
- `http://www.felderbooks.com/papers/psi.html` is an overview of the mathematical approach required to solve quantum mechanical problems. If you got lost somewhere between the second-order PDE, the Fourier transform, and the eigenstates, this may help you see how they fit together.

12.10.1 Discovery Exercise: The Quantum Harmonic Oscillator and Ladder Operators

The derivative d/dx is called an "operator," meaning it takes as input a function and produces as output another function. We will abbreviate that derivative as D. The operator x just multiplies any function by x. So for example, the operator $D - x$ acting on the function $f(x) = x^2$ produces the function $(D - x)f = 2x - x^3$. For this exercise we define the operators $\hat{a}_R = D - x$ and $\hat{a}_L = D + x$.

1. Calculate $\hat{a}_L \sin x$. (This should be very simple; we just want to make sure you're clear on the notation.)

   See Check Yourself #88 in Appendix L
2. Calculate $\hat{a}_L \hat{a}_R x^2$. (Read this as "act with $\hat{a}_R$ on x^2, then act with $\hat{a}_L$ on the result.")

The easiest way to do algebra with operators is to see what they do to an arbitrary function. For example, the operator Dx acting on a function f gives $(d/dx)(xf) = x(df/dx) + f$. We can then take out the f and write $Dx = xD + 1$, where the 1 represents the operator "multiply the function by 1."

3. Calculate $\hat{a}_L \hat{a}_R$. *Hint*: a second derivative is D^2 in operator notation.

   See Check Yourself #89 in Appendix L
4. The "commutator" of two operators $\hat{A}$ and $\hat{B}$ is defined as $\left[\hat{A}, \hat{B}\right] = \hat{A}\hat{B} - \hat{B}\hat{A}$. Calculate the commutator $\left[\hat{a}_L, \hat{a}_R\right]$.

12.10.2 Explanation: The Quantum Harmonic Oscillator and Ladder Operators

A particle that experiences a force $F = -kx$ is a simple harmonic oscillator. Its potential energy is $V = (1/2)m\omega^2 x^2$ where m is the particle's mass and $\omega = \sqrt{k/m}$. The quantum mechanical wavefunction for such an oscillator must obey Schrödinger's equation with that potential function.

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \frac{1}{2}m\omega^2 x^2\psi = E\psi \tag{12.10.1}$$

Here $\hbar$ is a constant of nature and m and ω are both constant for a particular physical system. The energy E, however, can take on different values. The solution $\psi(x)$ for any particular E is the wavefunction for the state where the particle has that value of energy. Classically the particle could have any energy from 0 to ∞, but quantum mechanically only certain values of E are possible. Those are the eigenvalues of Equation 12.10.1 subject to the boundary conditions that $\psi(x)$ must be finite at $x \to \pm\infty$.

In this section we will find the eigenvalues E and corresponding eigenfunctions $\psi(x)$ that represent the energy states of the quantum harmonic oscillator. In Section 12.11 Problem 12.162 you'll solve this problem using the method of power series, but it involves a few subtleties. In this section we'll solve it in a very different way, using the so-called "ladder operators" developed by Paul Dirac.

You'll show in Problem 12.148 that the substitutions $y = \sqrt{m\omega/\hbar}\, x$ and $\lambda = 2E/(\hbar\omega)$ let us rewrite Equation 12.10.1 as:

$$\frac{d^2\psi}{dy^2} - y^2\psi + \lambda\psi = 0 \tag{12.10.2}$$

We're going to solve this problem in two steps. First we'll show you a trick that will allow us to find the eigenfunctions and eigenvalues quickly and easily. In that part we'll present this trick with no justification, as if it had been handed to Dirac on stone tablets. Later we'll show where the trick came from, which will suggest how you might apply it to other problems.

The Trick with No Justification

Suppose $\psi_m(y)$ is an eigenfunction of Equation 12.10.2 with eigenvalue $\lambda = c_m$. One day, for no obvious reason, Paul Dirac asks us to see if $\psi_n = d\psi_m/dy + y\psi_m$ happens to be an eigenfunction of the same equation. Agreeably, we begin by finding its first derivative.

$$\psi_n = \frac{d\psi_m}{dy} + y\psi_m \quad \rightarrow \quad \frac{d\psi_n}{dy} = \frac{d^2\psi_m}{dy^2} + y\frac{d\psi_m}{dy} + \psi_m$$

Now remember that ψ_m is an eigenfunction of the differential equation with eigenvalue c_m, so we know that $d^2\psi_m/dy^2 = y^2\psi_m - c_m\psi_m$.

$$\begin{aligned}
\frac{d\psi_n}{dy} &= y^2\psi_m - c_m\psi_m + y\frac{d\psi_m}{dy} + \psi_m \\
&= y\frac{d\psi_m}{dy} + \left(y^2 - c_m + 1\right)\psi_m \\
\frac{d^2\psi_n}{dy^2} &= y\frac{d^2\psi_m}{dy^2} + \left(y^2 - c_m + 2\right)\frac{d\psi_m}{dy} + 2y\psi_m \\
&= \left(y^2 - c_m + 2\right)\frac{d\psi_m}{dy} + \left(y^3 + 2y - yc_m\right)\psi_m
\end{aligned}$$

Now plug that into $d^2\psi_n/dy^2 - y^2\psi_n$. If ψ_n is an eigenfunction then the result should be a constant (minus the eigenvalue) multiplied by ψ_n.

$$\frac{d^2\psi_n}{dy^2} - y^2\psi_n = \left(-c_m + 2\right)\frac{d\psi_m}{dy} + \left(2y - yc_m\right)\psi_m = -(c_m - 2)\left(\frac{d\psi_m}{dy} + y\psi_m\right) = -(c_m - 2)\psi_n$$

We've therefore shown that the function ψ_n is in fact an eigenfunction of Equation 12.10.2, and its eigenvalue is $c_m - 2$. In Problem 12.149 you'll go through a similar calculation to show that $d\psi_m/dy - y\psi_m$ is also an eigenfunction, with eigenvalue $\lambda = c_m + 2$. So all we need to do is find one eigenfunction and these two operators will generate as many more as we wish.

But Sturm-Liouville theory says there should be a lowest eigenvalue. If we keep applying $d\psi_m/dy + y\psi_m$ it seems that we will keep finding different eigenfunctions, each with a lower eigenvalue than the one before. The series will only terminate if we find that $d\psi_m/dy + y\psi_m$ gives us zero; then there will not be another valid eigenfunction, so there will be no lower eigenvalues.

Conveniently that fact lets us find the lowest eigenfunction. The solution to $d\psi/dy + y\psi = 0$ is $\psi = C_0 e^{-y^2/2}$, so that is the state with the lowest eigenvalue. In Problem 12.150 you'll derive that solution and show that its eigenvalue is $\lambda = 1$.

If we apply our trick to find the next eigenfunction we get $\psi = (d/dy)(C_0 e^{-y^2/2}) - yC_0 e^{-y^2/2} = -2C_0 y e^{-y^2/2}$ with eigenvalue $\lambda = 3$. Each eigenfunction can have a different arbitrary constant in front of it, however, so we replace $-2C_0$ with a new constant C_1.

The remaining solutions will have eigenvalues 5, 7, 9, and so on. The last step is to convert back to our original variables x and E instead of y and λ. Any constants that appear in front of the functions can be absorbed into the arbitrary constants, and the resulting first few eigenfunctions are as follows.

$\psi_0(x) = C_0 e^{-m\omega x^2/2\hbar}$	$E = (1/2)\hbar\omega$	The ground state
$\psi_1(x) = C_1 x e^{-m\omega x^2/2\hbar}$	$E = (3/2)\hbar\omega$	The first excited state
$\psi_2(x) = C_2[(2m\omega/\hbar)x^2 - 1]e^{-m\omega x^2/2\hbar}$	$E = (5/2)\hbar\omega$	The second excited state

The constants C_n are determined by the "normalization condition" $\int_{-\infty}^{\infty} \psi(x)^2 dx = 1$. See Problem 12.151.

Reframing the Problem in the Language of Operators

To see where that trick came from we need to talk about the problem in terms of "operators." We introduced operators in Chapter 10 but we'll recap the key ideas here. Becoming comfortable with operators will be more useful in the long run than anything you learn about quantum oscillators.

An operator acts on a function to produce another function. For instance if D is the operator "take the derivative with respect to y" then we can write $D(y^3) = 3y^2$.

When a constant is used as an operator it indicates multiplication. So the operator 7 turns y^3 into $7y^3$, and the operator 1 turns any function into itself.

All the operators that concern us in this section will be "linear operators." An operator $\hat{A}$ is linear if it obeys the following two rules.

- $\hat{A}(f + g) = \hat{A}f + \hat{A}g$
- $\hat{A}(kf) = k\hat{A}f$

The product of two operators $\hat{A}\hat{B}$ is defined as the operator "do $\hat{B}$, then $\hat{A}$." It's important to go right to left, just as with matrices, and for the same reason: so that $(\hat{A}\hat{B})f$ is the same function as $\hat{A}(\hat{B}f)$. (Formally this means that operator multiplication is "associative." Informally it means we can put parentheses anywhere we like in a string of operator multiplications.)

EXAMPLE **Operator Multiplication**

Define the operator D to mean "take the derivative with respect to y." The operator y means "multiply the function by y."

Question: What do the operators yD and Dy do to a function $f(y)$?

Answer:

$$yDf = y\left(\frac{df}{dy}\right)$$
$$Dyf = \frac{d}{dy}(yf) = y\left(\frac{df}{dy}\right) + f$$

We see that $Dyf = yDf + f$, an equation that relates two functions. We can rewrite that as an equation that directly relates two operators: $Dy = yD + 1$. (Remember what the operator "1" means!)

The example above illustrates the very general fact that operator multiplication is not commutative. If you want to switch the order of an operator multiplication you need to use a "commutator," as we discuss below. Note also that, because of this definition of operator multiplication, squaring an operator means doing that operator twice. So D^2 gives a second derivative, not a first derivative squared.

With this terminology we can rewrite Equation 12.10.2 as $(D^2 - y^2)\psi + \lambda\psi = 0$. We left the λ term separate to make it clear that this equation is asking for the eigenvalues and eigenfunctions of the operator $D^2 - y^2$. If these were numbers we could factor that into $(D + y)(D - y)$, or equivalently $(D - y)(D + y)$. Since these are operators instead of numbers those two expressions are not equivalent, and in fact neither one gives $D^2 - y^2$.

Let's see what they do give us. Since operators are defined by how they act on functions, it's easiest to manipulate them by putting a function after them, so we'll consider how these operators act on an arbitrary function $f(y)$.

$$(D + y)(D - y)f = \left(\frac{d}{dy} + y\right)\left(\frac{df}{dy} - yf\right) = \frac{d^2f}{dy^2} - \frac{d}{dy}(yf) + y\frac{df}{dy} - y^2f$$
$$= \frac{d^2f}{dy^2} - y\frac{df}{dy} - f + y\frac{df}{dy} - y^2f = (D^2 - 1 - y^2)f$$

We can now drop the f and write an operator equation.

$$(D + y)(D - y) = (D^2 - y^2 - 1)$$

In Problem 12.141 you'll do a similar calculation to show that $(D - y)(D + y) = (D^2 - y^2 + 1)$.

The key ideas in this section flow from calculations like the one above, so we urge you to go through it carefully. The -1 in our final result came from the difference between "multiplying-by-y-and-then-taking-a-derivative" and "taking-a-derivative-and-then-multiplying-by-y." The order of operator multiplication usually matters, as it does in this case. When you want to reverse the order you need the "commutator."

Definition: Commutator

The "commutator" of two operators is the difference between multiplying them in different orders.

$$\left[\hat{A}, \hat{B}\right] = \hat{A}\hat{B} - \hat{B}\hat{A}$$

If two operators "commute" (that is, $\hat{A}\hat{B} = \hat{B}\hat{A}$) then their commutator is zero. You should be able to easily convince yourself that every operator commutes with itself: $[\hat{A}, \hat{A}] = 0$. The second rule we gave above for linear operators can be expressed as "a linear operator commutes with a constant operator."

The list below summarizes some of the important arithmetic properties of linear operators.

Property	Name	Why it Works
$(\hat{A}\hat{B})\hat{C} = \hat{A}(\hat{B}\hat{C})$	Associative Property	This is true for all operators, because $\hat{A}\hat{B}$ is defined as the operator that makes this true.
$(\hat{A}+\hat{B})\hat{C} = \hat{A}\hat{C} + \hat{B}\hat{C}$	Distributive Property	This is true for all operators, because $\hat{A}+\hat{B}$ is defined as the operator that makes this true.
$\hat{A}(\hat{B}+\hat{C}) = \hat{A}\hat{B} + \hat{A}\hat{C}$	Distributive Property	This is part of the definition of a linear operator.
$\hat{A}k = k\hat{A}$	Commuting with a constant	This is part of the definition of a linear operator.
$\hat{A}\hat{B} = \hat{B}\hat{A} + \left[\hat{A}, \hat{B}\right]$	Commutator	This is the definition of the commutator.

The following example shows how to use these properties. Operator arithmetic is a useful skill in general, but the particular operators in the example will also be important for our purposes.

EXAMPLE Commutator

Let $\hat{a}_L = D + y$ and $\hat{a}_R = D - y$. (The reasons for the subscripts R and L will become clear later.)

Problem:
Find the commutator $\left[\hat{a}_R, \hat{a}_L\right]$.

Solution:
We found $\hat{a}_L\hat{a}_R$ above, and you will find $\hat{a}_R\hat{a}_L$ in Problem 12.141. All that remains is to subtract them.

$$\left[\hat{a}_R, \hat{a}_L\right] = \hat{a}_R\hat{a}_L - \hat{a}_L\hat{a}_R = 2$$

That means that we can replace $\hat{a}_R\hat{a}_L$ in any equation with $\hat{a}_L\hat{a}_R + 2$. (Every time an $\hat{a}_R$ passes right through an $\hat{a}_L$ it picks up a +2.) Equivalently, we can replace $\hat{a}_L\hat{a}_R$ with $\hat{a}_R\hat{a}_L - 2$. (Every time an $\hat{a}_R$ passes left through an $\hat{a}_L$ it picks up a −2.)

Problem:
Rewrite $\hat{P} = \hat{a}_L\hat{a}_L\hat{a}_R$ with the $\hat{a}_R$ on the left.

Solution:
Pay careful attention to what operator properties we use at each step of this solution.

$$\begin{aligned}\hat{P} &= \hat{a}_L(\hat{a}_L\hat{a}_R)\\ &= \hat{a}_L(\hat{a}_R\hat{a}_L - 2)\\ &= \hat{a}_L\hat{a}_R\hat{a}_L - \hat{a}_L 2\\ &= (\hat{a}_L\hat{a}_R)\hat{a}_L - 2\hat{a}_L\\ &= (\hat{a}_R\hat{a}_L - 2)\hat{a}_L - 2\hat{a}_L\\ &= \hat{a}_R\hat{a}_L\hat{a}_L - 2\hat{a}_L - \hat{a}_L 2\\ &= \hat{a}_R\hat{a}_L\hat{a}_L - 4\hat{a}_L\end{aligned}$$

If it's useful for your calculations you can rewrite this as $\hat{P} = (\hat{a}_R\hat{a}_L - 4)\hat{a}_L$.

Raising and Lowering Operators

We now return to Equation 12.10.2. Since we found that $(D+y)(D-y) = (D^2 - y^2 - 1)$ we can rewrite this equation in terms of our operators $\hat{a}_L$ and $\hat{a}_R$.

$$(\hat{a}_L\hat{a}_R + 1)\psi + \lambda\psi = 0 \tag{12.10.3}$$

We found above that $\left[\hat{a}_R, \hat{a}_L\right] = 2$ so we will replace $\hat{a}_R\hat{a}_L$ with $\hat{a}_L\hat{a}_R + 2$ in the middle of the following calculation, leading us to the same equation with a *different* eigenvalue.

$$\begin{aligned}(\hat{a}_L\hat{a}_R + 1)(\hat{a}_L\psi_m) + \lambda(\hat{a}_L\psi_m) &= \hat{a}_L\hat{a}_R\hat{a}_L\psi_m + (1+\lambda)\hat{a}_L\psi_m\\ &= \hat{a}_L(\hat{a}_L\hat{a}_R + 2)\psi_m + (1+\lambda)\hat{a}_L\psi_m\\ &= \hat{a}_L(\hat{a}_L\hat{a}_R\psi_m) + (3+\lambda)\hat{a}_L\psi_m\end{aligned}$$

We now use the fact that ψ_m is an eigenfunction to replace $\hat{a}_L\hat{a}_R\psi_m$ with $(-1 - c_m)\psi_m$, which gives:

$$(2 - c_m + \lambda)\hat{a}_L\psi_m = (2 - c_m + \lambda)\psi_n$$

We conclude that ψ_n is an eigenfunction with eigenvalue $\lambda = c_m - 2$, and a similar calculation leads us to conclude that $\hat{a}_R\psi_m$ is an eigenfunction with eigenvalue $\lambda = c_m + 2$. (See Problem 12.149.)

The operators $\hat{a}_R$ and $\hat{a}_L$ are called the "raising" and "lowering" operators for this problem. (Hence the subscripts.) The raising operator takes any eigenfunction and turns it into the one with the next highest eigenvalue, and the lowering operator turns it into the one with the next lowest eigenvalue. When you act with the lowering operator on the ground state you get 0, which leads to the alternative names "annihilation" and "creation" operators.[13] Together, $\hat{a}_R$ and $\hat{a}_L$ are called "ladder operators" because they generate a ladder of states with eigenvalues stretching upward from the ground state.

[13]These names make more sense in quantum field theory, where the eigenvalue represents the number of particles. Then the annihilation operator takes you from a state with n particles to one with $n-1$, and the creation operator takes you to one with $n+1$ particles.

Stepping Back

We just solved the quantum oscillator problem twice, once by manipulating differential equations and then again in the more abstract language of operators.[14] You may reasonably feel that the second approach made the problem look more confusing without giving you anything new, so we want to point out a few reasons why the operator approach is in the long run better. It's worth putting in the effort to learn to work with operators now, since they are used widely in quantum mechanics, optics, and a variety of other fields.

For one thing, operators make the calculations easier. Of course the operators are just shorthand for pieces of differential equations, but once you rewrite the equation in terms of $\hat{a}_R$ and $\hat{a}_L$ and calculate their commutator you've reduced a calculus problem to a simple algebra problem.

More importantly the operator formulation shows *why* this strange trick worked, and that allows you to apply it to other problems. When we plugged $\hat{a}_R\psi_m$ into Equation 12.10.3 we had to move the $\hat{a}_R$ to the left of $\hat{a}_L\hat{a}_R$ so we could simplify by acting with $\hat{a}_L\hat{a}_R$ on the eigenfunction ψ_m. Because $[\hat{a}_L, \hat{a}_R]$ is a constant, the result was to simply add a constant to the eigenvalue in the equation. In Problem 12.152 you'll try this trick on a similar problem where the commutator of the two operators is not a constant, and you'll see that it doesn't work. Problem 12.153 is another problem where it *does* work, finding the angular momentum values of the hydrogen atom.

Finally, a warning: if you take a course on quantum mechanics, expect to encounter different conventions from those we used here. The conventional ladder operators, usually denoted $\hat{a}_+$ and $\hat{a}_-$, are related to ours by $\hat{a}_- = \hat{a}_L/\sqrt{2}$ and $\hat{a}_+ = -\hat{a}_R/\sqrt{2}$. We therefore ended up with the commutator $[\hat{a}_R, \hat{a}_L] = 2$ rather than the more common $[\hat{a}_-, \hat{a}_+] = 1$.

12.10.3 Problems: The Quantum Harmonic Oscillator and Ladder Operators

The problems in this section assume the following definitions.

$$D = d/dy \qquad \hat{a}_R = D - y \qquad \hat{a}_L = D + y$$

You will also make frequent use of the following fact (derived in the Explanation above).

$$\left[\hat{a}_R, \hat{a}_L\right] = 2 \quad \rightarrow \quad \hat{a}_R\hat{a}_L = \hat{a}_L\hat{a}_R + 2$$

This means that you can pass $\hat{a}_R$ right through $\hat{a}_L$ and pick up a +2, or pass $\hat{a}_R$ left through $\hat{a}_L$ and pick up a −2.

[14] Remember when the differential equations themselves seemed hopelessly abstract? Ah, those innocent bygone days.

12.139 In the box on Page 8 we step through the process of rewriting the operator $\hat{a}_L \hat{a}_L \hat{a}_R$ with $\hat{a}_R$ on the left. Copy that solution (only the part about the three operators) and say what operator property justifies each step. *Hint*: None of the steps depends in any way on knowing what the operators $\hat{a}_L$ and $\hat{a}_R$ are. Some steps depend on knowing that both are linear operators, and other steps depend on knowing their commutator.

12.140 **Operator Algebra** Commutators are used to pass one operator "through" another, as in the example in example that starts on Page 8.

(a) Rewrite each of the following expressions so every $\hat{a}_R$ is to the right of every $\hat{a}_L$.

i. $\hat{a}_R \hat{a}_L \hat{a}_R$

ii. $\hat{a}_R \hat{a}_L \hat{a}_L$

(b) Suppose ψ is an eigenfunction of the operator $\hat{a}_R$ with eigenvalue λ, which means $\hat{a}_R \psi = \lambda \psi$. Use that fact and the commutator above to simplify the expression $\hat{a}_R \hat{a}_L \psi$ as much as possible.

(c) Suppose ϕ is an eigenfunction of the operator $\hat{a}_R \hat{a}_L$ with eigenvalue γ. Is ϕ an eigenfunction of $\hat{a}_L \hat{a}_R$? If not, explain why not. If so, find its eigenvalue.

12.141 Calculate each of the following. Give your answers in terms of D and y, not $\hat{a}_R$ and $\hat{a}_L$.

(a) $[D, y]$

(b) $\hat{a}_R^2$

(c) $\hat{a}_R \hat{a}_L$

12.142 Calculate each of the following. Simplify your answers as much as possible. You should be able to answer all of these just from knowing the commutator $[\hat{a}_R, \hat{a}_L]$, without having to use the definitions of $\hat{a}_R$ and $\hat{a}_L$.

(a) $[\hat{a}_L, \hat{a}_L^2]$

(b) $[\hat{a}_L, \hat{a}_L \hat{a}_R]$

(c) $[\hat{a}_L \hat{a}_R, \hat{a}_R \hat{a}_L]$

12.143 Prove that $[\hat{A}, \hat{B} + \hat{C}] = [\hat{A}, \hat{B}] + [\hat{A}, \hat{C}]$ for any linear operators $\hat{A}$, $\hat{B}$, and $\hat{C}$.

12.144 In the Explanation (Section 12.10.2) we listed the first three eigenfunctions of the quantum oscillator. Calculate the next two.

12.145 In the Explanation (Section 12.10.2) we listed the first three eigenfunctions of the quantum oscillator. Verify that the lowering operator $\hat{a}_L$ acting on ψ_2 (the "second excited state") gives you ψ_1.

12.146 It's possible to find the ground state energy of the quantum oscillator using operators rather than calculus.

(a) Rewrite Equation 12.10.3 in terms of $\hat{a}_R \hat{a}_L$ rather than $\hat{a}_L \hat{a}_R$.

(b) Let ψ_0 be the ground state eigenfunction. Don't use the formula we derived; just leave it as ψ_0 and put it into the equation you just wrote. What value does λ have to have in order for this equation to work? *Hint*: remember what $\hat{a}_L$ does when it acts on ψ_0.

12.147 In this problem you will examine the differential equation $\hat{B}\hat{A}\psi + \lambda\psi = 0$ where $\hat{A}$ and $\hat{B}$ are two linear operators about which you know only one thing: $[\hat{A}, \hat{B}] = 5$.

(a) We begin by assuming that we have already found one solution. Write an equation that asserts "ψ_m is an eigenfunction of this differential equation, with eigenvalue λ_m."

(b) Show that $\psi_n = \hat{B}\psi_m$ is an eigenfunction of the same differential equation and find its eigenvalue λ_n in terms of λ_m.

(c) Write a differential equation that you could solve to find the eigenfunction with the lowest eigenvalue. This differential equation will involve the unknown operators so you cannot solve it.

(d) Find the lowest eigenvalue for this equation, and use that to find what all the possible eigenvalues are. (If you're stuck on how to find the lowest eigenvalue you may find it helpful to look at Problem 12.146.)

12.148 Derive Equation 12.10.2 from Equation 12.10.1 using the substitutions given in the Explanation (Section 12.10.2).

12.149 In this problem you will prove that the raising operator works as advertised.

(a) Prove that if ψ_m is an eigenfunction of Equation 12.10.2 with eigenvalue c_m, then $d\psi_m/dy - y\psi_m$ is also an eigenfunction, with eigenvalue $c_m + 2$. Your proof should just involve functions and derivatives, with no operator notation.

(b) Prove that if ψ_m is an eigenfunction of Equation 12.10.3 with eigenvalue c_m, then $\hat{a}_R \psi_m$ is also an eigenfunction, with eigenvalue $c_m + 2$. Your proof should use operators and commutators, with no derivatives written out.

12.150 In the Explanation (Section 12.10.2) we said that the equation $d\psi_0/dy + y\psi_0 = 0$ would give us the ground state of the quantum harmonic oscillator.

(a) Explain how we know that the solution to this particular equation will give us the eigenfunction with the lowest eigenvalue.

(b) Solve the equation by separating variables.

(c) Plug your solution into Equation 12.10.2 to show that it works, and to find the corresponding eigenvalue.

12.151 In the Explanation (Section 12.10.2) we listed the first three eigenfunctions of the quantum oscillator, and we noted that the arbitrary constants in front are determined by the normalization condition $\int_{-\infty}^{\infty} \psi(x)^2 dx = 1$. Use that condition to calculate C_0. You may use the fact that $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$. (You solved a minor variation of that integral in Section 12.3 Problem 12.49).

12.152 A particle with potential energy $V = (1/4)\kappa x^4$ oscillates, but it is not a "simple harmonic oscillator." Schrödinger's equation for that particle is:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \frac{1}{4}\kappa x^4 \psi = E\psi$$

(a) Define a new independent variable y and a new eigenvalue λ and use them to rewrite this without any constants other than λ in it. Your answer should look similar (but not identical) to Equation 12.10.2.

(b) Define $\hat{a}_1 = D + y^2$ and $\hat{a}_2 = D - y^2$ and rewrite the differential equation in terms of these operators, with no explicit derivatives. (See Equation 12.10.3 for example.) There is more than one possible way to do this. *Hint*: begin by calculating $\hat{a}_1\hat{a}_2$ and $\hat{a}_2\hat{a}_1$.

(c) Calculate the commutator $[\hat{a}_1, \hat{a}_2]$.

(d) Assume $\psi_m(y)$ is an eigenfunction with eigenvalue c_m. Plug in $\psi_n = \hat{a}_1\psi_m$ and show that it is *not* an eigenfunction. As part of this process you should rearrange your equation (by using the commutator) so you can make use of the fact that ψ_m is an eigenfunction.

(e) Explain what went wrong. What was it about the commutator $[\hat{a}_1, \hat{a}_2]$ that meant the trick didn't work here the way it did for the simple harmonic oscillator equation?

12.153 A hydrogen atom consists of an electron orbiting about a proton. There is an operator $\hat{L}_z$ that corresponds to the z-component of angular momentum in the following sense. If the particle is in a state where it has a definite z-component of angular momentum μ, then the particle's wavefunction $\psi(x, y, z)$ obeys the eigenvalue equation $\hat{L}_z\psi = \mu\psi$. In other words the eigenfunctions of this equation represent the possible states of definite z-angular momentum. Similar equations hold for the operators $\hat{L}_x$ and $\hat{L}_y$.

It's possible to write these operators out explicitly in terms of x, y, z, $\partial/\partial x$, $\partial/\partial y$, and $\partial/\partial z$, but we're not going to bother. All you need to know about them is their commutation relations: $[\hat{L}_x, \hat{L}_y] = i\hbar\hat{L}_z$, $[\hat{L}_y, \hat{L}_z] = i\hbar\hat{L}_x$, and $[\hat{L}_z, \hat{L}_x] = i\hbar\hat{L}_y$. To find the eigenvalues of the $\hat{L}_z$ equation, we define the ladder operators $\hat{L}_+ = \hat{L}_x + i\hat{L}_y$ and $\hat{L}_- = \hat{L}_x - i\hat{L}_y$.

(a) Calculate $[\hat{L}_z, \hat{L}_+]$ and $[\hat{L}_z, \hat{L}_-]$. Simplify your answers as much as possible.

(b) Suppose ψ_m is an eigenfunction of $\hat{L}_z$ with z-angular momentum $\mu = c_m$. Show that $\hat{L}_+\psi_m$ and $\hat{L}_-\psi_m$ are also eigenfunctions of $\hat{L}_z$, and find their z-angular momenta.

12.11 Additional Problems

For Problems 12.154–12.159 solve the differential equation using either the method of power series or the method of Frobenius. Your answer should be in the form of a partial sum with five non-zero terms and two arbitrary constants.

12.154 $4y''(t) + y'(t) + t^3 y(t) = 0$

12.155 $y''(t) + (1+t)y(t) = t$

12.156 $ty''(t) + 2y'(t) + y(t) = 0$. (Your answer will only have one arbitrary constant, and will therefore not be the general solution.)

12.157 $y''(t) + y'(t) + y(t) = \sin t$

12.158 $e^t y''(t) + y(t) = 0$

12.159 $y''(t) + (3/2)(\cos t/t)y'(t) + y(t) = 0$ (You may just report the first four non-zero terms for this one.)

12.160 **Airy Functions.** The "Airy equation" arises in the quantum mechanical treatments of a triangular potential well and for a particle in a one-dimensional constant force field.

$$\frac{d^2y}{dx^2} - xy = 0$$

(a) Explain how we can tell by looking at this equation that a power series solution exists.

(b) Find the recurrence relation using the method of power series.

(c) What coefficients must be zero?

(d) Write the solution up to the seventh order for $c_0 = c_1 = 1$.

12.161 **Laguerre Polynomials.** The "Laguerre differential equation" appears in the quantum mechanical treatment of the Hydrogen atom.

$$x\frac{d^2y}{dx^2} + (1-x)\frac{dy}{dx} + \lambda y = 0$$

(a) Show that the Laguerre differential equation meets the requirements for the method of Frobenius, but does not meet the requirements for the method of power series.

(b) Begin applying the method of Frobenius. Stop after you reach and solve the indicial equation.

Tragedy strikes! You discover that r must be zero which means the solution will in fact be a power series. (The moral of that story is that analytic coefficients are sufficient, but not necessary, for the method of power series.) So we begin again.

(c) Assuming a power series solution of the form $y(x) = \sum_{n=0}^{\infty} c_n x^n$, find the recurrence relation for c_{n+1} in terms of c_n, n, and the constant λ.

(d) For any positive integer λ the resulting solution will be a finite polynomial. How can we tell this, and what will be the order of the polynomial?

(e) Find a solution for $\lambda = 0$ and $c_0 = 1$. This is the first "Laguerre polynomial" generally denoted $L_0(x)$.

(f) Find a solution for $\lambda = 1$ and $c_0 = 1$. This is the next "Laguerre polynomial" $L_1(x)$.

(g) Find $L_2(x)$ and $L_3(x)$.

(h) Look up the Laguerre polynomials. If the first four do not agree with your answers, figure out what went wrong!

The power series method only gave us one solution. More complicated methods can be used to find the other one, but it isn't typically relevant for physical applications.

12.162 **Exploration: The Quantum Harmonic Oscillator**

A harmonic oscillator is defined by the potential function[15] $V = (1/2)m\omega^2 x^2$. The quantum mechanical wavefunction for such an oscillator must obey Schrödinger's equation with that potential function.

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \frac{1}{2}m\omega^2 x^2 \psi = E\psi$$

If you've never heard of a wavefunction, don't let it scare you off: this is just a differential equation for a function that happens to be called $\psi(x)$. The constants m and ω correspond to a particular physical situation, and $\hbar$ is a universal constant. Your job is to find the values of the constant E for which solutions exist (the eigenvalues of this equation), and the solutions associated with them (the eigenfunctions). (Section 12.10

[15]You may be more used to the form $V = (1/2)kx^2$. Since ω is defined to be $\sqrt{k/m}$ the two are equivalent, but this form is more commonly used in quantum problems.

shows you how to solve this problem in a very different way from what we do here.)

(a) You can rewrite this ODE in the form $\psi''(y) - (y^2 - \lambda)\psi(y) = 0$ with a substitution of the form $y = cx$. Find the constants c and λ in terms of $\hbar$, m, ω, and E. What are the units of y and λ?

(b) We are going to guess that ψ can be written as $e^{-y^2/2}$ times a simpler function. (That guess can be physically motivated, but we're going to skip that step here. The real justification will be to try it and see if the resulting solution looks simple.) Using the substitution $\psi(y) = e^{-y^2/2}u(y)$, rewrite the differential equation.

(c) Solve this equation using the method of power series. Your answer should be in the form of a recurrence relation for the coefficients.

The recurrence relation gives two series, one with the even coefficients c_0, c_2, etc., and another with the odd coefficients. For any given value of λ, the coefficients c_0 and c_1 are the two arbitrary constants in the general solution, and in general each one is the beginning of an infinite series. However, non-terminating polynomials do not represent physically meaningful solutions to this problem.[16]

(d) For what values of λ will one of the two series be a *finite* polynomial, meaning all the coefficients beyond some value of n will equal zero? (Your answer gives the eigenvalues of this problem.)

(e) For the three lowest eigenvalues λ, write the solutions $\psi(x)$. Each one will be in the form of a finite polynomial (with an arbitrary constant) times an exponential.

(f) The constant E represents the oscillator's energy. Based on the allowed values of λ that you found, what are the possible values of E?

12.163 Exploration: A Different Kind of Series Solution

The following differential equation describes the motion of an object falling in the gravitational field of a planet.

$$\frac{d^2r}{dt^2} = -\frac{GM}{r^2} \tag{12.11.1}$$

Here r is the distance of the object from the center of the planet. The given constants in this problem are G (a universal constant), M (the mass of the planet, not of the falling object), and r_0 and v_0 (the initial position and velocity of the object).

(a) Explain why this equation does not lend itself directly to either the power series method, or the method of Frobenius, as discussed in this chapter.

Nonetheless, we can approach this problem by looking for the first few coefficients in a Maclaurin series solution:

$$r(t) = c_0 + c_1 t + c_2 t^2 + c_3 t^3 + c_4 t^4 + \ldots \tag{12.11.2}$$

(b) Using Equation 12.11.2 find the first two coefficients in terms of the given constants.

(c) Take the second derivative of both sides of Equation 12.11.2. Then use Equation 12.11.1 to replace d^2r/dt^2, and finally plug 0 into both sides to find c_2 in terms of the given constants.

(d) Write the solution to Equation 12.11.1 up to the second order. This solution should look like the introductory mechanics equation $x = x_0 + v_0 t - (1/2)gt^2$ with the constant g being a function of our given constants. (This g should come out as $9.8\ \text{m/s}^2$ if you use the mass and radius of the Earth as M and r_0.)

(e) To find the next term—the first correction to the introductory mechanics equation—take the derivative of both sides of Equation 12.11.1 with respect to time, and then plug in $t = 0$. Solve for c_3 in terms of the given constants.

(f) Write the solution to Equation 12.11.1 up to the third order.

(g) We have seen that your second-order formula replicates the introductory mechanics equation $r = r_0 + v_0 t - (1/2)gt^2$, which works well for objects that stay near the surface of the Earth ($r \approx R$). Does the third-order correction make the effective acceleration due to gravity higher, or lower, than $9.8\ \text{m/s}^2$? Answer based on your equation, but then explain why your answer makes sense physically. Assume

[16]We're not going to go through the proof of that claim here. See for example "Introduction to Quantum Mechanics" by David Griffiths (one of our personal heroes).

$r_0 = R$ and $v_0 > 0$ (such as a rocket taking off). (Your answer will depend on time.)

(h) A bullet is fired straight up into the air from the surface of the Earth with an initial speed of 1000 m/s. Look up values for G, M, and r_0, and use them to graph the height of the bullet using your answers to Parts (d) and (f). How does adding the third-order term change the motion of the bullet?

CHAPTER 13

Calculus with Complex Numbers (Online)

13.6 Integrating Along Branch Cuts and Through Poles

13.6.1 Explanation: Integrating Along Branch Cuts and Through Poles

We have seen that it is important to know if a closed contour encloses a pole. But what if the contour goes directly through a pole? Or what if a contour travels along a branch cut of the function being integrated?

There is a fundamental problem with evaluating an integral where the function itself is ill-defined. For instance, along the positive real axis you can consider ϕ to be 0 or 2π (or infinitely many other possibilities). So $\int_1^2 \phi \, dz$ could be 0 or 2π and so on. The "right answer" is just a matter of definition.

So we might declare all such integrals to be undefined, but there are compelling reasons to work with such tricky cases. For instance, we have seen that complex integrals along the real axis are used to evaluate real integrals. But the real axis includes the branch cuts for common functions such as $\sqrt{x}$ and $\ln x$. If we invalidate all integrals along branch cuts we lose a valuable approach to many important problems.

Such problems have an analogue in the real-valued world. You may recall that an integral such as $\int_0^1 \ln x \, dx$ is called an "improper integral" because the integrand has a vertical asymptote at $x = 0$. We approach such an integral as a limit.

$$\int_0^1 \ln x \, dx = \lim_{a \to 0^+} \int_a^1 \ln x \, dx$$

In words, we evaluate the integral in a region where the function is always well defined, and then take a limit as that region approaches the asymptote.

Analogously, when a contour integral goes directly through a pole or along a branch cut, we begin by drawing a contour that comes *near* that pole or branch cut. Along such a contour the function, and therefore the integral, is uniquely defined. Then we take a limit as the contour approaches the singularity.

The following example demonstrates most of what you need to know about this technique. (A few more issues are brought out in Problem 13.84.) But we hope you will notice that almost everything you need are the things you *already* know about contour integrals, just being put together in a new way.

At one key point we will have to use the following theorem, which will also prove useful in many of the problems in this section. (We state it here without proof but it makes a lot of sense if you think about it. You can also prove it reasonably easily with a Riemann sum.)

The modulus of a contour integral must be less than or equal to the maximum modulus of the integrand on the contour times the length of the contour.

$$\left|\int_C f(z)\,dz\right| \le |f(z)|_{max} \times arclength(C) \tag{13.6.1}$$

We will be using that theorem to prove that certain integrals must be zero by showing that the product (modulus times arclength) is zero.

Example: Integrating Along a Branch Cut

We begin with the integral of a real function along the real number line.

$$R = \int_0^\infty \frac{1}{\sqrt{x}(x+1)}\,dx \tag{13.6.2}$$

The problem has nothing to do with complex numbers, but we have seen that the easiest way to approach such problems often begins with a contour integral.

$$\int_0^\infty \frac{1}{\sqrt{z}(z+1)}\,dz \tag{13.6.3}$$

Are these two integrals the same? This question is subtler than you might think, because in Equation 13.6.2 everything is real and $\sqrt{x}$ is by definition a positive number. In Equation 13.6.3 we have to decide what branch of $\sqrt{z}$ we are using.

Remember that $\sqrt{z}$ has phase $\phi_z/2$. We usually use the principal branch of ϕ, which goes from $-\pi$ to π. In that case $\sqrt{z}$ has a branch cut on the negative real axis, while on the positive real axis $\phi = 0$ and $\sqrt{z}$ is positive. For this problem, however, we will choose to put the branch cut on the positive real axis. Thus when z is slightly above the positive real axis $\phi \approx 0$ and $\sqrt{z}$ is (approximately) a positive real number, but when z is slightly below the positive real axis $\phi \approx 2\pi$ and $\sqrt{z}$ is (approximately) a negative real number.

With this choice Equation 13.6.3 is not strictly defined, but if you shift the contour slightly up in the complex plane then it equals R, the solution to Equation 13.6.2. If you shift the contour slightly down the solution to Equation 13.6.3 equals $-R$.

It might not seem like any of this is getting us closer to evaluating Equation 13.6.2, but at this point we have all the ingredients we need. To see how this works, consider integrating $f(z)$ around the closed contour C shown in Figure 13.11. The contour avoids both the pole at $z = 0$ and the branch cut along the real axis. Contour C consists of four distinct pieces: a horizontal line just above the positive real axis, a horizontal line just below the positive real axis, a small circle around the origin (whose radius we will call ρ_1), and a large circle around the origin (radius ρ_2). The integral around the entire contour C must equal the sum of these individual integrals. In the limit as $\rho_1 \to 0$ and $\rho_2 \to \infty$ the two horizontal lines go from $x = 0$ to $x \to \infty$.

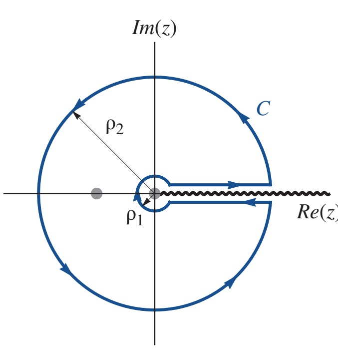


FIGURE 13.11 The contour C is made up of four parts, two horizontal lines just above and below the positive real axes, and two (almost) circles of radius ρ_1 and ρ_2. The gray dots indicate poles.

- **The entire contour.** The contour C is closed, and the integral around any closed contour depends on the enclosed poles. The pole at $z = 0$ is *not* enclosed by this contour! But for $\rho_1 < 1$ and $\rho_2 > 1$ the pole at $z = -1$ is enclosed. We can rewrite the function as this.

$$f(z) = \frac{1/\sqrt{z}}{z-(-1)}$$

That is the correct form for Cauchy's integral formula with $g(z) = 1/\sqrt{z}$ and $z_0 = -1$. The integral is therefore:

$$2\pi i g(z_0) = \frac{2\pi i}{\sqrt{-1}} = 2\pi$$

- **The small circle around the origin.** We are interested in the integral around this loop as $\rho_1 \to 0$. We will show that this integral must approach zero by invoking Equation 13.6.1. To find the modulus of $f(z)$ we multiply it by its complex conjugate and then take the square root.

$$f(z)f(z)^* = \frac{1}{\sqrt{z}(z+1)}\frac{1}{\sqrt{z^*}(z^*+1)} = \frac{1}{\sqrt{zz^*}(zz^*+z+z^*+1)}$$

The quantity $z + z^*$ is $2Re(z)$. On a circle of radius ρ_1 the quantity zz^* is always ρ_1^2.

$$f(z)f(z)^* = \frac{1}{\rho_1(\rho_1^2 + 2Re(z) + 1)}$$

We don't need calculus to maximize this quantity: we just have to minimize the denominator, which occurs on the negative real axis where $Re(z) = -\rho_1$.

$$f(z)f(z)^*_{\max} = \frac{1}{\rho_1(\rho_1^2 - 2\rho_1 + 1)} = \frac{1}{\rho_1(1-\rho_1)^2}$$

$$|f(z)|_{\max} = \sqrt{f(z)f(z)^*_{\max}} = \frac{1}{\sqrt{\rho_1}(1-\rho_1)}$$

(Within the world of positive real numbers, the square root always means a positive number. Because $\rho_1 < 1$ we therefore wrote $(1-\rho_1)$ for the square root of $(1-\rho_1)^2$.) Now we multiply $|f(z)|_{\max}$ by the arclength of the contour, which is $2\pi\rho_1$, to put an upper limit on the integral.

$$\left|\oint f(z)dz\right| \le \frac{2\pi\sqrt{\rho_1}}{1-\rho_1}$$

In the limit as $\rho_1 \to 0$ this approaches zero. We conclude that the integral itself must approach zero.

- **The outer circle.** The math is almost the same as for the inner circle with ρ_1 replaced with ρ_2. Because the values of ρ_2 that interest us are greater than 1, we replace $\sqrt{(\rho_2-1)^2}$ with $\rho_2 - 1$. Then we take the limit $\rho_2 \to \infty$, and we once again get 0.

$$\left|\oint f(z)dz\right| \le \lim_{\rho_2\to\infty}\frac{2\pi\sqrt{\rho_2}}{\rho_2 - 1} = 0$$

- **The horizontal lines.** In the limit where the horizontal lines approach the positive real axis the integral along the upper line becomes R, the real-valued integral that we are trying to find. Along the bottom horizontal line, as $\phi \to 2\pi$, all the values of $\sqrt{z}$ are the negative versions of those same real numbers. But we are integrating that line in the negative direction, which brings in another negative sign. Therefore the bottom line contributes exactly the same R as the top line.

We now have all the pieces to evaluate the integral we started with.

entire loop =	small (ρ_1) circle	+	big (ρ_2) circle	+	top line	+	bottom line
2π =	0	+	0	+	R	+	R

We finally conclude that $R = \pi$, and thus solve the problem we began with.

The example above showcases the general approach to such problems, along with most of the tricks that you will need along the way. Only one complication remains: what if there had been a pole on the positive x-axis, right in the middle of the region of integration? Our contour C would then have to circle around that pole. You will work such an example in Problem 13.84.

13.6.2 Problems: Integrating Along Branch Cuts and Through Poles

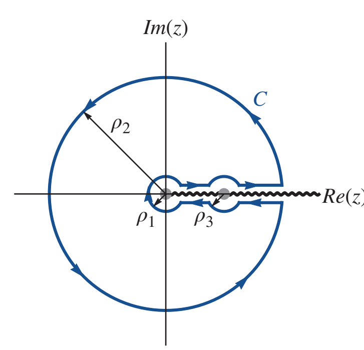
13.84 Walk-Through: Integration Through a Pole. In this problem you're going to evaluate the integral

$$R = \int_0^{\infty} \frac{1}{x^{1/3}(x-1)}\,dx$$

This looks similar to the problem we solved in the Explanation (Section 13.6.1) but there are two differences. The first is that we've used a cube root instead of a square root, which will give you a bit more practice thinking about branch cuts. The more important difference is that this integrand has a pole on the *positive* real axis, right on the region of integration. You can define the value of this improper integral as:

$$R = \lim_{\rho_1 \to 0^+, \rho_2 \to \infty, \rho_3 \to 0^+} \left[\int_{\rho_1}^{1-\rho_3} \frac{1}{x^{1/3}(x-1)}\,dx + \int_{1+\rho_3}^{\rho_2} \frac{1}{x^{1/3}(x-1)}\,dx \right]$$

Take the branch cut of $z^{1/3}$ to be along the positive real axis. To solve this problem, you're going to define $f(z) = 1/[z^{1/3}(z-1)]$ and integrate it on the closed contour C shown below. The small circle around the origin has radius ρ_1, the large circle has radius ρ_2, and the circle around the pole has radius ρ_3.



(a) Evaluate the contour integral $\oint_C f(z)dz$ along the entire contour C by considering the poles of $f(z)$.

(b) As in the Explanation, use Equation 13.6.1 to show that the integrals around the inner and outer loops vanish in the limits $\rho_1 \to 0$ and $\rho_2 \to \infty$.

(c) Now consider the loop of radius ρ_3 around $z = 1$. We can represent this entire loop as $z = 1 + \rho_3 e^{i\theta}$. (We didn't call it ϕ to avoid confusion with the phase of z.) As $\rho_3 \to 0^+$ the $z^{1/3}$ in the denominator of our fraction simply approaches $1^{1/3}$ (although even there you have to be careful to use the correct branch for each half-loop). But the $z - 1$ in the denominator approaches zero, and as you know zeroes in the denominator of a limit require extra care.

i. For the top half of the loop, the phase ("where-you-are-in-the-complex-plane" ϕ, not "where-you-are-in-our-little-loop" θ) is zero so $z^{1/3} = 1$. Rewrite $f(z)$, replacing $z^{1/3}$ with 1 and $z - 1$ with $\rho_3 e^{i\theta}$, and then integrate as θ goes from 0 to π. Then let $\rho_3 \to 0$ in your answer. *Hints*: Don't forget to calculate dz in terms of $d\theta$, and pay attention to the direction of integration shown in the picture above.

ii. For the bottom half of the loop, $\phi = 2\pi$. Rewrite $f(z)$, replacing $z^{1/3}$ with $e^{2\pi i/3}$ and $z - 1$ with $\rho_3 e^{i\theta}$, then integrate (the limits will be different for this half), and finally take the limit.

(d) As $\rho_1 \to 0$ and $\rho_2 \to \infty$ the upper horizontal contour equals R, the real integral you are trying to find. How is the lower horizontal contour related to R? *Hint*: The answer is neither R nor $-R$.

(e) Set the integral around the entire loop equal to the sum of all the contours that make it up and solve the resulting equation for R, the real integral we set out to find. Make sure your answer is a real number.

13.85 *[This problem depends on Problem 13.84.]* We plugged the integral from Problem 13.84 into a computer and got the (quite correct) answer that it diverges. Find a way to get your computer to evaluate this integral without using any complex numbers and get the answer from Problem 13.84, which is the Cauchy principal value.

Evaluate the integrals in Problems 13.86–13.89 by following the basic template outlined in the Explanation (Section 13.6.1). If the region of integration goes directly through a pole you will need to bend your contour around that pole as demonstrated in Problem 13.84. At the end you should write your answers in real form.

13.86 $R = \int_0^\infty dx/[x^{2/3}(x-1)]$.

13.87 $R = \int_0^\infty dx/[x^{1/4}(x+3)]$.

13.88 $R = \int_0^\infty dx/[\sqrt{x}(x^2-4)]$.

13.89 $R = \int_0^\infty dx/[x^k(x-x_0)]$ where $0 < k < 1$, $x_0 < 0$. (You'll do the case where $x_0 > 0$ in Section 13.11 Problem 13.153.) *Hint*: When you are putting your final answer in real form you may find it helpful to replace x_0 with $|x_0|e^{i\pi}$, but don't do this until you get to the end.

13.90 Try evaluating $\int_0^\infty \sqrt{x}/(x+1)dx$ using the techniques of this section. Explain why it doesn't work.

13.91 In this problem we introduce a trick for contour integrals involving logarithms. Our first use of this trick will be an integral that you can easily evaluate without complex numbers (which is a good way to check your answer); in Problem 13.92 you will apply the same trick to a harder problem. The question for this problem is $\int_0^{x_0} \ln x \, dx$ where $x_0 > 0$. Use R to represent the real-valued integral we are looking for.

(a) Start with the complex function $\ln z$, defining the branch cut along the positive real axis. The integral of $\ln z$ just above the positive real axis going forward from 0 to x_0 is R, the integral we want. How is the integral of $\ln z$ that goes backward from x_0 to 0 just *below* the positive real axis related to R? Given that answer, explain why directly applying the technique we have used in this section will *not* help us find the integral we want.

(b) Here is an apparently unrelated question: find the real and imaginary parts of the function $f(z) = (\ln z)^2$. (This is the function that we will define as $f(z)$ for the rest of this problem.)

(c) Now consider the integral of $f(z)$ along the two horizontal paths you used in Part (a). Write an expression for the sum of those two integrals. Your answer should be in terms of R, the original integral we wanted to find.

(d) Finish drawing a closed contour using those two horizontal lines and avoiding both the branch cut and the origin. This contour will look just like many of the ones we have used in this section, with the notable exception that the larger loop is *not* going to infinity. Find the integral of $f(z)$ around this entire closed loop.

(e) Argue that the integral of $f(z)$ around the inner loop is zero in the limit where the radius approaches zero. The following limit (which you can prove using l'Hôpital's rule) may prove helpful: $\lim_{x\to 0^+} x(\ln x)^2 = 0$.

(f) Along the outer circle $z = x_0 e^{i\phi}$. Evaluate the integral of $f(z)$ along the outer circle by converting to an integral over ϕ. (Earlier you broke $f(z)$ into its real and imaginary parts, which should have given you a total of three terms. You can do those three integrals separately and sum the results. Two of them will require integration by parts.)

(g) Use all the pieces you have found to solve for the original integral.

13.92 Let $R = \int_0^\infty [(\ln x)/[(x+1)(x+2)]\,dx$. To solve this you'll begin by defining $f(z) = (\ln z)^2/[(x+1)(x+2)]$, with the branch cut of $\ln z$ to be along the positive real axis.

(a) Draw a contour like the ones we've been using in this section, going above and below the branch cut and closing the contour with a small circle around the origin and a large circle that approaches infinity. Evaluate $\int f(z)\,dz$ around this entire contour.

(b) Find the sum of the integrals above and below the branch cut. Remember that the second one goes backwards and that $\ln z$ is different on these two contours. Your answer should be given in terms of R. *Hint*: at one point in these calculations you should find that you need to integrate $dx/[(x+1)(x+2)]$. You can do that by rewriting it as $dx/(x+1) - dx/(x+2)$. You could find that using partial fractions, but you can even more easily check that these formulas are equal.

(c) Prove that the integrals around the small and large loops both go to zero. The following limit (which you can prove using l'Hôpital's rule) may prove helpful: $\lim_{x \to 0^+} x(\ln x)^2 = 0$. (You will also need to use l'Hôpital's rule for another, more straightforward limit.)

(d) Put all of your answers together to find R.

13.10 Special Application: Fluid Flow

We have seen how analytic functions can be used to solve Laplace's equation under a variety of boundary conditions—first in simple regions (Section 13.3) and then, using conformal mapping, in more complicated regions (Section 13.9). Our applications for these techniques have all been steady-state temperature and electrostatic potential problems. In this section we apply those same techniques to the slightly more complicated problem of fluid flow.

Velocity Fields and Stream Functions

Figure 13.18 shows a rock in a stream. Our goal is to mathematically model the flow of the water around this rock. That is, we want to find the water's velocity field $\vec{v}(x, y)$.

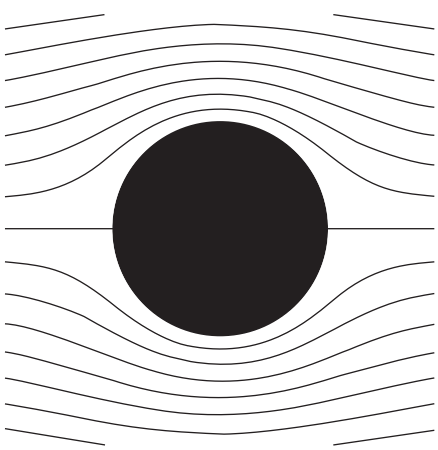


FIGURE 13.18 Flow of water around a circular obstacle.

If we neglect viscosity then the flow can be conceptually divided into thin curves of flowing liquid that exert no forces on each other. These curves, called streamlines, are shown in Figure 13.18. We could replace all the water below one of those streamlines with a solid boundary and the flow above that streamline would be unaffected.

This particular problem has two boundary conditions. The first is the rock. Because no water flows into or out of the rock, one streamline must lie directly along the top curve of the rock, and another streamline along the bottom curve. The second boundary condition is "at infinity"—far away the rock is irrelevant, so the streamlines are evenly spaced horizontal lines.

The technique we are going to present here works for this problem and many others like it, but let's start by laying out its limitations. We're going to assume throughout this section that the velocity field is divergenceless ($\vec{\nabla} \cdot \vec{v} = 0$) and irrotational ($\vec{\nabla} \times \vec{v} = \vec{0}$). We will also restrict ourselves to flow in two dimensions. While real fluid flow occurs in 3D, 2D flow is a good model for systems ranging from shallow streams to wind across an airplane wing. (Movement sideways to the airplane is not generally too significant.)

The Stream Function

Instead of directly finding the velocity function we will spend most of our efforts finding a scalar field called the "stream function" $\psi(x, y)$. We can find such a function, and then find $\vec{v}$ from it, provided $\vec{\nabla} \cdot \vec{v} = 0$ (one of the assumptions we mentioned above). We define this new function by its relationship to the vector we are looking for.

- If you have ψ and you want $\vec{v}$ you take derivatives.

$$\vec{v}(x, y) = \frac{\partial \psi}{\partial y}\hat{i} - \frac{\partial \psi}{\partial x}\hat{j} \tag{13.10.1}$$

  This equation may remind you of how we differentiate the potential V to find the electric field $\vec{E}$, although you should certainly note the differences as well as the similarities. You will see below that the way we integrate $\vec{v}$ to find ψ is reminiscent of the way we get from $\vec{E}$ to V, but again different in important ways. We do want to draw your attention to one important similarity between the two systems: what matters is the *change* in ψ

from one point to another, not its actual value, so you can choose any point you like to set $\psi = 0$.

- If you have $\vec{v}$ and you want ψ you begin by choosing an arbitrary point (x_0, y_0) as the place where $\psi = 0$. Then draw a curve C connecting this point to a second point (x, y). At this second point ψ is the flux of $\vec{v}$ through that curve—in other words the number of streamlines passing through the curve.

$$\psi(x, y) = \int_C \left(\vec{v} \cdot \hat{n}\right) ds \qquad (13.10.2)$$

You will show in Problem 13.141 that this flux is the same along any curve between (x_0, y_0) and (x, y). (This definition leaves the sign of ψ ambiguous since the flux through an open contour requires a decision about which direction is positive. When necessary this ambiguity can be removed using Equation 13.10.1.)

- Visually, the *contour lines* of ψ (the curves along which ψ is constant) are the *streamlines* of $\vec{v}$.

These rules are three different ways of expressing the same relationship between ψ and $\vec{v}$, but not obviously so. As one way to begin to see the connection, Figure 13.19 shows a streamline between two points and a curve C that lies directly along the streamline. Because C lies directly along a streamline, there is no flux of the stream through the curve. This means that $\psi(x_1, y_1)$ and $\psi(x_2, y_2)$ are the same as each other. So you can see why the streamlines of $\vec{v}$ become the curves along which ψ is constant.

In Problem 13.142 you will show how Equation 13.10.2 implies Equation 13.10.1.

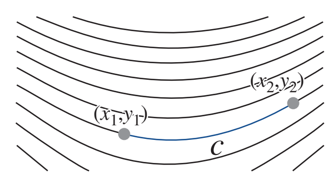


FIGURE 13.19 A curve lying directly along a streamline.

EXAMPLE The Stream Function and the Velocity Field

Question: For water flowing uniformly in the horizontal direction (such as the water far away from the rock in Figure 13.18) the streamlines are evenly spaced horizontal lines. What are $\vec{v}$ and ψ in that situation?

Answer:
In the picture $(0, 0)$ is the point we have arbitrarily chosen to represent $\psi = 0$ and (x, y) is the point where we want to find ψ. We have drawn a curve C between them. The flux through this curve—the amount of water that passes through it—is directly proportional to $y - y_0$, independent of the x-values of the two points. From Equation 13.10.2, then, $\psi(x, y) = ky$ fits this stream. We can use Equation 13.10.1 to go back the other way.

$$\psi(x, y) = ky \quad \rightarrow \quad \vec{v} = \frac{\partial\psi}{\partial y}\hat{i} - \frac{\partial\psi}{\partial x}\hat{j} = k\hat{i}$$

This correctly predicts a uniform horizontal flow.

Note that Figure 13.20, which we drew to represent the streamlines of the flow, is also a picture of the contour lines of $\psi(x, y) = ky$: evenly spaced horizontal lines.

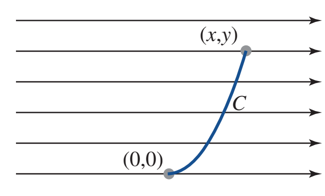


FIGURE 13.20

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As a final note on this example, the function $\psi(x,y) = ky + A$ works perfectly for any constant A. $\Delta\psi$ between any two points is still proportional to $y - y_0$ (the flux), and $\partial\psi/\partial y$ is unchanged. Choosing a constant A for this solution is equivalent to choosing an arbitrary streamline on which to set $\psi = 0$.

Question: How does all that change for $\psi = ky^2$?

Answer:
The contour lines of ψ are once again horizontal, but this time they are not evenly spaced. An identical curve at a higher y-value would have more streamlines passing through it, i.e. a higher flux. (Figure 13.21.)

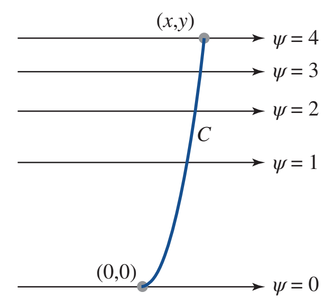


FIGURE 13.21

$$\psi(x,y) = ky^2 \quad \to \quad \vec{v} = \frac{\partial\psi}{\partial y}\hat{i} - \frac{\partial\psi}{\partial x}\hat{j} = 2ky\hat{i}$$

Finding the Stream Function

The two examples above make sense, both mathematically and visually, but there is a subtle difference between them that you probably didn't notice. The first velocity field, $\vec{v} = k\hat{i}$, is "irrotational": that is, $\vec{\nabla} \times \vec{v} = \vec{0}$. The second velocity field does not have a zero curl.

At the beginning of this section we said that we were going to restrict our discussion to irrotational fields. As you will prove in Problem 13.140, the assumption of irrotational fluid flow corresponds to the restriction that $\psi(x,y)$ must be a harmonic function—that is, it must obey Laplace's equation. In the examples above, ky is a solution to Laplace's equation and ky^2 is not.

You may have been wondering what all this is doing in a chapter on complex analysis, and now we are finally ready to make the connection. Under the assumptions we started with—divergenceless, irrotational, two dimensional fluid flow—we can find the stream function by solving Laplace's equation, and then find the velocity field from the stream function. Our strategy for finding harmonic functions is to treat the real plane as if it were the complex plane. (This is why this technique is limited to two dimensions.) We carefully define our boundary conditions. We choose an analytic function, knowing that both its real and imaginary parts must be harmonic. There can be only one harmonic function that fits our boundary conditions, and that is the stream function we're looking for. We can then use Equation 13.10.1 to find the velocity field.

EXAMPLE Return of the Rock in the Stream

We have now built up the tools to analyze the problem that began this section, the stream flowing around a rock in Figure 13.18. We begin as always with our boundary conditions. Since no water flows into or out of the rock, the circular boundary of the rock must be a streamline. By symmetry, the x-axis (the real axis on our complex plane) must also be a streamline. Finally, the water far away from the rock (in any direction) has to look like uniform horizontal flow, which we calculated above as $\psi = ky$.

The streamlines of $\vec{v}$ are the contour lines of ψ, so we need a function $\psi(x, y)$ that is constant on the real axis and on a circle of radius R centered on the origin. For simplicity we can let that constant value be 0. (Remember that we can choose any streamline we want as $\psi = 0$.)

So we are looking for a real function $\psi(x, y)$ that solves Laplace's equation, and that happens to be zero along the border of the rock and the real axis. The obvious choice is $\psi = 0$ (still water) but that does not meet our third boundary condition of horizontal flow far away from the rock, so we have to find something else.

We will begin by choosing an analytic function $f(z)$. Both its real and imaginary parts will be harmonic functions, and we will choose one of them to be our $\psi(x, y)$. On our two contour lines we need $f(z)$ to be a pure imaginary function (and then its real part will be zero) or a pure real function (so its imaginary part is zero).

It's easy to find analytic functions that are real on the real axis, as long as you avoid logs and square roots. (For instance the function $z^2 e^{\sin z}$ is real everywhere on the real axis—obvious when you think about it, isn't it?) The hard part is finding a function that is also real-valued on the edge of the rock.

The boundary of our rock is defined by $|z| = R$, which we can also write as $zz^* = R^2$, so on this circle $z^* = R^2/z$. And now comes a nifty trick: the sum of any function and its complex conjugate is real, so $f(z) = z + R^2/z$ is real on the boundary of the rock. With a little algebra you can find that $Im(f) = y[1 - R^2/(x^2 + y^2)]$.

That's almost the stream function we were looking for. It matches the right boundary condition on the x-axis and the edge of the rock. We know $Im(f)$ is harmonic because $f(z)$ is analytic. (You might notice an exception at $z = 0$ but that point does not concern us at all. Do you see why?) Finally, we need to get $\psi = ky$ far away from the rock. To do that we simply multiply f by the constant k. (Take a moment to convince yourself that this doesn't mess up the other boundary conditions or the fact that f is analytic.)

$$\psi = ky\left(1 - \frac{R^2}{x^2 + y^2}\right) \tag{13.10.3}$$

Finding ψ is the hard part; from there it's easy to find $\vec{v}$ if you want it.

$$\vec{v}(x, y) = \frac{\partial\psi}{\partial y}\hat{i} - \frac{\partial\psi}{\partial x}\hat{j} = k\left(1 - R^2\frac{x^2 - y^2}{(x^2 + y^2)^2}\right)\hat{i} - \frac{2kxyR^2}{(x^2 + y^2)^2}\hat{j}$$

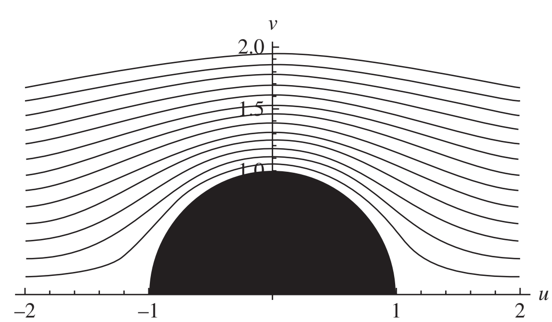
The resulting streamlines are the ones plotted in Figure 13.18. The streamline along the x-axis simply arrives at the obstacle and stops. That streamline is the dividing point between fluid that flows above and below the obstacle, and the point where it reaches the obstacle is called the "stagnation point."

Conformal Mapping and Inverse Problems

Hopefully you were able to follow everything we did in the previous examples, but you might not have come up with the function $f(z) = z + R^2/z$ on your own. Honestly, we probably wouldn't have either. As with electrostatics and steady-state temperature, however, once you know the solutions to a few simple problems you can use conformal mapping to extend those to more complicated ones. And as in those cases, the most useful solutions often come from applying a mapping to a solved problem and then figuring out what harder and hopefully interesting problem you've just solved. In that spirit we end with the following example.

EXAMPLE **Flow Around a Complicated Obstacle**

Question: The figure below shows a horizontal flow in the upper half-plane going around a semicircular obstacle. From our calculations above we know that the stream function for this flow can be written as $\psi(u, v) = v[1 - 1/(u^2 + v^2)]$. (We are using u and v for our axes because we want to reserve x and y for the more complicated scenario we are going to map this to, and we are setting $k = 1$ for simplicity.) Use the mapping $z(f) = \sqrt{f+1}$ to map this to the stream function for flow around a differently shaped obstacle.



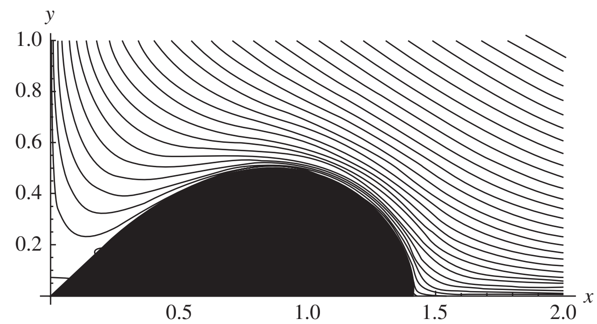
Answer:
There are a number of ways to plot the mapped region. The brute force method is to have a computer take a large array of points in the original region, apply the mapping, and plot their new coordinates. In Problem 13.145 you'll go through analytic calculations to find the mapped region for this problem. Either way it ends up looking like the figure below. The upper half-plane in uv space has mapped to the first quadrant in xy space, and the semicircular obstacle has mapped to a half-teardrop shape.

The process of finding the stream function $\psi(x, y)$ is the same for any conformal mapping problems.

1. *Find the mapping from xy space to uv space.* Since we were given the mapping the other way we need to invert it, giving $f(z) = z^2 - 1$.
2. *Find the real and imaginary parts of the mapping.* This just requires writing $f = u + iv$ and $z = x + iy$, which immediately gives $u(x, y) = x^2 - y^2 - 1$, $v(x, y) = 2xy$.
3. *Plug $u(x, y)$ and $v(x, y)$ into the original stream function.* This gives the final answer.

$$\psi(x, y) = 2xy\left[1 - \frac{1}{(x^2 - y^2 - 1)^2 + 4x^2y^2}\right]$$

The contour lines of this stream function are shown below. This represents flow around a corner with an obstacle.



This example may seem contrived, but the process it illustrates is useful for a variety of problems. In Problem 13.136 you'll use this method to find flow around a corner without a strangely shaped obstacle. In Problem 13.146 you'll use a similar transformation to find the flow around a shape called an "airfoil," which is used to simulate flow around airplane wings.

Stepping Back

The techniques we've used in this section apply to divergenceless, irrotational, laminar fluid flow in two dimensions. ("Laminar" means you can divide the flow into non-interacting streamlines.) Even for that restricted class of flows, there are techniques well beyond what we've covered in this section. In addition to the stream function $\psi(x, y)$ people often calculate a "velocity potential" $\phi(x, y)$ related to $\vec{v}$ by $\vec{\nabla}\phi = \vec{v}$. The functions ψ and ϕ are "harmonic duals," meaning they are the real and imaginary parts of an analytic function $\Omega = \phi + i\psi$, called the "complex potential" of the velocity field. Once you know ψ or ϕ you can find the other one, for example using the Cauchy-Riemann equations, and of course given either one you can find $\vec{v}$. You can use techniques from electrostatics such as the "method of images" to find potential functions for certain flows. Once you have solved a given problem you can reverse the roles of ψ and ϕ (since they are both harmonic) and have the solution to a different fluid flow problem. We don't discuss those methods here, but this section should give you a good introduction to techniques that can be used to solve for fluid flow in certain circumstances.[11]

13.10.1 Problems: Fluid Flow

In all the problems in this section you should assume fluid flow is divergenceless and irrotational unless otherwise specified.

13.134 In the ride "The Cyclone Zone"® at "Wet 'n Wild"® water park in North Carolina riders are carried around by water flowing in circles in a doughnut shaped region between two concentric circles of radius R_1 and R_2. Find a stream function and velocity function for the flow.

[11] For a longer discussion of the use of complex potentials for fluid flow see e.g. "Visual Complex Analysis" by Tristan Needham, Clarendon Press, Oxford, 1997. Chapter 11 introduces the complex potential and Chapter 12, "Fluid Flows and Harmonic Functions"…well, you can figure out what that discusses.

13.135 In the Explanation (Section 13.10.1) we solved for the flow around a circular obstacle, assuming the x-axis was a streamline. If we drop that assumption then any stream function that is constant on the surface of that circle represents a possible flow.

- **(a)** What common analytic function has a constant real part on a circle centered on the origin? Use your answer to find a simple possible stream function around a circular obstacle.
- **(b)** To get a more interesting stream function, add your answer to Part (a) to the stream function we derived in the Explanation. The result will be a new stream function that is harmonic and meets the right boundary conditions. For simplicity you can take $R = k = 1$.
- **(c)** Plot the contours of this new stream function. How is this flow similar to the one we derived in the Explanation and how is it different?

The moral of this story is that the flow around an obstacle can take many forms, depending on the boundary conditions away from the obstacle. That makes sense physically; the same rock in different streams will have different flows around it. To determine the flow for a given physical situation you always need to specify the boundary conditions. When we specified that the flow was horizontal and uniform far from the rock we were led to the unique solution Equation 13.10.3.

13.136 In the boxed example on Page 11 we found the stream function for flow around a corner with an obstacle.

- **(a)** Repeat the process to find the stream function for flow around a corner without the obstacle present.
- **(b)** Sketch the contour lines of the stream function (first quadrant only).

13.137 In the Explanation (Section 13.10.1) we solved for flow bounded by the horizontal axis and a circle centered on the origin.

- **(a)** Use the mapping $z(f) = f^{1/3}$ to map this stream function to a more complicated flow problem.
- **(b)** Find the velocity field $\vec{v}(x, y)$.
- **(c)** Sketch the region and the contour lines of $\psi(x, y)$.
- **(d)** Sketch the region and the vector field $\vec{v}(x, y)$.

13.138 Horizontal flow in the upper half of the uv-plane is described by the stream function $\psi = v$, which is the imaginary part of the analytic function $f = u + iv$. In this problem you're going to use the mapping $z(f) = \sqrt{f^2 - 1}$ to map this simple solution to a more interesting one.

- **(a)** What region in the xy-plane is the boundary $v = 0$ mapped to? In words, what fluid flow problem are you solving with this mapping?
- **(b)** Find the stream function $\psi(x, y)$. The following identity may help.

$$\sqrt{a + bi} = \frac{1}{\sqrt{2}}\sqrt{a + \sqrt{a^2 + b^2}} + \frac{i}{\sqrt{2}}\sqrt{-a + \sqrt{a^2 + b^2}}$$

- **(c)** Plot contours of this new stream function. This solution describes horizontal flow around a boundary. Describe the shape of that boundary.

13.139 In the Explanation (Section 13.10.1) we solved for flow around a circular barrier. Use the mapping $z(f) = (1/2)(3f + 1/f)$ to map this problem to a more complicated one. Sketch the mapped region, find the mapped stream function, and sketch its contours. *Nothing in this problem requires a computer, but the algebra and the sketching are kind of ugly without one.*

13.140 Assume a velocity field is irrotational ($\vec{\nabla} \times \vec{v} = \vec{0}$) and prove that the stream function $\psi(x, y)$ is harmonic.

13.141 We defined the stream function $\psi(z)$ as the flux through a curve drawn from a reference point z_0 to the point z.

- **(a)** Sketch a region with two points z_0 and z and draw two different curves connecting them. Use the divergence theorem to argue that the flux through the two curves must be the same if $\vec{\nabla} \cdot \vec{v} = 0$ everywhere in the region, and will not generally be the same otherwise.
- **(b)** Suppose you have defined a stream function $\psi(z)$ using a reference point z_0, but then you change your mind and decide to define a new stream function $\chi(z)$

using a different reference point z_1. Write an equation relating $\psi(z)$ and $\chi(z)$.

13.142 The Explanation (Section 13.10.1) described the relationship between the stream function ψ and the velocity field $\vec{v}$ in several ways that are not obviously connected to each other. In this problem you will draw the key connection by starting with Equation 13.10.2 and proving Equation 13.10.1 (up to the sign ambiguity mentioned in the Explanation).

(a) Explain why $\vec{\nabla}\psi$ must be perpendicular to $\vec{v}$.

(b) Next you need to relate the magnitude of $\vec{\nabla}\psi$ to the magnitude of $\vec{v}$. Consider two streamlines separated by a line segment of length ds. If the streamlines are close enough, that line segment can be perpendicular to both streamlines. Let $d\psi$ be the difference in the values of ψ between the two streamlines.

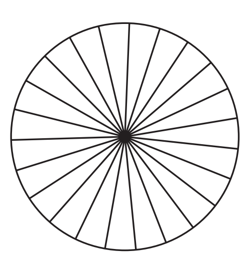
i. Express $d\psi$ in terms of ds and the magnitude of $\vec{\nabla}\psi$.

ii. Express $d\psi$ in terms of ds and the magnitude of $\vec{v}$.

iii. How are the magnitudes of $\vec{\nabla}\psi$ and $\vec{v}$ related?

(c) Write a vector that is perpendicular to $\vec{\nabla}\psi$ and has the same magnitude as $\vec{v}$. Your answer should only contain ψ, not $\vec{v}$. *Hint*: recall that two vectors $\vec{c}\hat{i} + d\hat{j}$ and $d\hat{i} - c\hat{j}$ are perpendicular to each other.

13.143 Water is falling onto the middle of a circular table and flowing out to the edge at radius R. One boundary condition is that the streamlines must be normal to the edge of the table. The other is that the streamlines must point directly outward from the point at the center. (If that's a "boundary condition" then what is the "boundary"? Imagine the water is coming straight out of a small circular area of radius r in the center, and then take the limit of your final answer as $r \to 0$.)



(a) Find a harmonic function that meets these boundary conditions and thus write the stream function for this flow. Your answer should have one undetermined constant in it.

(b) Find $\vec{v}$.

(c) Water has a density of 1 g/cm^3. If the layer of water on the table is 1 cm thick then its mass per unit *area* on the table is 1 g/cm^2. Assuming water is being added to the center at 5 kg/s, find the constant in your expression for $\vec{v}$.

13.144 *[This problem depends on Problem 13.143.]* In Problem 13.143 you found the fluid flow on a circular table with a source at the center.

(a) Verify that this flow is divergenceless everywhere except at the origin.

(b) Argue using the divergence theorem that the divergence cannot be zero at the origin.

Because the domain in which the flow is divergenceless isn't "simply connected" (meaning there's a hole in the middle of it), the stream function is not single-valued. To demonstrate this, consider the points $P = (1/2, 0)$ and $Q = (0, 1/2)$.

(c) Draw an arc going counterclockwise from P to Q. Find the flux through this arc.

(d) Draw an arc going clockwise from P to Q. Find the flux through this arc.

(e) Take P as the reference point for the stream function. Find ψ if the curve you draw from P to any point z is always a combination of a counterclockwise arc and a radial line segment.

(f) Using the same reference point, find ψ assuming each curve is a combination of a clockwise arc and a radial line segment.

(g) Show that these two stream functions describe the same flow. (Your two velocity functions may differ in sign because of the direction ambiguity of flux through a non-closed curve, but they should otherwise be identical.)

13.145 In the boxed example on Page 11 we used the mapping $z(f) = \sqrt{f+1}$ to map flow in the upper half-plane around a semicircular obstacle to a more complicated problem. In this problem you'll work out the mapping we used there.

(a) What region does the transformation $z = f + 1$ map the upper half-plane to? *Hint*: This is trivial if you think about it.

(b) Now apply the transformation $z = \sqrt{f}$ to the region you found in Part (a) to find the region that $z = \sqrt{f+1}$ maps the upper half-plane to. In answering this part you should assume you are using the principal branch of the square root function.

(c) The lower boundary of the semicircle is the line segment from $(-1, 0)$ to $(1, 0)$. What does this line segment get mapped to?

(d) The top boundary of the semicircle satisfies $u^2 + v^2 = 1$. Using the formulas for $u(x, y)$ and $v(x, y)$ we derived in the example, write an equation for the curve in xy space that defines the upper boundary of the new obstacle.

(e) Make a sketch of the curve you just found. You can do this with a computer or by hand, but if you do it by hand explain how you know the basic shape.

13.146 **Exploration: The Joukowski Airfoil** In the Explanation (Section 13.10.1) we argued that the function $z + R^2/z$ is real on a circle of radius R centered on the origin. An equivalent way of saying that is that the mapping $z(f) = f + R^2/f$ maps this circle to part of the x-axis. In 1908 Nikolai Joukowski applied this mapping to circles with other centers and found that they produced interesting shapes. You'll work with one such example here.

(a) Apply the mapping $z = f + 1/f$ to a circle that passes through the point $f = -1$ but is centered on $f = .1 + .2i$. Plot the resulting shape. This shape is called an "airfoil," and is often used to model airplane wings.[12]

(b) Invert the mapping to find $f(z)$. You should get two solutions. For now you'll just hold onto both of them.

For each of the two inverse functions $f(z)$ you found, have the computer define functions $u(x, y)$ and $v(x, y)$ and use them to define a function $\psi(x, y)$. *Don't copy this function down. You don't even have to print it on your screen. It's pretty ugly.*

(c) Make two plots showing the contours of $\psi(x, y)$ around the airfoil, one for each stream function you defined. You should find that each of them fits around the airfoil perfectly in some parts of the plot and not in others.

(d) Define a new function equal to the appropriate stream function in each part of the domain. Plot the contours of this piecewise function around the airfoil to see the flow. You should be able to see the stagnation point.

[12]See e.g. *Theoretical Aerodynamics, 4th ed.* by L. M. Milne-Thomson, Macmillan and Company, 1966.

13.11 Additional Problems

13.147 **(a)** Write the real and imaginary parts of $\ln z$ as functions $u(x, y)$ and $v(x, y)$ (where $z = x + iy$).

(b) Show that $\ln z$ is analytic by showing that u and v satisfy the Cauchy-Riemann equations. It helps to know that the derivative of $\tan^{-1} x$ is $1/(1 + x^2)$.

13.148 The function $f(z) = (\cos z)/(z - \pi/4)^2$ has a singularity at $z = \pi/4$.

(a) Find the residue at this singularity using the formula given in Section 13.4.

(b) Find the Laurent series for $f(z)$ about the point $z = \pi/4$. *Hint*: Start by writing $\cos z = \cos[(z - \pi/4) + \pi/4]$ and use the cosine addition rule.

(c) Is the singularity at $z = \pi/4$ removable, essential, or a pole? If it's a pole, what order is it? If it's a pole or an essential singularity, what is the residue? Explain how you know the answers based on the Laurent series.

(d) Find the contour integral of $f(z)$ around a unit circle centered on the origin.

13.149 The multi valued function $f(z) = \sqrt{1 + z^2}$ can be made single-valued with two branch cuts. The starting point is to rewrite $f(z)$ as $\sqrt{z - i}\sqrt{z + i}$.

(a) First consider $\sqrt{z - i}$. Where is the branch cut for this function (using the principal phase of $z - i$)?

(b) Where is the branch cut for $\sqrt{z + i}$?

(c) Draw a z-plane with a small circle around $z = i$. Sketch the curve this circle maps to on the f-plane. What happens to $f(z)$ as the z circle crosses a branch cut?

13.150 *[This problem depends on Problem 13.149.]* In this problem you will make plots of mappings of the function $f(z) = \sqrt{1 + z^2}$ using the branch that has two branch cuts, as discussed in Problem 13.149.

(a) Have a computer make a z-plane with the circle $|z| = 2$ and an f-plane with the curves this circle maps to.

(b) What are the values of f just before and after crossing each branch cut in this mapping?

(c) Repeat Parts (a)–(b) for the circle $|z - i| = 1$. The curve will only cross one branch cut this time.

13.151 In this problem you will define $\sqrt{z - i}$ and $\sqrt{z + i}$ with branch cuts extending vertically downward from their poles. For example, $\sqrt{z - i}$ will have a branch cut going from $z = i$ to $z \to -i\infty$. Your goal is to find the branch cut(s) for the function $f(z) = \sqrt{1 + z^2} = \sqrt{(z - i)(z + i)}$.

(a) Start by drawing a complex plane. Throughout the problem you are going to mark points z and $f(z)$ on it.

(b) For $z_1 = 2i$ find the phases of $\sqrt{z + i}$, $\sqrt{z - i}$, and $f(z)$. If any of these phases is ambiguous due to the branch cut, find instead the phases of $z_1 + \epsilon$ and $z_1 - \epsilon$ where ϵ is a small positive real number. Mark and label the points z_1 and $f(z_1)$ on your plot. (If it lands at different places depending on whether you add or subtract ϵ then mark them both.)

(c) Repeat Part (b) for $z_2 = i/2$.

(d) Repeat Part (b) for $z_3 = -i/2$.

(e) Repeat Part (b) for $z_4 = -2i$.

(f) A branch cut for a function $f(z)$ is a line where the position of $f(z)$ is different depending on which side you approach that line from. Looking at the plot you made, where is the branch cut for this $f(z)$?

13.152 Let $f(z) = 1/(1 + z)$.

(a) Let $z = x + iy$ and $f = u + iv$. Find $u(x, y)$ and $v(x, y)$.

(b) What is the (constant) value of $u(x, y)$ on the unit circle $x^2 + y^2 = 1$ (not counting the pole at $(-1, 0)$)?

(c) What is the value of $u(x, y)$ in the limit $|z| \to \infty$?

(d) Write a boundary value Laplace's equation problem that $u(x, y)$ is the solution to.

13.153 Evaluate the following (real-valued) integral.

$$R = \int_0^\infty \frac{1}{x^k(x - x_0)}\,dx \text{ where } 0 < k < 1,\ x_0 > 0$$

Hint: Your final answer will probably look complex. You should be able to simplify it by replacing x_0 with $|x_0|e^{i\pi}$, but don't do this until you get to the final answer.

13.154 Let $f(z) = 1/\sin z$.

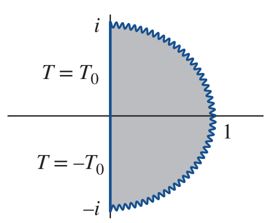
(a) What is the order of the pole at $z = 0$?

(b) Calculate the residue of f at $z = 0$ using the formula on Page 730.

(c) Find the first three terms of the Laurent series for $f(z)$ about $z = 0$.

(d) Use your Laurent series to find the residue at $z = 0$ and confirm that you get the same answer you did in Part (b).

13.155 The picture below shows half the unit disk in the uv-plane. The straight boundary is split in two, with a temperature $T = -T_0$ on the bottom half and $T = T_0$ on the top. The curved boundary is insulating, meaning the derivative of T normal to that boundary is zero.



(a) The steady-state temperature in the region obeys Laplace's equation. Find that steady-state temperature $T(u, v)$.

(b) The transformation $z(f) = e^f$ maps this half-disk to a different region. Make a rough sketch of that region. Mark on your mapped region where the $T = -T_0$ boundary, the $T = T_0$ boundary, and the insulating boundary from the original region mapped to.

(c) Solve Laplace's equation in the mapped region subject to the boundary conditions you just added to your sketch.

13.156 An infinite wire is at potential $V = V_L$. A circle of radius R is tangent to that wire, and is held at $V = V_C$. Define a useful set of axes for this problem, find a Möbius transformation that maps the circle and line to two parallel lines, and use that mapping to solve Laplace's equation for the potential in the region bounded by the circle and the line. (That region is shaped like a half plane with a disk cut out of it.)

13.157 Sometimes the best way to solve a difficult problem is to find an easier problem that's similar enough that it gives a good approximate answer. In this problem you'll solve for the steady-state temperature in a rectangle stretching from the origin to the point $(\pi/2, 5)$, with the left edge fixed at $T = 1$ and the bottom and right edges fixed at $T = 0$. The top edge is insulating, which means the derivative of T normal to that boundary is zero.[13]

(a) Write $f(z) = \sin(x + iy)$ in the form $f = u(x, y) + iv(x, y)$. This is a bit of a mess, but you can do it by using Euler's formula $e^{iz} = \cos z + i \sin z$ and the related formula $e^{-iz} = \cos z - i \sin z$.

(b) Use the function $\sin z$ to map the left, bottom, and right boundaries of the rectangle to the uv-plane. You should find that they all map to line segments.

(c) Find the curve that the top boundary maps to. You should be able to simplify its formula to the form $f = p \sin a + iq \cos a$, where p and q are real constants and a is a parameter that varies along the curve. Find the numbers p and q and specify the range of values for a.

If p equaled q this curve would be an arc (part of a circle). If that were the case it would be relatively easy to solve Laplace's equation to find $T(u, v)$ in the mapped region. Since $p \neq q$ it's actually an ellipse, and it's not easy to find $T(u, v)$. However, since $p \approx q$, you can get a good approximation by pretending that it's an arc.

(a) Assuming the curved boundary is an arc, write the solution $T(u, v)$ that solves Laplace's equation and meets all the boundary conditions in the mapped region.

(b) Use the functions u and v to map your solution $T(u, v)$ back to the xy-plane. You should get an answer in the form $T(x, y)$.

(c) Your answer should correctly solve Laplace's equation and meet the left, bottom, and right boundary conditions. It will *not* meet the boundary condition at the top, which is $\partial T/\partial y = 0$. Calculate $\partial T/\partial y$ at the point $(\pi/4, 5)$ and verify that it is not zero, but is very close. As a point of comparison calculate $\partial T/\partial y$ at the point $(\pi/4, 1)$.

13.158 **Exploration: The Uniqueness Theorem for Laplace's Equation**

Throughout this chapter you've been solving the Dirichlet problem for Laplace's equation: you are given the value of a

[13]If you worked Section 13.9 Problem 13.131 you may recognize this as a finite version of the same problem. As it often does, making the domain of the problem finite makes it harder.

function $V(x, y)$ on the boundaries of a region and asked to find a harmonic function $V(x, y)$ that meets those boundary conditions. In this problem you will prove that each such problem has a unique solution. First you'll prove another remarkable property of harmonic functions: the average value of a harmonic function $V(x, y)$ on a circle centered on the point (x_0, y_0) equals $V(x_0, y_0)$.

(a) Consider an analytic function $g(z)$. In Cauchy's integral formula, Equation 13.4.1, let the contour be a circle of radius R around centered on z_0. Such a circle can be described as $z = z_0 + Re^{i\phi}$. By rewriting the integral in that form, show that the average value of $g(z)$ on this circle equals $g(z_0)$.

(b) Use your result from Part (a) to argue that the average value of a harmonic function $V(x, y)$ on a circle of radius R must equal the value of V at the center of that circle.

(c) Use your result from Part (b) to argue that a function that is harmonic everywhere in a closed region cannot take on a global maximum or minimum value anywhere in the interior of that region unless it also takes on that value on the boundary.

(d) Suppose you knew the values of a function $V(x, y)$ everywhere on the boundary of a region and you found two harmonic solutions V_1 and V_2 satisfying those boundary conditions. Prove that $V_1 - V_2 = 0$. That completes the proof that the solution to a boundary-value Laplace's equation problem must be unique.

A corollary of this result is Liouville's Theorem, which says that an entire function whose magnitude is bounded must be constant. We will leave it to you to think about how that follows from what you derived in this problem.

APPENDIX M

Answers to Odd-Numbered Problems

Chapter 1

1.1(a) $e^{\sin} x \times \cos x$
1.3 (b), (c)
1.5 (b), (d), (e)
1.7 (a), (c), (e)
1.9 All of them work
1.11 (a), (d), (e)
1.13 (d)
1.15 (c)
1.17 (a) I. (b) V. (c) II. (d) III. (e) IV.
1.19(a) $d^2x/dt^2 = -g$
(b) $m(d^2x/dt^2) = -k|x|/x^3$
(c) **(ii)** $m(d^2x/dt^2) = k - b(dx/dt)$
(d) $m(d^2x/dt^2) = (b - rt) - k$
1.21(a) $dM/dt = 30 + M/10$
(c) D is in dollars, k in years
1.23(d) $dM/dt = S_0 1.05^t - E_0 1.02^t + .04M$
1.25(a) **(ii)** negative
(b) $dm/dt = -k^2 m^{2/3}$
(d) **(ii)** 125
1.27(a) 0
(b) -4
(c) 2
1.29(a) $Q = 0$ and $Q = 5$
1.31(a) **(ii)** e^t or e^{-t}
(b) **(ii)** $\cos t$ or $\sin t$
(c) **(ii)** e^{3t} or e^{-3t}
(d) **(ii)** $x = \cos(3t)$ or $x = \sin(3t)$
1.33(b) 2π
(c) **(ii)** phase
(d) **(ii)** $x = \sin(2t)$ or $x = \cos(2t)$, Period = π
(e) **(ii)** $2\pi/\omega$
(f) **(ii)** $\pi/5$ seconds
1.35(a) $M(t) = (\$1.05)^t M_0$
(b) $M(1) = (\$1.005)^{10} M_0$, $M(t) = (\$1.005)^{10t} M_0$
(c) **(ii)** $M = M_0 e^{.05t}$
(e) **(ii)** $R = 15e^{10t}$
1.37(a) $-e^{-x}$
(e) **(ii)** $y = 3$
(f) **(ii)** $y = -2e^{-x} + 3$

1.39(b) **(ii)** $P(t) = 7e^{-10t}$
1.41 $y = -1/(2e^{\sqrt{x}} - 11/5)$
1.43 $y = (x - 3/2)(x^2 + 1)$
1.45 (b) $u(t) = t^3 + 4e^{2t} + e^{-2t}$
1.47(d) $C = 0$
(e) $C = (1/2)\ln 5$
1.49(b) **(ii)** $f(z) = 4e^{z/2} + 5ze^{z/2}$
1.51(a) $dx/dt = k$
(b) $x = kt + C$
1.53(a) **(ii)** $dS/dt = -10$
(b) **(ii)** $S(t) = -10t + 30$
(c) **(ii)** $dS/dt = -S/3$
(d) **(ii)** $S(t) = 30e^{-(1/3)t}$
1.55(a) $dS/dt = 2 - .02S$
(c) $S(t) = Ce^{-.02t} + 100$
(d) **(ii)** $S(t) = 100(e^{-.02t} + 1)$
(e) **(ii)** $\lim_{t\to\infty} S(t) = 100$
1.57(a) **(ii)** $A = 1/2$ and $B = -1/2$
(b) **(ii)** $C = 1$ and $D = 0$
1.59(a) $dx/dt = 55$
(b) $x(t) = 55t + C$
(c) **(ii)** $x(t) = 55t - 5$
1.61(a) **(ii)** positive
(b) **(ii)** kg/s
(c) **(ii)** $m(dv/dt) = -bv$
(d) **(ii)** $v(t) = C_1 e^{-(b/m)t}$
(e) $dx/dt = C_1 e^{-(b/m)t}$
(f) $x(t) = -(m/b)C_1 e^{-(b/m)t} + C_2$
(g) **(ii)** $x(t) = (m/b)v_0(1 - e^{-(b/m)t}) + x_0$
1.63 $C = \sqrt{x_0^2 + v_0^2/\omega^2}$ and $\phi = \tan^{-1}\left[-v_0/(\omega x_0)\right]$.
1.65(e) 0
1.73 (c) $y = -2$ stable
1.75 (c) no equilibrium solutions
1.79(e) Unstable
(f) Stable
1.81(a) $m = mx + b - 2x$
(b) $2x + m = mx + b$
(c) $m = 2$ and $b = m = 2$
1.83(b) $S = 0$ and $S = P$
(d) P
1.85(a) $dV/dt = 3V - 12000$
(c) $V = 4000$
(d) **(ii)** unstable
1.87(a) **(ii)** $dQ/dt = k(Q_r - Q)$
(c) $dQ/dt = t + 20 - Q$
(e) **(ii)** $Q = t + 19$
1.89 **(ii)** $y = x^2 + C$
1.91 **(ii)** $y = Ae^{x^2/2}$
1.93 **(ii)** $z = Ae^{-5t} + 8/5$
1.95 **(ii)** $x = \pm\sqrt{Ae^t - 1}$

1.97 **(ii)** $r = \left[\ln\left(\frac{1}{4}\sin(2\theta) + C\right)\right]^2$

1.99 **(ii)** $s = (At^{3t}/e^{3t} - 3)^{1/3}$

1.101 **(ii)** $y = 3/(3\cos\theta - \cos^3\theta + C)$

1.103(a) For $R(0) = 10$, $R = -10e^{5t} + 20$. For $R(0) = 20$, $R = 20$. For $R(0) = 30$, $R = 10e^{5t} + 20$.

(d) $R = 20$; unstable

1.105(a) $dv/dt = -9.8 - 2v$

(b) $v = -4.9$

(c) $v(t) = C_1 e^{-2t} - 4.9$

(d) **(ii)** -4.9

(e) $x(t) = -(C_1/2)e^{-2t} - 4.9t + C_2$

(f) **(ii)** $h(3) = 2988$ m

1.107(a) **(ii)** $a = (3\beta\sqrt{k}\,t/2 + C)^{2/3}$

(b) **(ii)** $a = \sqrt{2\beta\sqrt{k}\,t + C}$

(c) **(ii)** $a = Ae^{\beta\sqrt{\rho}\,t}$

1.109(d) **(ii)** $y(x) = Ce^{x^2/2+x}$

1.111 **(ii)** $y = Ae^{3x} + Be^{4x}$

1.113 **(ii)** $s = Ae^{-(5+\sqrt{41})t/2} + Bte^{-(5-\sqrt{41})t/2}$

1.115 **(ii)** $y(x) = A\sin\left(\sqrt{k^2+p^2}\,x\right) + B\cos\left(\sqrt{k^2+p^2}\,x\right)$

1.117 **(ii)** $y(x) = A\sin x + B\cos x + Ce^x + De^{-x}$

1.119 **(ii)** $u(x) = Ax^{-4} + Bx - 1$

1.121(e) $y = Ae^{3x} + Bxe^{3x} + 1/3$

1.123(d) $y = Ae^x + 2x + 2$

1.125(b) **(ii)** $y = (1/4)x^2$

1.127(a) $ap(p-1)(t-b)^{p-2} = -GM_E(t-b)^{-2p}/a^2$

(b) **(ii)** $r(t) = (9GM_E/2)^{1/3}(t-b)^{2/3}$

(c) Yes, and no

1.129(a) **(ii)** The equation is linear and homogeneous and all three solutions work.

(b) **(ii)** The equation is linear and inhomogeneous; y_2 and $y_1 + y_2$ work, but y_1 does not.

(c) **(ii)** The equation is non-linear. The solutions y_1 and y_2 work but their sum does not.

1.133(a) $y'y$

1.135 **(ii)** solution

1.137 **(ii)** solution

1.139 **(ii)** Not a solution

1.141(a) $dO/dt = k_1(O+E)$, $dE/dt = k_2(O+E)$

(b) $dO/dt = -k_1E$, $dE/dt = -k_2O$

(c) $dO/dt = k_1(O-E)$, $dE/dt = k_2(E-O)$

(d) $dO/dt = k_1E$, $dE/dt = -k_2O$

1.143(a) $dG_A/dt = r(1 - G_A/V_A)$, $dG_B/dt = r(G_A/V_A - G_B/V_B)$

1.145(a) Spring 2 pulls m_1 to the right. Springs 2 and 3 pull m_2 to the left.

(b) **(iv)** $x_1''(t) = -(k_1/m_1)x_1 + (k_2/m_1)(x_2 - x_1)$

(c) $x_2''(t) = (k_2/m_2)(x_1 - x_2) - (k_3/m_2)x_2$

1.147(a) $GM/(x^2+y^2)$

(b) **(ii)** $a_x = -GMx/(x^2+y^2)^{3/2}$, $a_y = -GMy/(x^2+y^2)^{3/2}$

(c) $d^2x/dt^2 = -GMx/(x^2+y^2)^{3/2}$, $d^2y/dt^2 = -GMy/(x^2+y^2)^{3/2}$

(d) **(ii)** Circle

1.149 **(ii)** $f(x) = Ae^{\sqrt{18}x} + Be^{\sqrt{18}x}$, $g(x) = A\sqrt{18}e^{\sqrt{18}x}/3 - B\sqrt{18}e^{\sqrt{18}x}/3$

1.151 **(ii)** $f(x) = Ae^{k_1 x} + Be^{k_2 x}$, $g(x) = (A/b)(k_1 - a)e^{k_1 x} + (B/b)(k_2 - a)e^{k_2 x}$ where $k_{1,2} = (1/2)\left(a + d \pm \sqrt{a^2 - 2ad + d^2 + 4bc}\right)$.

1.153(a) $T_A' = k_A(T_B - T_A)$, $T_B' = k_B(T_A - T_B)$

(d) **(ii)** $T_A = C_1 e^{-2(k_A+k_B)t} + C_2$, $T_B = -C_1\left(2k_B/k_A + 1\right)e^{-2(k_A+k_B)t} + C_2$

1.155(a) $T_A' = k_A(T_B - T_A)$, $T_B' = k_B(T_A + T_R - 2T_B)$

(c) **(ii)** $T_A = T_R + C_1 e^{-p_1 t} + C_2 e^{-p_2 t}$ where $p_{1,2} = (1/2)\left(k_A + 2k_B \pm \sqrt{k_A^2 + 4k_B^2}\right)$

1.157 **(ii)** $R\left(d^2R/dt^2\right) - (dR/dt)^2 - \left(\gamma R^2 - \delta R\right)(dR/dt) + \alpha\gamma R^3 - \alpha\delta R^2 = 0$

1.165 $x = 1.3$

1.169(c) **(ii)** $y = n\pi$ where n is any integer

1.173(a) $y(x) = C_1 J_0(k\rho) + C_2 Y_0(k\rho)$

(b) $y(x) = C_1 J_0(k\rho)$

(c) $k = 24.0\ \text{m}^{-1}$, $k = 55.2\ \text{m}^{-1}$, and $k = 86.6\ \text{m}^{-1}$

(d) $f = 383$ Hz, $f = 879$ Hz, and $f = 1377$ Hz

1.175(a) $\psi(r) = Aj_l(r) + By_l(r)$ where j_l and y_l are “spherical Bessel functions” of the first and second kind respectively.

(b) $\psi(r) = Aj_l(r)$

(c) For $l = 0$, $r = 3.14$. For $l = 1$, $r = 4.49$. For $l = 2$, $r = 5.76$.

1.177(c) $y'(1/3) = 4/3$

(d) **(ii)** $16/9$

(e) $y'(2/3) = 16/9$

(f) **(ii)** $64/27$

(h) **(ii)** $y(1) = e$

1.179 (a) IV. (b) III. (c) V. (d) I. (e) II.

1.181 **(ii)** $r = (\sin\theta + 3\cos\theta)/10 + (7/10)e^{-3\theta}$

1.183 **(ii)** $y = 10^{x/2+1} - x$

1.185 **(ii)** $V = x^3 + (4/15)\left((\ln 2)^3 - 1\right)\left(e^{-2x} - e^{2x}\right)$

1.187 **(ii)** $y_{general}(t) = Ae^{kt}$, $y_{specific}(t) = pe^{kt}$

1.189 **(ii)** $u_{general}(x) = Ae^x - 3$, $u_{specific}(t) = 3e^x - 3$

1.191 **(ii)** $f_{general}(x) = Ae^{2x} + Be^{3x}$, $f_{specific}(x) = (e^{2x} - e^{3x})/(e^2 - e^3)$

1.193(a) $dP/dt = 120P - 12000$

(b) **(ii)** $P = 100$; unstable

(c) **(ii)** $P = 25e^{120t} + 100$

1.195(a) **(ii)** $y(x) = -\ln(C - e^x)$

(b) **(ii)** $y(x) = -\ln(e^2 - e^x + 1)$

1.201 $B(10) = 200e^{10} + 1000$

1.203(b) $dv/dt = -kv^2/m$

(c)

(d) The limit is $v = 0$ in both cases.

(e) **(ii)** $v(t) = 1/\left[(k/m)t + (1/v_0)\right]$

(f) **(ii)** $t_{1/10} = 9m/(kv_0)$

(g) **(ii)** $x(t) = (m/k)\ln(v_0 kt/m + 1)$

1.205(a) $m(t) = m_0 - kt$

(b) **(ii)** $d^2x/dt^2 = F/(m_0 - kt)$

(c) **(ii)** $x(t) = (F/k^2)(m_0 - kt)\ln(m_0 - kt) + C_1 t + C_2$

(d) **(ii)** $x(t) = \left(\dfrac{Fm_0}{k^2} - \dfrac{F}{k}t\right)\ln\left(\dfrac{m_0 - kt}{m_0}\right) + \dfrac{F}{k}t$

(e) **(ii)** $v(50\text{ s}) = 7.6 \times 10^4$m/s, $x(50\text{ s}) = 1.6 \times 10^6$m.

1.207 $P = C(T_0 - kz)^{mg/(Rk)}$

1.209(b) 0
(c) $M(t) = M_0 e^{-kt}$
(d) **(ii)** $t_{1/2} = (\ln 2)/k$
(e) **(ii)** 22 days
1.211(b) **(ii)** $ds/dt = -5$ and $dp/dt = 5$
(c) $dp/dt = k(f_0 - p)(s_0 - p)$
(d) **(ii)** $p(t) = f_0 - 1/(kt + C)$
(e) **(ii)** $p(t) = \dfrac{f_0 Ce^{k(s_0 - f_0)t} - s_0}{Ce^{k(s_0 - f_0)t} - 1}$
(f) **(ii)** f_0 for $s_0 > f_0$ and $s_0 = f_0$, s_0 for $s_0 < f_0$.
1.213(a) **(ii)** $P(t) = Ce^{2t}/(Ce^{2t} + 1)$
(b) $P(t) = e^{2t}/(7 + e^{2t})$
(c) 1

Chapter 2

2.1(a) 20.5
(b) 19.5
(c) 23
(d) None of the above
2.3(b) Yes
(c) Low
(d) $g(x)$
2.5 60.3°C
2.7(a) $f(25) = 5$
(b) $1/10$
(c) **(i)** $1/10$
(ii) y decreases by $1/10$
(iii) $(1/10)\Delta x$
(iv) $\sqrt{23} \approx 4.8$
(d) $g(x) = 0.1x + 2.5$
(e) $f(23) \approx 4.8$
2.9 8
2.11 0.2
2.13 0.1
2.15 $3/e$
2.17(a) $22.5e^4 - 16e^4 = 6.5e^4 \approx 354.89$
2.19 $y = -0.75x + 6.25$
2.21(a) $\sin\theta \approx \theta$
(b) $d^2\theta/dt^2 + (g/L)\theta = 0.$
(c) $\theta(t) = C\cos\left(\sqrt{g/L}\,t\right) + D\sin\left(\sqrt{g/L}\,t\right)$
(e) Values near θ=0
2.23(a) Zero. ϕ will stay constant for a while.
(b) Decrease
(c) Increase
(e) $-\lambda v^2\phi + \lambda v^3$
(f) $\phi(t) = C\sin(\sqrt{\lambda}vt) + D\cos(\sqrt{\lambda}vt) + v$
(g) $t = 2\pi/\left(\sqrt{\lambda}v\right)$
2.25(a) $y = -e^{-t} + 1$
(c) $y = \ln 2$
(d) $-2y + 2\ln 2$
(e) $y = -\ln 2e^{-2t} + \ln 2$
2.27 $f(b) \approx f(a) + (b - a)f'(a)$

2.29(a) $4096a + 64b + c = 8$

(b) $128a + b = 1/16$

(c) $2a = -1/2048$

(d) $y = -(1/4096)x^2 + (3/32)x + 3$

(f) 0.071%

(g) 0.00274%

(h) 0.664%

2.31

Equation	Plug in $x = 0$
$e^x = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \ldots$	$c_0 = 1$
$e^x = c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + \ldots$	$c_1 = 1$
$e^x = 2c_2 + 6c_3 x + 12c_4 x^2 + \ldots$	$c_2 = 1/2$
$e^x = 6c_3 + 24c_4 x + \ldots$	$c_3 = 1/6$
$e^x = 24c_4 + \ldots$	$c_4 = 1/24$

2.33 $1 - x + x^2/2$

2.35 $1 + x - x^2/2$

2.37 $1 - x^2/2 - x^3$

2.39 $e + ex + ex^2$

2.41 $x + x^3/3 + 2x^5/15$

2.47(a) $1 + kx + k(k-1)x^2/2 + k(k-1)(k-2)x^3/6 + k(k-1)(k-2)(k-3)x^4/24$

(b) $k(k-1)(k-2)(k-3)...(k-n+1)x^n/n!$

(c) $k!/(k-n)!x^n/n!$

2.49(a) $1 + x^2/2$

(b) $E = mc^2 + mv^2/2$

2.51(a) R_E

(b) $R_E - gx^2/(2v_0^2)$

(c) $y(x) = R_E - g/(2v_0^2)x^2$

2.53(a) $1 - x^2 + x^4/3 - 2x^6/45 + x^8/315$

(b) $\cos^2 x = 1 + \sum_{n=1}^{\infty}((-1)^n 2^{2n-1} x^{2n})/(2n)!$

(c) $\sum_{n=1}^{\infty}((-1)^{n-1} 2^{2n-1} x^{2n})/(2n)!$

2.55(a) $c_0 = 0$

(b) $-2\sin(2x) = c_1 + 2c_2(x - \pi/4) + 3c_3(x - \pi/4)^2 + 4c_4(x - \pi/4)^3 + \ldots$

(c) $c_1 = -2$

(d) $c_2 = 0,\ c_3 = 4/3,\ c_4 = 0$

(e) $-2(x - \pi/4) + (4/3)(x - \pi/4)^3$

(f) $\cos(2) = -0.416147$ and the fourth-order approximation is -0.416026, so they match to three decimal places.

2.57(a) $\sum_{n=0}^{\infty}(-1)^n(1/2)(x-\pi/3)^{2n}/(2n)! + (-1)^{n+1}(\sqrt{3}/2)(x-\pi/3)^{2n+1}/(2n+1)!$

(b) $\cos(1) \approx 0.540317$

(c) 0.0028%

2.59 $-2\sqrt{\pi}(x-\sqrt{\pi}) - (x-\sqrt{\pi})^2 + (4/3)\pi^{3/2}(x-\sqrt{\pi})^3$

2.61 $(5/2) + (3/2)(x-\ln 2) + (5/4)(x-\ln 2)^2 + (1/4)(x-\ln 2)^3$

2.63 $(x-\pi/2)^2$

2.65 $-1-(1/2)(x-\pi)^2$

2.67 $(1/2\pi)(x-2\pi) - (1/4\pi^2)(x-2\pi)^2 + [(3-2\pi^2)/24\pi^3](x-2\pi)^3$

2.69(b) Relative minimum.

(c) $g(x) = 5 + (x+2)^3/12 + (x+2)^4/36$

2.71(b) $V(r) \approx Db^2(r-r_0)^2$

(c) $1.326 \times 10^{14}\ \mathrm{s}^{-1}$

2.73(b) within 18%

2.75 $\displaystyle\sum_{n=0}^{\infty}(-1)^n\left(\frac{x^{2n+1}}{(2n+1)!} + \frac{x^{2n}}{(2n)!}\right) = 1 + x - \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$

2.77 $\displaystyle\sum_{n=0}^{\infty}\frac{x^{2n+1}}{n!} = x + x^3 + \frac{x^5}{2!} + \frac{x^7}{3!} + \frac{x^9}{4!} + \ldots$

2.79 $\displaystyle\sum_{n=0}^{\infty}(-x)^n/2^{n+1} = 1/2 - x/4 + x^2/8 - x^3/16 + x^4/32\ldots$

2.81 $2 + x^2/4 - x^4/64 + x^6/512 - 5x^8/16384$

2.83 $(1+x)^2 = 1 + 2x + x^2$

2.85 $x - x^3/3 + x^5/10$

2.87 $1 + x + x^2/2 - x^4/8 - x^5/15$

2.89(a) $\displaystyle\ln x = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{(x-1)^n}{n}$

(c) $\displaystyle\ln x = \sum_{n=1}^{\infty}2(-1)^{n+1}\frac{(x-1)^n}{n}$

2.91(a) $\displaystyle\frac{d^2x}{dt^2} = \frac{\kappa}{md^2}\left(\frac{1}{(1+(x/d))^2} - \frac{1}{(1-(x/d))^2}\right)$

(b) $\displaystyle\frac{d^2x}{dt^2} = -\frac{4\kappa}{md^3}x$

(c) $T = \pi\sqrt{md^3/\kappa}$

2.93(a) $\displaystyle\sum_{0}^{\infty}(-1)^n x^{2n} = 1 - x^2 + x^4 - x^6 + \ldots$

(b) $\displaystyle\tan^{-1}x = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$

(c) $\displaystyle\tan^{-1}(1) = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots = \frac{\pi}{4}$

(d) $\pi \approx 3.340$

2.95(a) $S_{22} = 1$

(b) This series diverges.

2.97(a) 10^{19}

(b) $\left(10^{20}-1\right)/9$

(c) $\sum_{n=1}^{20} 10^{n-1}$

2.99(a) $1/5$

(b) $100 \times (1/5)^{99}$

(c) 125

(d) $\sum_{n=1}^{100} 100 \times (1/5)^{n-1}$

2.101(a) $S_{10} \approx 175.312$

2.103(a) $\sum_{n=1}^{\infty} 2(1.1)^{n-1}$

(b) 12.2102

(c) 114.54999

(d) $-20 + 20(1.1)^n$

(e) $+\infty$

2.107(a) No guarantee

(b) $\ln(1/4)$

(c) $-\ln(n+1)$

(d) Diverges

2.111(a) $3/10$, $3/100$, $3/1000$

(b) $1/10$

(c) $1/3$

2.113 $0.\overline{9} = 1$

2.115(a) $2^{63} = 9,223,372,036,854,775,808$

(b) $2^{64} - 1 = 18,446,744,073,709,551,615$

2.117(a) 19

(b) $1,179,648,000$

(c) $2,359,291,500$

(d) $\approx 2.669 \times 10^{10}$

2.119(a) 112

(b) $\sum_{n=0}^{3} 2(2n)^2$

(c) 34%

(d) $\sum_{n=1}^{6} (1/2)(\ln((n+1)/2) = 2.183$

(e) $\sum_{n=1}^{50} 1.2^{n-1} \approx 45,497.191$

2.121(a) $\sum (1/3)(3/4)^n$

(b) $a_1 = 1/4$, and $r = 3/4$

(c) It converges to 1

2.123(a) $2 \times 3 \times 4 \times 5 \times 6 = 720$

(b) $\prod_{n=0}^{3} 2(3^n)$

(c) $\ln\left(\prod_{n=1}^{50} n\right)$

2.125(a) $\$24,290$

(b) $\$70,429$

2.127 1.0815

2.129 converges for $-1/2 < x < 1/2$

2.133 Convergent

2.135 Divergent

2.137 Inconclusive

2.139 Inconclusive

2.141 Converges for all x

2.143 Converges for $5/2 < x < 7/2$

2.145 Converges for all x.

2.149 Converges

2.151 Converges

2.153 Diverges

2.155 Converges

2.157 Converges

2.159 Diverges

2.161 Converges

2.163 Absolutely converges

2.165 Diverges by the n^{th}-term test

2.167 Diverges

2.169 Converges

2.171 Converges

2.173 Converges

2.175 diverges

2.177 $-2 \le x \le 0$

2.181(b) $\sum_{n=1}^{4} f(n)$

2.187(a) $\int_x^{\infty} e^{-t^3}\,dt = \left(\frac{1}{3x^2} - \frac{2}{9x^5}\right)e^{-x^3} + \frac{10}{9}\int_x^{\infty} \frac{1}{t^6} e^{-t^3}\,dt$

(b) approaching ∞

2.189(a) $\operatorname{erfc} x = \frac{e^{-x^2}}{\sqrt{\pi}\,x}\left(1 - \frac{1}{2x^2}\right) + \frac{3}{2\sqrt{\pi}}\int_x^{\infty} \frac{1}{t^4} e^{-t^2}\,dt$

(b) $a_{n+1} = \frac{c_n e^{-x^2}}{2x^{2n+1}}$, $R_{n+1} = -\frac{c_n(2n+1)}{2}\int_x^{\infty} \frac{e^{-t^2}}{t^{2(n+1)}}\,dt$

(c) $|a_{n+2}/a_{n+1}| = (2n+1)/(2x^2)$

(d) greater when $n > x^2 - 1/2$

2.193 $1 + x + (3/2)x^2 + (1/6)x^3 + (1/24)x^4$

2.195 $1 - x + (3/2)x^2 - (11/6)x^3 + (19/8)x^4$

2.197 $2\ln 2 + (x-2) - (1/4)(x-2)^2 + (1/12)(x-2)^3 - (1/32)(x-2)^4$

2.199 $-(x-\pi/2) - (x-\pi/2)^2 - (4/3)(x-\pi/2)^3 - (5/3)(x-\pi/2)^4$

2.201 $x^3 - (x^9/6) + (x^{15}/120)$

2.203 $\ln 5 + x$

2.209 $f(x) = A\sin x + B\cos x$

2.211 $x(t) = A\sin t + B\cos t + 1$

2.213 $x(t) = A\sin(\sqrt{k}t) + B\cos(\sqrt{k}t)$

2.215 Diverges

2.217 Converges

2.219 Diverges

2.221 Converges

2.223 Converges

2.225 $-1 < x < 1$

2.227 All x

2.229 All x

2.231 $0 \le x \le 2$

2.233 $f(0.1) = 2.1 \pm 0.016$

2.235(a) $\sin(1.5) \approx 0.9375$

(b) $\sin(1.5) \approx 0.997494$

2.237 $f(0.1) = 0.91 \pm .001$

2.239(a) $2x + 1$

(b) $x(t) = Ce^{2t} - 1/2$

(c) no

2.241(a) $\dfrac{E}{kq/x^2} = \dfrac{1}{(1-d/x)^2} - \dfrac{1}{(1+d/x)^2}$

(b) $4(d/x) + 8(d/x)^3$

2.243(a) $F = \dfrac{\lambda^2}{4\pi\epsilon_0}\ln\left(\dfrac{(L/D+1)^2}{(2(L/D)+1)}\right)$

(b) $F = \dfrac{\lambda^2}{4\pi\epsilon_0}(L/D)^2$

(c) $F = Q^2/(4\pi\epsilon_0 D^2)$

2.245 $B(T) = b - A/RT$, $C(T) = b^2$

2.247(a) $c_0 = 0$

(b) $c_1 = 1$

(c) $2c_2 + (2\times 3)c_3 x + (3\times 4)c_4 x^2 + \ldots = -(c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots)$

(d) $c_2 = 0$

(e) $c_3 = -1/(3!)$

(f) $y = x - x^3/(3!) + x^5/(5!) - x^7/(7!)$

(g) $y = \sin x$, yes it does

(h) $y(x) = 1 + x^3/6 + x^6/180$

Chapter 3

3.1 $Re = 3$, $Im = 0$

3.3 $Re = 0$, $Im = 0$

3.5 $Re = 15$, $Im = -3$

3.7 $Re = 1/2$, $Im = 3/4$

3.9 $Re = 49$, $Im = 0$

3.11 $Re = 0$, $Im = 3$

3.13 $Re = 39$, $Im = -80$

3.15 $Re = 5$, $Im = -31$

3.17 $Re = 15/169$, $Im = -36/169$

3.19 $Re = a^2 + b^2$, $Im = 0$

3.21 $x = \pm 10i$

3.23 $2 \pm i$

3.25

i^1	i^2	i^3	i^4	i^5	i^6	i^7	i^8	i^9	i^{10}	i^{11}	i^{12}
i	-1	$-i$	1	i	-1	$-i$	1	i	-1	$-i$	1

3.27 1

3.29 -1

3.31 $(-1 - i)z^{i-2}$

3.33 $-1/z^2$

3.35 $y(x) = Ae^{[-(1/5)+(2/5)i]x} + Be^{[-(1/5)-(2/5)i]x}$

3.37 $r = -2/3$, $s = 5$

3.39 $2 + 2i$, $-2 - 2i$

3.41 $2 + i$, $-2 - i$

3.45(a) $e^{ix} = \left(1 - x^2/2! + x^4/4! + \dots\right) + \left(x - x^3/3! + x^5/5! + \dots\right) i$

(c) $e^{x+iy} = \left(1 + x + \frac{x^2}{2} - \frac{y^2}{2} + \frac{x^3}{6} - \frac{xy^2}{2}\right) + i\left(y + xy + \frac{x^2 y}{2} - \frac{y^3}{6}\right)$

3.47(a) $|3 - 3i| = 3\sqrt{2}$, $\phi_1 = 315°$

(b) $|-1 - 5i| = \sqrt{26}$, $\phi_2 = 259°$

(c) $|(3 - 3i)(-1 - 5i)| = 6\sqrt{13}$, $\phi_{product} = 574° = 214°$

(d) $-18 - 12i$

3.49(a) 13

(b) 169

3.51(a) $1/\sqrt{a^2 + b^2}$

(c) $1/k$

3.55(c) First quadrant for $p < 0$ and fourth quadrant for $p > 0$.

3.57(a) $3 + 4i$ is one possible answer

(b) No c exists

(c) No c exists

3.59 $\text{Re} = \sqrt{3}/2$, $\text{Im} = 1/2$

3.61 $\text{Re} = 1$, $\text{Im} = 0$

3.63 Re $= e^{-2}/2$, Im $= e^{-2}\sqrt{3}/2$

3.65(a) $7e^0$

(b) $7e^{i\pi}$

(c) $e^{i\pi/2}$

(d) $\sqrt{2}e^{i\pi/4}$

(e) $13e^{1.2i}$

(f) $13e^{-1.2i}$

3.71 A

3.73(a) $\rho = \sqrt{a^2+b^2}$, $\phi = \tan^{-1}(b/a)$

3.75 $\cos(3x) = \cos^3 x - 3\cos x \sin^2 x$, $\sin(3x) = 3\cos^2 x \sin x - \sin^3 x$

3.77(b) $k = (3 \pm 6i)/5$

(c) $y_1 = e^{(3+6i)x/5}$, $y_2 = e^{(3-6i)x/5}$

(d) $y(x) = Ae^{(3+6i)x/5} + Be^{(3-6i)x/5}$

(e) $y(x) = e^{(3/5)x}\left(Ae^{(6i/5)x} + Be^{-(6i/5)x}\right)$

(f) $y(x) = e^{3x/5}\left[A\cos(6x/5) + iA\sin(6x/5) + B\cos(-6x/5) + iB\sin(-6x/5)\right]$

(g) $y(x) = e^{3x/5}\left((A+B)\cos(6x/5) + i(A-B)\sin(6x/5)\right)$

(h) $y(x) = e^{3x/5}\left(A\cos(6x/5) + B\sin(6x/5)\right)$

3.79 $x(t) = e^{-2t}(A\cos(2t) + B\sin(2t))$

3.81 $x(t) = e^{t/3}(A\cos(t/3) + B\sin(t/3))$

3.83(a) Doesn't oscillate; increases toward ∞

(b) Doesn't oscillate; decreases toward 0

(c) Oscillates with a period $2\pi/5$; neither

(d) Oscillates with a period $2\pi/5$; amplitude increases toward ∞

(e) Oscillates with a period $2\pi/5$; amplitude increases toward ∞

(f) Oscillates with a period $2\pi/5$; decreases to 0

3.85(a) $x(t) = e^{-(3/2)t}[A\cos(t/2) + B\sin(t/2)]$

(b) $x(t) = Ae^{-t} + Be^{-(5/2)t}$

(c) $\sqrt{40}$ kg/s

3.87(a) $8(d^2Q/dt^2) + R(dQ/dt) + Q = 0$

(b) $Q(t) = Ae^{(-R/16+\sqrt{R^2-32}/16)t} + Be^{(-R/16-\sqrt{R^2-32}/16)t}$

(c) $Q(t) = Ae^{(-1/4)t} + Be^{(-1/2)t}$

(d) $Q(t) = e^{(-1/4)t}(A\cos(t/4) + B\sin(t/4))$

(e) 0

3.89(f) $e^{-ix} = \cos x - i\sin x$

3.91(d) $x = A_2e^{3t} + B_2e^{-3t}$

3.93 $y(x) = -(1/2)\cos x$

3.95 $y(x) = -(1/2)\cos x$

3.97(a) $\mathbf{X}(0) = 3e^{i\pi/4}$, $x(0) = 3/\sqrt{2}$

(c) $\mathbf{X}(t) = 3e^{i\pi/4}e^{i(\pi/8)t}$

3.101(a) $|X| = |3-4i| = 5$, $\omega = 3$

(b) $\phi_{0x} = \tan^{-1}(-4/3) \approx -.93$

(d) behind by ϕ_{0x}

3.103(a) $\mathbf{X}''(t) + 2\mathbf{X}'(t) + 2\mathbf{X}(t) = 10e^{it}$

(b) $X_0 = 2-4i$

(c) $|X_0| = 2\sqrt{5}$

(d) 2π

(e) $x(0) = 2$

3.105 $(e^x/2)(\cos x + \sin x) + A$

3.107(d) $x(t) = (21/5)\cos(2t) - (1/5)\cos(3t)$

3.109(b) $f(x) = \pm(1/2)\cos(3x)$

3.111(a) $\mathbf{X_1}(t) = (13i)e^{(2\pi/3)it}$

(b) $\mathbf{X_2}(t) = (5+12i)e^{60\pi it}$

(c) $\mathbf{X_3}(t) = (5-12i)e^{-it^2}$

3.113(a) $Re(\mathbf{X}(t)) = a\cos(\omega t) - b\sin(\omega t)$, $Im(\mathbf{X}(t)) = a\sin(\omega t) + b\cos(\omega t)$

3.115(c) $Re\left[e^{i(\omega t+\delta+\phi_0)}\left(1-e^{iN\delta}\right)/(1-e^{i\delta})\right]$

3.117(a) $|z| = 1$, $\phi = \pi/8$

3.118(a) $\sqrt{y^2 + [x+(n-1)\delta]^2}$

(b) $\sqrt{y^2+x^2} + \dfrac{x(n-1)}{\sqrt{y^2+x^2}}\delta$

(c) $\dfrac{2\pi}{\lambda}\left(\sqrt{x^2+y^2} + \dfrac{x(n-1)}{\sqrt{x^2+y^2}}\delta\right)$

(d) $\displaystyle\sum_{n=1}^{N} A\cos\left[\omega t + \frac{2\pi}{\lambda}\left(\sqrt{x^2+y^2} + \frac{x(n-1)}{\sqrt{x^2+y^2}}\delta\right)\right]$

(e) $c = \dfrac{2\pi}{\lambda}\left(\sqrt{x^2+y^2} - \dfrac{x\delta}{\sqrt{x^2+y^2}}\right)$ and $d = \dfrac{2\pi x\delta}{\lambda\sqrt{x^2+y^2}}$

(f) $Ae^{i\omega t}e^{i(c+d)}\left(1-e^{idN}\right)/\left(1-e^{id}\right)$

3.119(a) $Z = 2000 + 5500i$

(b) $\mathbf{I} = (1.8\times10^{-4} - 4.8\times10^{-4}i)e^{500it}$

(c) 5.1×10^{-4}

(d) $1.2\text{rad} \approx 70°$

(f) The amplitude of the current and the phase lag would remain the same. The plot of current as a function of time would be identical except shifted in time.

3.121 $Z = 1000 - 333i$, $I_0 \approx .474$, $\phi \approx -18.435°$

3.123(a) $-89.8°$

(b) approaches 0

(c) $R_2 = 66114\Omega$

3.125(a) **(i)** 0

(ii) Always ≥ 0.

(b) The power is never positive.

(c) The power will be positive half of the time.

3.127(a) $R - i/(\omega C)$

(b) $\mathbf{I_1} = (V_1 e^{i\omega_1 t})/(R - i/(\omega_1 C))$

(c) $\mathbf{I_2} = (V_2 e^{i\omega_2 t})/(R - i/(\omega_2 C))$

3.129 $Re(Z) = (5p-1)/26$, $Im(Z) = 5(1-5p)/26$, $|Z| = |5p-1|/\sqrt{26}$, $\phi_Z = -\tan^{-1} 5$

3.131 $Re(Z) = -7.36 \times 10^8$, $Im(Z) = 8.22 \times 10^8$, $|Z| = 1.10 \times 10^9$, $\phi_Z = 2.30$

3.133 $Re(Z) = 3/2$, $Im(Z) = -3\sqrt{3}/2$, $|Z| = 3$, $\phi_Z = -\pi/3$

3.135 $Re(Z) = \cos(p/2)$, $Im(Z) = \sin(p/2)$, $|Z| = 1$, $\phi_Z = p/2$

3.137 $Re(Z) = (3/2)\ln(2)$, $Im(Z) = -\pi/4$, $|Z| = \sqrt{(9/4)(\ln 2)^2 + \pi^2/16}$,
$\phi_Z = -\tan^{-1}\left[\pi/(6\ln 2)\right]$

3.139(a) $\sin x = i\left(e^{-ix} - e^{ix}\right)/2$

(b) $\cos x = \left(e^{ix} + e^{-ix}\right)/2$

(d) $\sinh(ix) = i\sin x$, $\cosh(ix) = \cos x$

(e) $\sin(ix) = i\sinh x$, $\cos(ix) = \cosh x$

3.141 $r = 0$, $s = \pm 3$

3.143(a) a circle

(b) a ray (half a line) starting at the origin

3.149(a) $2\pi/k$

(b) $2\pi/\omega$

(c) $\pi/600$ s

3.151 $y(x) = e^{-(2/5)x}(A\cos(6x/5) + B\sin(6x/5))$

3.153 $x(t) = e^{-(5/4)t}(A\cos(\sqrt{3}t/4) + B\sin(\sqrt{3}t/4))$

3.155 $(i/2)\left(e^{2x}/2 - x + C\right)$

3.157(a) $x(t) = Ae^{\left(-a+\sqrt{a^2-8}\right)t/2} + Be^{\left(-a-\sqrt{a^2-8}\right)t/2}$

(b) $-2\sqrt{2} < a < 2\sqrt{2}$

(c) Approach 0 $(a > 2\sqrt{2})$ or $\pm\infty$ $(a < 2\sqrt{2})$

3.159 $x(t) = (3/17)\sin(2t) - (12/17)\cos(2t) + e^{-t}[A\cos(2t) + B\sin(2t)]$

Chapter 4

4.1(a) positive

(b) positive

4.7 $\partial f/\partial x = 1/y$, $\partial f/\partial y = -x/y^2$, $\partial^2 f/\partial x\partial y = -1/y^2$

4.9 $\partial f/\partial x = \cos x/\cos y$, $\partial f/\partial y = (\sin x \sin y)/(\cos^2 y)$, $\partial^2 f/\partial x\partial y = (\cos x \sin y)/(\cos^2 y)$

4.11 $\sin x/\cos^2 y$

4.13(a) Uphill

(b) Downhill

(c) Downhill

(d) Uphill

4.15(a) $1/R$

(b) $-V/R^2$

(c) $-1/R^2$

4.17(a) Negative

(b) Positive

(c) At $(0, 1)$, $\partial\phi/\partial x < 0$ and $\partial\phi/\partial y = 0$
At $(-1, 1)$, $\partial\phi/\partial x < 0$ and $\partial\phi/\partial y < 0$
At $(-1, -1)$, $\partial\phi/\partial x > 0$ and $\partial\phi/\partial y < 0$

(d) $\partial\phi/\partial x(1,, 1) = -1/2$, $\partial\phi/\partial y(1, 1) = 1/2$

(e) 1

4.19(a) $\dfrac{2h^2 v^4 e^{\frac{hv}{kT}}}{kT^2 c^2 \left(e^{\frac{hv}{kT}} - 1\right)^2}$

(b) $\dfrac{\partial I}{\partial v} = \left(\dfrac{2hv^2}{kTc^2}\right) \dfrac{(3kT - hv)\, e^{\frac{hv}{kT}} - 3kT}{\left(e^{\frac{hv}{kT}} - 1\right)^2}$

4.21(a) $5/2$

(b) $\left(z_{42} - z_{22}\right)/2$

4.23 (b) and (c) only

4.25 (a), (b) and (d)

4.27(a) $V = (Swl - w^2 l^2)/(2w + 2l)$

(b) $\left(\dfrac{\partial V}{\partial w}\right)_l = \dfrac{l^2(S - w^2 - 2wl)}{2(w + l)^2}$

(c) $\left(\dfrac{\partial V}{\partial w}\right)_h = \dfrac{h^2(2S - 8wh - 2w^2)}{(w + 2h)^2}$

(d) $(\partial V/\partial w)_l = 1/4\ \text{m}^2$, $(\partial V/\partial w)_h = 0$

4.31(b) $dz/dt = 4(4t + 5)^{e-1} e^{3-t} - (4t + 5)^e e^{2-t}$

4.33(b)

$$\frac{dB}{dt} = \frac{\partial B}{\partial \rho_b}\left(\frac{\partial \rho_b}{\partial h}\frac{dh}{dt} + \frac{\partial \rho_b}{\partial T}\frac{dT}{dt}\right) + \frac{\partial B}{\partial \rho_{atm}}\frac{d\rho_{atm}}{dh}\frac{dh}{dt}$$

4.35(b) $ds/dt = 3(dp/dt) + 2(dg/dt) + (df/dt)$

(c) 29/15 points per minute

4.37 $dP/dt \approx 10^{13}$ J/s

4.39 $-0.027c$ /s

4.41 $\dfrac{dq}{dt} = \dfrac{2q}{r}\dfrac{dr}{dt} - \dfrac{r^2}{kQ}\dfrac{dF}{dt}$

4.43 $dF = m\,da + a\,dm$

4.51(a) $2xy\dfrac{dx}{dt} + x^2\dfrac{dy}{dt} - e^x\dfrac{dx}{dt} + 2zt\dfrac{dz}{dt} + z^2 = 0$

(b) $dx/dt = (e^x - 2xy)^{-1}[x^2(dy/dt) + 2zt(dz/dt) + z^2]$

(c) 0.7

4.53 $df/dt = [h^2((dh/dt) + f\sin(gh)(g(dh/dt) + h(dg/dt))) + fe^{f/h}(dh/dt)](h^2\cos(gh) + he^{f/h})^{-1}$

4.55 $df/dt = \left[f - af^2(dg/dt) - bf^2(dh/dt)\right] / \left(cf^2 + t\right)$

4.57(b) $dy/dx = -(2x^3 + x)/(2y^3 + y)$

(c) $dy/dx = -0.13$

(d) $dy/dx = 0.13$

4.59(c) $d^2y/dx^2 = -25/y^3$

4.61(a) $m(x,y) = (e^x + 1)/(3y^2 + 1)$

4.63 $dy/dx = (y-1)/(1-x)$, $d^2y/dx^2 = 2(y-1)/(1-x)^2$

4.65(a) $x = e^y$

(b) $1 = e^y(dy/dx)$

(c) $dy/dx = 1/e^y$

(d) $dy/dx = 1/x$

4.67(a) $x = \sin y$

(b) $1 = \cos y(dy/dx)$

(c) $dy/dx = 1/\cos y$

(d) $dy/dx = 1/\sqrt{1 - \sin^2 y}$

(e) $dy/dx = 1/\sqrt{1 - x^2}$

4.69 $dy/dx = 1/(x^2 + 1)$

4.71(a) $D^2 = h^2 + s^2 - 2sh\cos(130°)$

(b) $dD/dt = (1/D)\,(s - h\cos 130°)\,ds/dt$

(c) $dD/dt = 569.5$ km/h

4.73(a) $di/dt = -50/9$ cm/s

4.75(a) $\dfrac{dK}{dt} = \dfrac{dY/dt - A\alpha L^{\alpha-1}K^{\beta}(dL/dt)}{A\beta L^{\alpha}K^{\beta-1}}$

(b) $\dfrac{dK}{dt} = \dfrac{dY/dt - A\alpha L^{\alpha-1}K^{\beta}(dL/dt) - AL^{\alpha}K^{\beta}(\ln L)(d\alpha/dt)}{A\beta L^{\alpha}K^{\beta-1}}$

4.77(a) $D_{\vec{u}}f(\pi,1,1) = -1$

(b) $D_{\vec{u}}f(\pi,1,1) = 0$

(c) $D_{\vec{u}}f(\pi,1,1) = 1$

(d) $D_{\vec{u}}f(\pi,1,1) = 1/\sqrt{2}$

4.79(a) $D_{\hat{u}}f(3,0,-1)=\sqrt{115}$

(b) $D_{\hat{u}}f(3,0,-1)=103\sqrt{93}/93$

(c) $D_{\hat{u}}f(3,0,-1)=127\sqrt{141}/141$

4.81(a) $df/dt=(\partial f/\partial x)(dx/dt)+(\partial f/\partial y)(dy/dt)+(\partial f/\partial z)(dz/dt)$

4.83(a) $\hat{u}=\left(0.3\hat{i}+0.1\hat{j}+0.2\hat{k}\right)/\sqrt{0.14}$, $D_{\hat{u}}f=-1/\sqrt{0.14}$

4.85 $y=x, +y, +x$

4.87(a) $\hat{u}=(\cos\phi)\hat{i}+(\sin\phi)\hat{j}$

(b) $D_{\hat{u}}\rho=e^{-x^2+y^2}\left(\sin(\pi y^2/2)\cos\phi\left[(\pi/2)\cos(\pi x/2)-2x\sin(\pi x/2)\right]\right.$
$\left.+y\sin(\pi x/2)\sin\phi\left[\pi\cos(\pi y^2/2)+2\sin(\pi y^2/2)\right]\right)$

(c) $D_{\hat{u}}\rho(1,1)=e^{-3}\left(-2\cos\phi+2\sin\phi\right)$

(d) At $\phi=3\pi/4$ we get $D_{\hat{u}}\rho\approx 0.141$.

4.95(a) $\vec{\nabla}f(0,0)=\vec{0}$, $\vec{\nabla}f(0,1)=2\hat{j}$, $\vec{\nabla}f(0,-1)=-2\hat{j}$, $\vec{\nabla}f(1,0)=2\hat{i}$, $\vec{\nabla}f(-1,0)=-2\hat{i}$

4.97(a) $2\sqrt{13}$

(b) $-2\sqrt{13}$

(c) $-4\hat{i}-6\hat{j}$

(d) $\hat{u}=(3/\sqrt{13})\hat{i}-(2/\sqrt{13})\hat{j}$

(e) one other

4.99(a) $-\left(x/\sqrt{25-x^2-y^2}\right)\hat{i}-\left(y/\sqrt{25-x^2-y^2}\right)\hat{j}$

(b) $2x\hat{i}+2y\hat{j}+2z\hat{k}$

4.101(a) $t=3\pi/4$

(b) $t=3\pi/2$

4.105 $-2\hat{i}-4\hat{j}+6\hat{k}$

4.107(a)
$$\vec{E}=\left(\frac{k(x-x_0)}{\left[(x-x_0)^2+y^2\right]^{3/2}}-\frac{k(x+x_0)}{\left[(x+x_0)^2+y^2\right]^{3/2}}\right)\hat{i}+\left(\frac{ky}{\left[(x-x_0)^2+y^2\right]^{3/2}}-\frac{ky}{\left[(x+x_0)^2+y^2\right]^{3/2}}\right)\hat{j}$$

(b) In the positive x-direction

(c) 0.6 radians

4.109(a) $\partial f/\partial x=14$, $\partial f/\partial y=-8$

(b) $14x-8y$

(c) $\vec{u}=-14\hat{i}+8\hat{j}$

(d) $\partial f/\partial x=(\alpha d-\beta b)/(ad-bc)$, $\partial f/\partial y=(\alpha c-\beta a)/(ad-bc)$

(e) $ab-cd=0$ means parallel vectors

4.111(a) $g(x,y)=7+(1/14)(x-40)+(3/7)(y-3)$

(c) $f(41,2.9)=7.02922$, $g(41,2.9)=7.02857$

(d) $f(50,5)=8.66$, $g(50,5)=8.57$

4.113(a) $\left(\dfrac{\partial^{11} f}{\partial x^7 \partial y^4}(0,0)\right) \dfrac{1}{7! \times 4!} x^7 y^4$

(b) $\left(\dfrac{\partial^{11} f}{\partial x^7 \partial y^4}(-3,\pi)\right) \dfrac{1}{7! \times 4!} (x+3)^7 (y-\pi)^4$

4.115 $f(1/2, 1/2) \approx -0.613$

4.117 $z(x,y) \approx 3 + (1/2)(x-6) - (3/2)(y-2)$

4.119 $\sin(x+y^2) \approx x + y^2 - x^3/6 - x^2y^2/2$

4.121(a) $e^{x+2y} = 1 + x + 2y + (1/2)x^2 + 2xy + 2y^2 + (1/6)x^3 + x^2y + 2xy^2 + (4/3)y^3 + \ldots$

(b) $e^{x+2y} = 1 + x + 2y + (1/2)x^2 + 2xy + 2y^2 + \ldots$

(c) $2e^{x+2y} = 2 + 2x + 4y + x^2 + 4xy + 4y^2 + \ldots$

4.123(a) $f(x,y,z) = \displaystyle\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{p=0}^{\infty}\left(\frac{\partial^{n+m+p} f}{\partial x^n \partial y^m \partial z^p}(x_0, y_0, z_0)\right)\frac{1}{n!m!p!}(x-x_0)^n(y-y_0)^m(z-z_0)^p$

(b) $f(x,y,z) = y$, $f(x,y,z) = y + x^2 + (5/2)yz$

(c) $f(0.01, 0.02, -0.01) = 0.0191245$, first-order approximation 0.02, second-order approximation 0.0196

4.125(a) $v_{AC} \approx v_{BC} + v_{AB}$

(b) $v_{AC} \approx c - (v_{AB} - c)(v_{BC} - c)/(2c)$

4.127(a) $f(\theta_1, \theta_2, \dot{\theta}_2, \ddot{\theta}_1, \ddot{\theta}_2) \approx 2(g/L)\theta_1 + 2\ddot{\theta}_1 + \ddot{\theta}_2$

(b) $2\ddot{\theta}_1 + \ddot{\theta}_2 + (2g/L)\theta_1 = 0$, $\ddot{\theta}_1 + \ddot{\theta}_2 + (g/L)\theta_2 = 0$

(c) $A = -\left(1/\sqrt{2}\right) B$

(e) $5\sqrt{2}\,^\circ$

4.129 $(-3,-4)$ is a local maximum, $(5/2, 4)$ is a local minimum, $(5/2, -4)$ and $(-3, 4)$ are saddle points

4.133(a) saddle point.

4.135 Absolute maximum: $2\pi - 2$. Absolute minimum: $2 - 6\pi$.

4.137 Absolute maximum: 48. Absolute minimum: $-16/729$.

4.139 $64C_0D^2/49$

4.141 $(2/3, 5/3, -1/3)$

4.145 $\left(1/2, 1/\sqrt{8}, 1/2\right)$

4.147 $(13/9, 15/9)$

4.149(a) $(x_3 = 4/5, y_3 = 3/5, x_4 = -3/5, y_4 = 4/5)$, and $(x_3 = -4/5, y_3 = 3/5, x_4 = 3/5, y_4 = 4/5)$.

4.151(a) $y = (y_1 - y_0)x + y_0$

(b) $y = (1/2)(y_2 - y_0)x + (1/6)(5y_0 + 2y_1 - y_2)$

4.155(a) $\vec{\nabla} f = \left(2x + 4y + 2\right)\hat{i} + \left(6y + 4x\right)\hat{j}$

(b) $g(x,y) = x - y = 13$ so $\vec{\nabla} g = \hat{i} - \hat{j}$

(c) $2x + 4y + 2 = \lambda$, $6y + 4x = -\lambda$

(d) $(8, -5)$

4.157 $W = L = 260/7$, $H = 390/7$

4.159 $(2/3, 5/3, -1/3)$

4.161 $(0.393, 0.786, 0.772)$

4.163 $(1/2, 1/\sqrt{8}, 1/2)$

4.167 $(6, 0)$, $(3/2, \pm\sqrt{135}/2)$

4.169 $(0, 0, 0, 0)$ and $(2/9, 4/27, -10/27, -26/81)$

4.173(a) $\sqrt{(x-50)^2+(y-10)^2+(z-20)^2}+\sqrt{(x-30)^2+(y-20)^2+(z-30)^2}$

(b) $x^2+y^2+z^2=100$

4.175(a) $v_x = v_y = \sqrt{E/m}$

(b) $\lambda = 2/(mg)$

(c) $2(dE)/(mg)$

4.177(a) $L = \left(\dfrac{s+c}{T(\alpha+\beta)}\right)^{1/(\alpha+\beta-1)}$

(b) $\lambda = (c\alpha - s\beta)/(\alpha+\beta)$

(c) $w(s\beta - c\alpha)/(\alpha+\beta)$

(d) \$1200

4.179(a) $-P$

4.181(a) 0 (not changing)

(b) $dS/dt = (nR/V)(dV/dt)$

4.183(a) $dT/dt = (P/nR)(dV/dt)$

(b) $dU/dt = (3P/2)(dV/dt)$

(c) $dS/dt = (5P/2T)(dV/dt)$

(d) entering

4.185(a) Six

(b) 8715 J

4.187(a) $C_V = (f/2)nR$

(b) $C_P = \dfrac{f}{2}nR + \dfrac{nRV\left(an^2+PV^2\right)}{PV^3-an^2+2abn^3}$

4.193(a) $3nR/(2T)$

(b) $5nR/(2T)$

(d) First expression: $\dfrac{\partial S}{\partial T} = \dfrac{n^2R^2}{PV+nRT}+\dfrac{3nR}{2T}$

Second expression: $\dfrac{\partial S}{\partial T} = \dfrac{2n^2R^2}{PV+2nRT}+\dfrac{3nR}{2T}$

4.195(d) $(\partial f/\partial x)_y = 0$

(e) $(\partial f/\partial x)_\rho = -x/y$

(f) $(\partial f/\partial x)_\phi = y/x$

4.197(a) **(i)** $\partial a_1/\partial x = 1+y^2$ and $\partial a_1/\partial y = 2xy$.

(ii) $da_1 = \left(1+y^2\right)dx + 2xydy$.

(iii) $da_1 = \left(1+5x^4\right)dx$.

(b) **(i)** $\partial a_2/\partial x = 3x^2y$ and $\partial a_2/\partial y = (1/2)y^{-1/2}+x^3$.

(ii) $da_2 = 3x^2ydx + \left((1/2)y^{-1/2}+x^3\right)dy$.

(iii) $da_2 = \left(5x^4+1\right)dx$.

4.199(a) $dH = T\,dS + V\,dP$

4.201 $(\partial T/\partial P)_S = (\partial V/\partial S)_P$

4.203 $\partial f/\partial x = 2xy^3$, $\partial^2 f/\partial x\partial y = 6xy^2$, $\vec{\nabla} f = 2xy^3\hat{i} + 3x^2y^2\hat{j}$, $D_{\vec{u}} f = 3x^2y^2$

4.205 $\partial f/\partial x = y/(x+y)^2$, $\partial^2 f/\partial x\partial y = (x-y)/(x+y)^3$, $\vec{\nabla} f = \left[y/(x+y)^2\right]\hat{i} - \left[x/(x+y)^2\right]\hat{j}$, $D_{\vec{u}} f = \left[y\cos(70°) - x\sin(70°)\right]/(x+y)^2$

4.207 (a), (b)

4.209(a) $\partial k/\partial T = AE_a e^{-E_a/RT}/(RT^2)$

(b) $\partial k/\partial E_a = -Ae^{-E_a/RT}/RT$

4.211(b) $dI/dt = (\partial I/\partial V)(dV/dt) + (\partial I/\partial R)(dR/dT)(dT/dt) + (\partial I/\partial L)(dL/dt)$

4.213(a) $dc/dt = \partial c/\partial t + (\partial c/\partial x)(dx/dt)$

4.215(a) $\vec{\nabla} z = -8\hat{i} - 8\hat{j}$

(d) $\vec{\nabla} f = 8\hat{i} + 8\hat{j} + \hat{k}$

4.219(a) $f(x,y) \approx x$, $f(0.01, 0.01) \approx 0.01$

(b) $f(x,y) \approx x - 2x^2 - xy$, $f(0.01, 0.01) \approx 0.0097$

(c) $f(0.01, 0.01) = 0.00971$

(d) $f(x,y) \approx (1/4) + (1/8)(x-1) - (1/16)(y-1)$, $f(1.01, 0.9) \approx 0.2575$

(e) $f(x,y) \approx (1/4) + (1/8)(x-1) - (1/16)(y-1) - (1/16)(x-1)^2 + (1/64)(y-1)^2$, $f(1.01, 0.9) \approx 0.25766$

(f) $f(1.01, 0.9) \approx 0.257653.$

4.221(a) $R_1 + R_2 + R_3$

(b) $50R_1 + 20R_2 + 10R_3 \le 1000$, $2R_1 + 5R_2 + 10R_3 \le 200$, $R_1 \ge 0$, $R_2 \ge 0$, $R_3 \ge 0$

(d) 42.9 L

(e) 42.9 L

(f) 42.9 L

4.223(a) $\partial W/\partial v = -5.72v^{-0.84} + 0.0684Tv^{-0.84}$

4.225(a) $U \approx GM\left[-\dfrac{1}{R} - \dfrac{J_2}{2R} + \left(\dfrac{2+3J_2}{2R^2}\right)(r-R) - \left(\dfrac{1+3J_2}{R^3}\right)(r-R)^2 + \dfrac{3J_2}{2R}l^2\right]$

(b) $U \approx GM\left[-\dfrac{1}{R} - \dfrac{J_2}{2R} + \left(\dfrac{2+3J_2}{2R^2}\right)(r-R) - \left(\dfrac{1+3J_2}{R^3}\right)(r-R)^2 + \dfrac{3J_2}{2R^3}y^2\right]$

(c) $\vec{F} \approx -\dfrac{GM}{R^2}\left(3J_2\dfrac{y}{R}\hat{y} + \left[1 + \dfrac{3}{2}J_2 - 2\left(1+3J_2\right)\dfrac{r-R}{R}\right]\hat{r}\right)$

Chapter 5

5.1(a) 20

(c) $dm = 3x\ dx$

(d) $\int_1^3 3x\ dx$

(e) 12

5.3(a) $2/n$

(b) $1 + 2(i-1)/n$

(c) $3 + 6(i-1)/n$

(d) $6/n + 12(i-1)/n^2$

(e) $12 - 6/n$

(f) 12

5.5(a) $2/3$ m

(b) $4/3$ m

5.7 $M = 2cL/\pi$, $x_{COM} = L/2$, $I = (cL^3/\pi^3)(\pi^2 - 4)$

5.9 $M = (a/b)\left(1 - e^{-bL}\right)$
$x_{COM} = (1/b)\left(1 - (1+bL)e^{-bL}\right) / \left(1 - e^{-bL}\right)$
$I = (a/b^3)\left(2 - \left(2 + 2bL + b^2L^2\right)e^{-bL}\right)$

5.11 $V = kcL^2/2$

5.15(a) 20

(b) $m(10)\ dx$

(c) $\int_{10}^{13} m(x)\ dx$

5.17(a) 6 inches

(b) $g(t)dt$

(c) $\int_7^{10} g(t)dt$

5.19(a) 24 Coulombs

(b) $I(t)\ dt$

(c) $1/10\pi$ Coulombs

(d) 0 Coulombs

5.21 $(1/6)\rho g W H^3$

5.23(a) mgh

(b) $kL^2/2$

(c) $Gm_A m_E/R$

5.25(c) $2\pi\rho a/\left(\rho^2 + b^2\right)\ d\rho$

(d) $\pi a \ln\left(\left(R^2/b^2\right) + 1\right)$

(e) a: mass; b: distance

5.29(a) 18

(b) 36

5.31(d) $y_{com} = 12/5$

5.33 $x_{COM} = 3L/4$, $y_{COM} = (3/10)cL^2$

5.35(a) $ka^3/2$

(b) $x_{COM} = 2a/3$, $y_{COM} = a/2$

(c) $I = ka^5/4$

(d) $I = k\left(a^5/4 - 2ba^4/3 + b^2a^3/2\right)$

5.37 $4(e-1)\pi\rho_0 R^3/(3e)$

5.41 $V = \pi H^3$

5.43(a) $\rho(h) = 1.2e^{-1.16\times10^{-4}h}$

(b) 10344.8 kg

5.45(a) rectangular solid with square base

(b) $dV = w^2\ dy$

(c) $w = L(H-y)/H$

(d) $V = L^2H/3$

5.47 $MR^2/2$

5.51(b) $\sqrt{\rho^2+h^2}$

(c) $dV = \dfrac{2kQ\rho\ d\rho}{R^2\sqrt{\rho^2+h^2}}$

(d) $(2kQ/R^2)\left(\sqrt{R^2+h^2} - h\right)$

(e) kQ/h

5.53 $3R/8$

5.55 horizontal slices, $1/12$

5.57 horizontal slices, $\left(\alpha/k^2\right)\left(e^{kH}-1\right)\left(e^{kW}-1\right)$

5.59 vertical slices, 0

5.63(a) $4/\pi\ dx$

(b) $2\sqrt{2}/\pi\ dy$

(c) $-16/\pi^2$

5.65(a) $\int_0^{10}\int_0^7 kye^{xy}\ dx\ dy = 3.59\times10^{29}\ k$

(b) $k\int_0^7\int_0^{10} ye^{xy}\ dy\ dx = 3.59\times10^{29}\ k$

5.67 kW^2H

5.69(a) 0

(b) Half the surface is above the xy-plane and half is below.

5.71(a) $250(e^3 - e^{-3})/3$

(b) $250(e^3 - e^{-3})/3$

(c) $250(2e^3 - 2)/3$

5.75(a) $(k/\beta\alpha)\left[\cos(\alpha L) - 1\right]\left[e^{-\beta H} - 1\right]$

(b) 6000 g

5.77 97.2

5.79(a) $g(y)\left(F(b) - F(a)\right)$

(b) $\left(G(d) - G(c)\right)\left(F(b) - F(a)\right)$

(c) $\left(G(d) - G(c)\right)\left(F(b) - F(a)\right)$

5.81 $R^4/8$

5.83 329

5.85 $1/2$

5.87 $\int_4^6 \int_3^7 f(x,y)\,dy\,dx$

5.89 $\int_0^{16} \int_0^{x/2} f(x,y)\,dy\,dx$

5.91 $\int_0^1 \int_0^{\cos^{-1}(y)} f(x,y)\,dx\,dy$

5.93 $\int_0^5 \int_{\sqrt{25-y^2}}^{\sqrt{25-y}} f(x,y)\,dx\,dy$

5.95 $\int_0^3 \int_y^{(-y+9)/2} f(x,y)\,dx\,dy$

5.97(a) $\int_0^2 \int_0^{x^3} dx\,dy = 4$

(b) $\int_0^2 x^3\,dx = 4$

5.101(a) $V = (4kQ/\pi R^2) \int_0^R \int_0^{\sqrt{R^2-x^2}} 1/\sqrt{x^2+y^2}\,dy\,dx$

(b) $I = (4M/\pi R^2) \int_0^R \int_0^{\sqrt{R^2-x^2}} x^2+y^2\,dy\,dx$

5.105 $2cH^7/105$

5.107 $4R^3H/(3\pi)$

5.111(a) $\left(4cL^2/k\pi^2\right)\left(1-e^{-kL}\right)$

(b) $\left(\dfrac{\pi}{k}\right)\dfrac{1-e^{-kL}(kL+1)}{1-e^{-kL}}$

5.113 $M = 7500$, $x_{COM} = 3/2$, $y_{COM} = 0$, $z_{COM} = 125/7$

5.117(a) $-\sqrt{1-z^2} \le y \le \sqrt{1-z^2}$, $-1 \le z \le 1$

(b) $-\sin x \le y \le \sin x$, $0 \le x \le \pi$

5.119 $(1/6)ML^2$

5.121 $kL^4/3$

5.123 $(2/5)ML^2$

5.129 0

5.131 0

5.135 $R^4/8$

5.137 The potential is $2kQ/R$ and the moment of inertia is $(1/2)MR^2$.

5.139 $ck\int_{-R}^R \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} x^2/\left[\left(x^2+y^2\right)\sqrt{x^2+y^2+h^2}\right]\,dy\,dx$

5.141 $kR^4/2$

5.143(a) Arbitrarily choosing to place the flat edge on the x-axis, $\sigma = ky$

(b) $(2/3)kR^3$

(c) $\sigma = k\rho\sin\phi$

(d) $(2/3)kR^3$

5.147(c) ρ: 0 to $\sin(3\phi)$, ϕ: 0 to $\pi/3$

(d) $\pi/4$

5.149(b) $\rho = 3$, $\rho = 6\cos\phi$

(c) $\pm\pi/3$

(e) $(9\sigma/2)(\pi/3 - \sqrt{3}/2)$

(f) $3\pi\sigma$

(g) $(6\pi - 9\sqrt{3}/2)\sigma$

5.151 $(k/6)\left(2^{3/2}-1\right)$

5.155 1

5.157(b) 6

(d) $v=1$

(e) $u=1$

(f) $y=0$ becomes $u=0$ for $-1\le x\le 0$ and $v=0$ for $0\le x\le 1$

(h) $4u^2+4v^2$

(i) 6

(j) 4

5.159 $(e-2)/3$

5.161 $(2/3)\tanh^{-1}(1/2)$

5.163(a) $(\rho,\phi,z)=(\sqrt{2},-\pi/4,4),(r,\theta,\phi)=(\sqrt{18},0.339,-\pi/4)$

(b) $(\rho,\phi,z)=(\sqrt{2},3\pi/4,4),(r,\theta,\phi)=(\sqrt{18},0.339,3\pi/4)$

(c) $(x,y,z)=(-2.5,2.5*\sqrt{3},-1),(r,\theta,\phi)=(\sqrt{26},1.77,2\pi/3)$

(d) $(x,y,z)=(0,0,1),(\rho,\phi,z)=(0,0,1)$

5.165(a) $r=5/\cos\theta$

(b) Cartesian: $z=\sqrt{2}\sqrt{x^2+y^2}$, Cylindrical: $z=\sqrt{2}\rho$

(c) Cylindrical: $\rho=2$, Spherical: $r=2/\sin\theta$

5.167(a) $r(t)=4000,\ \theta(t)=\pi/3,\ \phi(t)=\pi t/12$

(b) $x(t)=2000\sqrt{3}\cos(\pi t/12),\ y(t)=2000\sqrt{3}\sin(\pi t/12),\ z(t)=2000$

5.169(a) c is 1/distance. k is mass/volume.

(d) $z=c\rho^2$, $D=k\rho/z$.

(e) $(k\rho/z)\rho\ d\rho\ d\phi\ dz$

(f) 0 to 2π

(h) $\int_0^H\int_0^{\sqrt{z/c}}\ d\rho\ dz$

(i) $(4\pi k/9)(H/c)^{3/2}$

(j) mass

5.171 $\left(\pi H^3/6\right)\left(e^{R^2}-1\right)$

5.173 $4\pi R^5/15$

5.175 $(\pi/10)\left(2H^3R^2+HR^4\right)$

5.177 $4\pi R^5/5$

5.179 For the hemisphere, $I=(2/5)MR^2$ and $V=3kQ/(2R)$. For the sphere, $I=(4/5)MR^2$ and $V=3kQ/R$.

5.181(a) r

(b) $r\ d\theta$

(c) radius: $r\sin\theta$, arc length: $r\sin\theta\ d\phi$

(d) $r^2\sin\theta\ dr\ d\theta\ d\phi$

5.183 $\pi kR^3H/6$

5.185(a) $R=\sqrt{3/2}\ H$

(b) $M=k\pi R^4(1-\sqrt{2/3})/2$

5.187(a) $(2M/5)\left(R_2^5 - R_1^5\right) / \left(R_2^3 - R_1^3\right)$

(b) $2MR_2^2/5$

(c) $(2/3)MR_1^2$

5.189 $\pi DHR^4/10$

5.191(a) $D = D_0 e^{k_1\rho} e^{-k_2 z}$

(b) D_0: density, k_1 and k_2: distance^{-1}

(c) $D_0 \int_0^H \int_0^R \int_0^{2\pi} e^{k_1\rho} e^{-k_2 z} \rho^3 \, d\phi \, d\rho \, dz$

(d) $\left(2\pi D_0/(k_2 k_1^4)\right)\left(1 - e^{-k_2 H}\right)\left(e^{k_1 R}\left[k_1^3 R^3 - 3k_1^2 R^2 + 6k_1 R - 6\right] + 6\right)$

5.193(c) $y^2 + z^2 = 1$

(d) z goes from $-\sqrt{1 - \rho^2 \sin^2\phi}$ to $\sqrt{1 - \rho^2 \sin^2\phi}$

(f) $I_z = \int_0^{2\pi} \int_0^1 \int_{-\sqrt{1-\rho^2\sin^2\phi}}^{\sqrt{1-\rho^2\sin^2\phi}} \rho^3 \, dz \, d\rho \, d\phi / \left[\int_0^{2\pi} \int_0^1 \int_{-\sqrt{1-\rho^2\sin^2\phi}}^{\sqrt{1-\rho^2\sin^2\phi}} \rho \, dz \, d\rho \, d\phi\right]$

5.195(a) 0 to 2π

(c) $r = 1/\sqrt{1 + \cos^2\theta}$

(d) $\int_0^{\pi} \int_0^{1/\sqrt{1+\cos^2\theta}} \int_0^{2\pi} r^4 \sin^2\theta \, d\phi \, dr \, d\theta$

5.197(a) 70

(b) -70

(c) 0

(d) 28

5.199(a) 60π

(b) -60π

(c) 0

(d) 0

5.201 8

5.203 $77/10 + 16\sqrt{2}/5 - 1/\sqrt{2}$

5.205 0

5.209 $\ln 2$

5.211 $\left(\sqrt{14}/8\right)\sinh 2 \approx 1.7$

5.215 $t = 125\left[\left(1 + (36/25)k^2 x_f^{2/5}\right)^{3/2} - 1\right] / \left[108\sqrt{2gk^5}\right]$

5.217(b) -33

(c) 65

(d) -65

(e) $(13/5)(4\,\hat{i} - 3\,\hat{j})$

5.219 $k/R_2 - k/R_1$

5.221 $k/(H - 4\pi p/\omega) - k/H$

5.223 0

5.225(a) $2\pi\rho|\vec{B}|$

(b) $|\vec{B}| = \mu_0 I/(2\pi\rho)$

5.229(b) $2|\vec{B}|L$

(c) $|\vec{B}| = \mu_0 J/2$

5.235(c) $\rho\, d\phi$

(d) $\sqrt{(\rho'(\phi))^2 + \rho^2}\, d\phi$

5.237(a) $x = 2\cos u + v^2, y = v + 3, z = \sin u$

(b) $x = 2\cos u - v^2, y = v, z = \sin u$

(c) $x = \cos u + v^2,\ y = v,\ z = (1/2)\sin u$

5.245(a) $x^2 + y^2 \le 9,\ z = 5$

(b) $\rho \le 3,\ z = 5$

(c) $x = v\cos u,\ y = v\sin u,\ z = 5,\ 0 \le v \le 3,\ 0 \le u \le 2\pi$

5.247(a) $x = 7\sin\theta\cos\phi,\ y = 7\sin\theta\sin\phi,\ z = 7\cos\theta,\ 0 \le \phi \le 2\pi,\ 0 \le \theta \le \pi$

(b) $x = 7\sin\theta\cos\phi + 3,\ y = 7\sin\theta\sin\phi + 7,\ z = 7\cos\theta - 4,\ 0 \le \phi \le 2\pi,\ 0 \le \theta \le \pi$

5.251(a) $x = x,\ y = \sin x\cos\theta,\ z = \sin x\sin\theta,\ 0 \le \theta \le 2\pi$

5.255(a) 60

(b) -60

(c) 60

5.257(a) 0

(b) 0

(c) $1000\pi/3$

(d) $500\pi/3$

(e) $500\pi/9$

5.259 $-44/15$

5.261 $\ln 2\,(1 - \cosh 2) + (\pi/9)\left(\cosh^3 2 - 1\right) \approx 16$

5.263(a) 1 kg

(b) $<$ 1 kg

(c) $>$ 1 kg

5.265(a) $4\pi/3$

5.267(a) 27π

(b) 108π

5.271(a) width $|\vec{a}|$, height h

(b) $|\vec{a}|\,|\vec{b}|\,\sin\theta$

5.273(a) Q/ε_0

(b) $4\pi r^2\left|\vec{E}\right|$

(c) $\left|\vec{E}\right| = Q/\left(4\pi\varepsilon_0 r^2\right)$

5.275(a) $\left|\vec{E}\right| = Q/2\pi\varepsilon_0 r$

(b) $\propto r^{-9}$

5.277(b) $\sigma\pi R^2$

(d) $\left|\vec{E}\right| 2\pi r^2$

(e) 0

(f) σ/ε_0

5.279(a) $|B_0|\pi\omega R^2 \sin\theta$

5.281(a) $\mu_0 IL \ln 2/(2\pi)$

(b) $-\mu_0 I_f L \ln 2/(2\pi\tau)$

(c) 6.16×10^{-11}s

5.283 $(GmM)/(w(w+L))\hat{i}$

5.293(a) $-3GMm/R^2$

(b) $\left[R^2 + L^2 - 2Lz\right]^{1/2}$

(c) $3GMm/R^2$

(d) $-GMm/L^2$

(e) $-GMm/L^2$

5.295 $\vec{B}_z = \mu_0 IR^2/\left[2\left(H^2 + R^2\right)^{3/2}\right]$

5.297(a) 150

(b) 375

(c) 225

(d) 1125/2

5.299 $kL^4/96$

5.301 $(1/2)\ln(5/3)$

5.303 0

5.305 $(1/160) + (1/4\pi)\cos(\pi/8) - (2/\pi)\sin(\pi/8)$

5.307 $\pi^2 R^4/4$

5.311(a) $(1/4)\left(4e + e^2 - 1\right)$

(b) 2π

5.313 $4\pi R^3/3$

5.315 $\pi\left(2R^2H + (4/3)RkH^3 + (2/5)k^2H^5\right)$

5.317 $\pi R^3/6$

5.319 $M = a\pi^2 R^5/5$, $Q = \pi bR^2(\ln(16) - 2)$, center of mass at origin, $I_z = 3a\pi^2 R^7/28$, $V = 4\pi kbR(1 - \ln 2)$

5.321 $x_{com} = y_{com} = 0$, $z_{com} = H/4$, $I_z = (2/5)ML^2$, $V = (4/3)k\beta HL^2$

5.323 $4W^2H/5$

5.327(b) $(4, -2)$ and $(4, 2)$

(d) $464/15$

5.329 $2kR/3$

5.331(b) mgH

5.333(b) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2}e^{-y^2}\,dx\,dy$

(c) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dx\,dy$

(e) $\int_{0}^{\infty}\int_{0}^{2\pi} e^{-\rho^2}\,\rho\,d\phi\,d\rho$

(f) $S=\sqrt{\pi}$

(h) $\sqrt{\pi}/2$

Chapter 6

6.1(c) $x_1 = 0$, $x_2 = 1$

6.3(a) yes

(b) no

(c) yes

(d) no

6.7(a) $x_1 = -1$, $x_2 = 1$

(b) $a = b = 1/2$

6.9(a) yes, yes

(b) no, yes

(c) no, no

6.17(a) $1/2$

(d) $-1/4$

(e) yes

(g) $x_1(t) = 2\cos t - 4\cos(\sqrt{7}t)$, $x_2(t) = \cos t + \cos(\sqrt{7}t)$

(h) $x_1(t) = 6\cos t + 4\cos(\sqrt{7}t)$, $x_2(t) = 3\cos t - \cos(\sqrt{7}t)$

6.19(a) $4{\times}5$

(b) $\begin{pmatrix} 7 & p & 3 & 8 & 15 \end{pmatrix}$

(c) $\mathbf{M} = \begin{pmatrix} 3/2 & 1 & -2 & 9/2 & 0 \\ 3 & 5 & -3/2 & 6 & x/2 \\ 7/2 & p/2 & 3/2 & 4 & 15/2 \\ 1 & 23/2 & 3y/2 & 10 & f/2 \end{pmatrix}$

(d) 6

6.21(a) $\begin{pmatrix} 6 & 4 \\ 10 & -10 \end{pmatrix}$

(b) illegal

(c) illegal

(d) 37

6.23(a) $w = 0$, $x = -2$, $y = -6$, $z = -8$

(b) impossible

(c) impossible

6.25 $M_{ij} = i + 2j$

6.27(b) $\begin{array}{l} \textit{meat} \\ \textit{vegetables} \end{array} \begin{pmatrix} 43/20 \\ 4/3 \end{pmatrix}$

6.29 $\begin{pmatrix} 68 + (2/3)x \\ -44 \end{pmatrix}$

6.31 (1)

6.33 Illegal

6.35(a) $\begin{array}{l} \textit{Icelanders} \\ \textit{Swedes} \end{array} \begin{pmatrix} I \\ S \end{pmatrix}$

(b) $\begin{array}{l|cc} & \textit{Icelanders} & \textit{Swedes} \\ \hline O & .56 & .38 \\ A & .31 & .44 \\ B & .11 & .12 \\ AB & .02 & .06 \end{array}$

$$\begin{array}{c} \\ O \\ A \\ B \\ AB \end{array} \begin{array}{c} \textit{Icelanders} \quad \textit{Swedes} \\ \begin{pmatrix} .56 & .38 \\ .31 & .44 \\ .11 & .12 \\ .02 & .06 \end{pmatrix} \end{array}$$

(c) $\begin{array}{l} O \\ A \\ B \\ AB \end{array} \begin{pmatrix} .56I + .38S \\ .31I + .44S \\ .11I + .12S \\ .02I + .06S \end{pmatrix}$

6.37 $\begin{array}{l} \textit{oribatid mites} \\ \textit{gamasid mites} \end{array} \begin{pmatrix} 1,736,000 \\ 1,074,000 \end{pmatrix} =$

$$\begin{array}{l} \\ \textit{oribatid mites / m}^2 \\ \textit{gamasid mites / m}^2 \end{array} \begin{array}{c} \textit{forest floor} \quad \textit{grass clippings} \\ \begin{pmatrix} 240,000 & 127,000 \\ 86,000 & 102,000 \end{pmatrix} \end{array} \quad \begin{array}{l} \textit{forest floor (m}^2\textit{)} \\ \textit{grass clippings (m}^2\textit{)} \end{array} \begin{pmatrix} 3 \\ 8 \end{pmatrix}$$

6.41(a) $\begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2/3 & -1/3 \end{pmatrix} \begin{pmatrix} -10 \\ 3 \end{pmatrix}$

(b) $\begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} -7 \\ -23/3 \end{pmatrix}$

6.45 $\begin{array}{c} \hat{i} \\ \hat{j} \end{array} \begin{pmatrix} 5 \\ 1 \end{pmatrix}$

6.47(a) $\begin{array}{c} \hat{i} \\ \hat{j} \end{array} \begin{pmatrix} -24 \\ -3 \end{pmatrix} = \begin{array}{c} \\ \hat{i} \\ \hat{j} \end{array} \begin{array}{c} \vec{A} \quad \vec{B} \\ \begin{pmatrix} 1 & -4 \\ 2 & 1 \end{pmatrix} \end{array} \quad \begin{array}{c} \vec{A} \\ \vec{B} \end{array} \begin{pmatrix} a \\ b \end{pmatrix}$

(b) $a = -4$, $b = 5$

6.49(a) $\begin{pmatrix} 4x \\ -y \end{pmatrix}$

6.51(a) **i** $\begin{pmatrix} 6 \\ -5 \end{pmatrix}$

iii $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$

6.53(e) The natural basis consists of the vectors $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

(f) $\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$

6.55(a) $a = \dfrac{V_x B_y - V_y B_x}{A_x B_y - A_y B_x}$, $b = \dfrac{V_x A_y - V_y A_x}{A_x B_y - A_y B_x}$

(b) $B_y / B_x = A_y / A_x$

6.59(b) $\hat{j}$

(c) $2\hat{i} + 1\hat{k}$

6.61 (a) position and velocity

(b) 4

6.63 (a) $\begin{array}{c} x_1 \\ x_2 \end{array}\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{array}{c} x_1 \\ x_2 \end{array}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

(b) $\begin{array}{c} A \\ B \end{array}\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{array}{c} A \\ B \end{array}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

6.65 $x_1 = a + b$, $x_2 = (2a - b)/3$

6.67 (a) $\mathbf{B} = \begin{array}{c} \\ \textit{chocolates} \\ \textit{flowers} \\ \textit{cards} \end{array} \begin{array}{c} \begin{array}{cc} \textit{regular} & \textit{deluxe} \end{array} \\ \begin{pmatrix} 10 & 20 \\ 3 & 5 \\ 1 & 1 \end{pmatrix} \end{array}$

(b) $\mathbf{S} = \begin{array}{c} \\ \textit{regular} \\ \textit{deluxe} \end{array} \begin{array}{c} \begin{array}{cc} \textit{drug} & \textit{grocery} \end{array} \\ \begin{pmatrix} 20 & 50 \\ 10 & 30 \end{pmatrix} \end{array}$

(c) $\mathbf{BS} = \begin{array}{c} \\ \textit{chocolates} \\ \textit{flowers} \\ \textit{cards} \end{array} \begin{array}{c} \begin{array}{cc} \textit{drug} & \textit{grocery} \end{array} \\ \begin{pmatrix} 400 & 1,100 \\ 110 & 300 \\ 30 & 80 \end{pmatrix} \end{array}$

(d) You have 8,200 chocolates, 2,240 flowers and 600 cards.

6.69 (a) $\mathbf{F} = \begin{array}{c} \\ \textit{sportscars} \\ \textit{minivans} \\ \textit{smartcars} \end{array} \begin{array}{c} \begin{array}{cc} \textit{young} & \textit{old} \end{array} \\ \begin{pmatrix} 2 & 1 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} \end{array}$

(b) $\mathbf{C} = \begin{array}{c} \\ \textit{cylinders} \\ \textit{seats} \end{array} \begin{array}{c} \begin{array}{ccc} \textit{sportscars} & \textit{minivans} & \textit{smartcars} \end{array} \\ \begin{pmatrix} 6 & 4 & 3 \\ 4 & 7 & 2 \end{pmatrix} \end{array}$

(c) **CF**

(d) $\mathbf{CF} = \begin{array}{c} \\ \textit{cylinders} \\ \textit{seats} \end{array} \begin{array}{c} \begin{array}{cc} \textit{young} & \textit{old} \end{array} \\ \begin{pmatrix} 12 & 13 \\ 8 & 13 \end{pmatrix} \end{array}$

6.71 $\begin{pmatrix} 33 & 65 \\ 23 & 43 \end{pmatrix}$

6.73 $\begin{pmatrix} 0 & 6 & 7 & 0 \\ 0 & -11 & 0 & 0 \end{pmatrix}$

6.75 (44)

6.77 $\begin{pmatrix} 3 & 1 & 0 \\ 5 & 2 & 4 \\ 10 & 9 & 8 \end{pmatrix}$

6.79 $\begin{pmatrix} 0 & 1 & 3 \\ 4 & 2 & 5 \\ 8 & 9 & 10 \end{pmatrix}$

6.83(a) $\begin{pmatrix} 10 & -4 \\ 2 & 16 \\ 8 & 16 \end{pmatrix}$

(b) $\begin{pmatrix} 10 & -4 \\ 2 & 16 \\ 8 & 16 \end{pmatrix}$

6.85(b) $\begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}$

(c) stretches by 3 in the x-direction

(d) stretches by 3 in the y-direction

(e) stretches by 3 both in x and the y-directions

(f) diagonal stretch

6.89(a) $M_{11} = 2$, $M_{21} = 1$

(b) $M_{12} = -2$, $M_{22} = 2$

6.91(a) $\begin{pmatrix} 9 & 51 \\ -84 & 96 \\ -35 & 53 \end{pmatrix}$

(b) $\begin{pmatrix} 9 & 51 \\ -84 & 96 \\ -35 & 53 \end{pmatrix}$

6.93(a) $\mathbf{X} = \begin{array}{c c} & \begin{array}{cc} \textit{basic} & \textit{deluxe} \end{array} \\ \begin{array}{c} \textit{regular} \\ \textit{edge} \end{array} & \begin{pmatrix} 1,100 & 3,140 \\ 250 & 670 \end{pmatrix} \end{array}$

(b) $\begin{array}{c} \textit{regular} \\ \textit{edge} \end{array} \begin{pmatrix} 117,800 \\ 25,900 \end{pmatrix}$

(c) $\mathbf{X}^{-1} = \begin{pmatrix} -67/4800 & 157/2400 \\ 1/192 & -11/480 \end{pmatrix}$

(e) 52 basic, 73 deluxe

6.97(b) $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$

(d) $\begin{pmatrix} 1/2 & 0 \\ 0 & 1 \end{pmatrix}$

6.101(d) 4

6.103(a) $\mathbf{I}$

(b) $\mathbf{X}$

(c) $\mathbf{X}$

6.105(a) $\begin{array}{c c} & \begin{array}{cc} \textit{swank} & \textit{trashy} \end{array} \\ \begin{array}{c} \textit{vampires} \\ \textit{zombies} \end{array} & \begin{pmatrix} 50 & 3 \\ 3 & 200 \end{pmatrix} \end{array}$

6.107(a) $\mathbf{A} = \begin{pmatrix} 3 & 2 \\ -12 & 5 \end{pmatrix}$, $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 6 \\ 28 \end{pmatrix}$

(b) left

(c) $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$

(d) $\begin{pmatrix} -2/3 \\ 4 \end{pmatrix}$

(e) $x = -2/3$, $y = 4$

6.109 $x = -23/3$, $y = 10/9$

6.113(a) $\mathbf{B}^{-1}\mathbf{A}^{-1}\mathbf{D}$

(b) $\mathbf{D}\mathbf{C}^{-1}\mathbf{B}^{-1}$

(c) $\mathbf{A}^{-1}\mathbf{D}\mathbf{C}^{-1}$

6.115(a) $\begin{pmatrix} 1/3 & 1 \\ 2/3 & -1 \end{pmatrix}$

(c) $x_1(t) = -(1/3)\cos(\sqrt{2}t) + (25/3)\cos(\sqrt{5}t)$, $x_2(t) = -(2/9)\cos(\sqrt{2}t) - (25/9)\cos(\sqrt{5}t)$

6.117(a) $\begin{array}{c} \\ \hat{i} \\ \hat{j} \end{array}\begin{array}{c} \begin{matrix} \vec{P}_1 & \vec{P}_2 \end{matrix} \\ \begin{pmatrix} 3 & 2 \\ -4 & 1 \end{pmatrix} \end{array}$

(b) $\begin{array}{c} \\ \hat{i} \\ \hat{j} \end{array}\begin{array}{c} \begin{matrix} \vec{Q}_1 & \vec{Q}_2 \end{matrix} \\ \begin{pmatrix} 1 & 0 \\ 5 & 4 \end{pmatrix} \end{array}$

(c) $\begin{array}{c} \\ \vec{Q}_1 \\ \vec{Q}_2 \end{array}\begin{array}{c} \begin{matrix} \hat{i} & \hat{j} \end{matrix} \\ \begin{pmatrix} 1 & 0 \\ -5/4 & 1/4 \end{pmatrix} \end{array}$

(d) $\begin{array}{c} \\ \vec{Q}_1 \\ \vec{Q}_2 \end{array}\begin{array}{c} \begin{matrix} \vec{P}_1 & \vec{P}_2 \end{matrix} \\ \begin{pmatrix} 3 & 2 \\ -19/4 & -9/4 \end{pmatrix} \end{array}$

(e) $\begin{array}{c} \vec{Q}_1 \\ \vec{Q}_2 \end{array}\begin{pmatrix} 1 \\ -5/2 \end{pmatrix}$

6.119(a) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

(b) Matrix $\mathbf{X}$ interchanges the axes. If you apply this transformation twice, you get your original points back. Every point (x, y) first becomes (y, x), then again (x, y).

6.123(a) $\begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} w & y \\ x & z \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

(b) $aw + cx = 1$, $bw + dx = 0$, $ay + cz = 0$, $by + dz = 1$

(c) $w = d/(ad - bc)$, $x = b/(bc - ad)$, $y = c/(bc - ad)$, $z = a/(ad - bc)$

(d) $\dfrac{1}{ad - bc}\begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$

(e) $ad - bc = 0$

6.125(a) $x = y = 0$

(b) $x = y = 0$

(c) $x = y = 0$

(e) $\begin{pmatrix} 2 & 1 \\ 3 & 5-\lambda \end{pmatrix}$

(f) $7-2\lambda$

(g) $7/2$

6.127 4

6.129 -46

6.131 unique solution, not homogeneous

6.133 inconsistent or linearly dependent, not homogeneous

6.135 unique solution, not homogeneous

6.139 -3

6.141 1 and 2

6.145(a) 12

(b) $aei+bfg+cdh-ceg-bdi-afh$

(c) $aei-ahf-bdi+bfg+cdh-ceg$

6.149(a) yes

(b) no

(c) no

6.153 $a=2$, $b=6$, $c=1$

6.155 Inconsistent or linearly dependent.

6.157 45 mol/min C_2H_4 and 10 mol/min CH_4

6.159 Inconsistent or linearly dependent

6.161 $x_1 = A\cos(\sqrt{2}t)+B\cos(\sqrt{6}t)$, $x_2 = -(1/2)A\cos(\sqrt{2}t)+(1/2)B\cos(\sqrt{6}t)$

6.163(b) $50\vec{V}_1$

(c) $-10\vec{V}_2$

(d) $30\vec{V}_1+5\vec{V}_2$

(e) $5a\vec{V}_1+b\vec{V}_2$

6.165(a) not an eigenvector

(b) $\lambda=1$

(c) $\lambda=5$

(d) not an eigenvector

6.171(a) $\lambda_1 = 4+i$

(b) $\begin{pmatrix} 3-i \\ 5 \end{pmatrix}$

6.173(c) 2, 2, 1

6.175 $\vec{V}_1 = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ with $\lambda_1 = 10$, $\vec{V}_2 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ with $\lambda_2 = -1$

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6.177 $\vec{V}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ with $\lambda_1 = 8$, $\vec{V}_2 = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$ with $\lambda_2 = 1$

6.179 only one eigenvector

6.181(a) $\begin{pmatrix} 8 & 0 \\ 0 & 3 \end{pmatrix}$

(c) $\begin{pmatrix} 16/5 \\ -9/5 \end{pmatrix}$

6.183 $\begin{pmatrix} -146/7 & 130/7 \\ -156/7 & 153/7 \end{pmatrix}$

6.185(a) $\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = \begin{pmatrix} -8 & -8 \\ -1 & -6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

(b) $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$ for $\lambda = -4$, $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$ for $\lambda = -10$.

(c) $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} A\cos(2t) + B\sin(2t) \\ (-1/2)A\cos(2t) - (1/2)B\sin(2t) \end{pmatrix}$

(d) $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} C\cos(\sqrt{10}t) + D\sin(\sqrt{10}t) \\ (1/4)C\cos(\sqrt{10}t) + (1/4)D\sin(\sqrt{10}t) \end{pmatrix}$

(e) $x_1 = A\cos(2t) + B\sin(2t) + C\cos(\sqrt{10}t) + D\sin(\sqrt{10}t)$
$x_2 = (-1/2)A\cos(2t) - (1/2)B\sin(2t) + (1/4)C\cos(\sqrt{10}t) + (1/4)D\sin(\sqrt{10}t)$

(f) $x_1 = -\cos(2t) + 2\cos(\sqrt{10}t)$, $x_2 = (1/2)\cos(2t) + (1/2)\cos(\sqrt{10}t)$

6.187 $x_1 = -(1/9)e^{2t} + (10/9)e^{11t}$ and $x_2 = (1/3)e^{2t} + (5/3)e^{11t}$

6.189 $x_1 = -\sin(\sqrt{2}t) + (1/\sqrt{14})\left(e^{\sqrt{7}t} - e^{-\sqrt{7}t}\right)$, $x_2 = (2/3)\sin(\sqrt{2}t)$
$+ 1/(3\sqrt{14})\left(e^{\sqrt{7}t} - e^{-\sqrt{7}t}\right)$

6.193(a) $\begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix} = \begin{pmatrix} -6 & 1 \\ -4 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

(b) Eigenvector $\begin{pmatrix} 1 \\ 4 \end{pmatrix}$ has eigenvalue -2, eigenvector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ has eigenvalue -5.

(c) $\ddot{x} = -5x$ when $y = x$

(d) $y = x = A\cos(\sqrt{5}\,t) + B\sin(\sqrt{5}\,t)$

(e) $x = C\cos(\sqrt{2}\,t) + D\sin(\sqrt{2}\,t)$ and $y = 4x$

(f) $x = A\cos(\sqrt{5}\,t) + B\sin(\sqrt{5}\,t) + C\cos(\sqrt{2}\,t) + D\sin(\sqrt{2}\,t)$
$y = A\cos(\sqrt{5}\,t) + B\sin(\sqrt{5}\,t) + 4C\cos(\sqrt{2}\,t) + 4D\sin(\sqrt{2}\,t)$

6.195(a) $\begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix} = \begin{pmatrix} -3 & -4 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

(b) Eigenvector $\begin{pmatrix} 1 \\ 1/2 \end{pmatrix}$ has eigenvalue -5, eigenvector $\begin{pmatrix} 1 \\ -3/2 \end{pmatrix}$ has eigenvalue 3.

(c) $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = f(t)\begin{pmatrix} 1 \\ 1/2 \end{pmatrix} + g(t)\begin{pmatrix} 1 \\ -3/2 \end{pmatrix}$

(d) $\ddot{f} = -5f$, $\ddot{g} = 3g$

(e) $f(t) = A\sin(\sqrt{5}t) + B\cos(\sqrt{5}t)$, $g(t) = Ce^{\sqrt{3}t} + De^{-\sqrt{3}t}$

(f) $x(t) = A\sin(\sqrt{5}t) + B\cos(\sqrt{5}t) + Ce^{\sqrt{3}t} + De^{-\sqrt{3}t}$, $y(t) = (1/2)A\sin(\sqrt{5}t) + (1/2)B\cos(\sqrt{5}t) - (3/2)Ce^{\sqrt{3}t} - (3/2)De^{-\sqrt{3}t}$

6.197 $x(t) = A\sin(\sqrt{5}t) + B\cos(\sqrt{5}t) + Ce^{\sqrt{3}t} + De^{-\sqrt{3}t}$, $y(t) = (1/2)A\sin(\sqrt{5}t) + (1/2)B\cos(\sqrt{5}t) - (3/2)Ce^{\sqrt{3}t} - (3/2)De^{-\sqrt{3}t}$

6.199 $x(t) = (7/4)\cos t - (3/4)\cos(\sqrt{5}\ t)$, $y(t) = -(7/2)\cos t - (3/2)\cos(\sqrt{5}\ t)$

6.201 $x(t) = (3/5)e^{6t} + (7/5)e^{t}$, $y(t) = (12/5)e^{6t} - (7/5)e^{t}$

6.205 $\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \lambda = -1, -4$

6.207 $\dot{f} = -5f$, $\dot{g} = 2g$

6.209(a) $\dot{v} = -7x - 6y$

6.211 $\begin{pmatrix} A\sin(\sqrt{10}\ t) + B\cos(\sqrt{10}\ t) + C\sin(\sqrt{10/3}\ t) + D\cos(\sqrt{10/3}\ t) \\ -A\sin(\sqrt{10}\ t) - B\cos(\sqrt{10}\ t) + C\sin(\sqrt{10/3}\ t) + D\cos(\sqrt{10/3}\ t) \end{pmatrix}$

6.213 $\begin{pmatrix} A\sin(\sqrt{6}\ t) + B\cos(\sqrt{6}\ t) + C\sin(\sqrt{11}\ t) + D\cos(\sqrt{11}\ t) \\ 2A\sin(\sqrt{6}\ t) + 2B\cos(\sqrt{6}\ t) - (C/2)\sin(\sqrt{11}\ t) - (D/2)\cos(\sqrt{11}\ t) \end{pmatrix}$

6.215 $\begin{pmatrix} A\sin(\sqrt{12}\ t) + B\cos(\sqrt{12}\ t) + C\sin(\sqrt{10}\ t) + D\cos(\sqrt{10}\ t) + E\sin(2t) + F\cos(2t) \\ -A\sin(\sqrt{12}\ t) - B\cos(\sqrt{12}\ t) + 3E\sin(2t) + 3F\cos(2t) \\ A\sin(\sqrt{12}\ t) + B\cos(\sqrt{12}\ t) - C\sin(\sqrt{10}\ t) - D\cos(\sqrt{10}\ t) + E\sin(2t) + F\cos(2t) \end{pmatrix}$

6.217(a) $2\ddot{\theta}_1 + \ddot{\theta}_2 + 2(g/L)\theta_1 = 0$, $\ddot{\theta}_2 + \ddot{\theta}_1 + (g/L)\theta_2 = 0$

(b) $\ddot{\theta}_1 = -2(g/L)\theta_1 + (g/L)\theta_2$, $\ddot{\theta}_2 = 2(g/L)\theta_1 - 2(g/L)\theta_2$

(c) $\omega = \sqrt{(2 \pm \sqrt{2})(g/L)}$

6.219(b) In one normal mode $I_2 = I_1$ oscillating at a frequency $\sqrt{(1/L)(1/C_1 + 2/C_2)}$. In the other normal mode $I_2 = -I_1$ oscillating at frequency $\sqrt{1/LC_1}$.

6.221 $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

6.223(a) Yes

(c) $\begin{array}{cc} & \begin{array}{cc} a & b \end{array} \\ \begin{array}{c} x \\ y \end{array} & \begin{pmatrix} -1 & 2 \\ -1 & 0 \end{pmatrix} \end{array}$

(d) $\begin{array}{cc} & \begin{array}{cc} x & y \end{array} \\ \begin{array}{c} a \\ b \end{array} & \begin{pmatrix} 0 & -1 \\ 1/2 & -1/2 \end{pmatrix} \end{array}$

(e) $\begin{pmatrix} -4 \\ -1/2 \end{pmatrix}$

6.225(a) $\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{pmatrix}$

(b) $\begin{pmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & -1 & 0 \end{pmatrix}$

(c) $\begin{pmatrix} x/\sqrt{x^2+y^2} & -y/\sqrt{x^2+y^2} & 0 \\ y/\sqrt{x^2+y^2} & x/\sqrt{x^2+y^2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$

6.229(a) $I_3 = I_1 + I_2$

(e) $I_1(t) = (5/2)e^{-4\times10^{-6}t} + (1/2)e^{-2\times10^{-6}t}$, $I_2(t) = (5/2)e^{-4\times10^{-6}t} - (1/2)e^{-2\times10^{-6}t}$

Chapter 7

7.1(a) $\begin{pmatrix} 8 & 0 \\ 0 & 3 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

(c) $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

(d) $\begin{pmatrix} 0 & -3 \\ -8 & 0 \end{pmatrix}$

(e) $\begin{pmatrix} 0 \\ -8 \end{pmatrix}$

(f) $\begin{pmatrix} 0 & 8 \\ 3 & 0 \end{pmatrix}$

(g) $\begin{pmatrix} 0 \\ 3 \end{pmatrix}$

(h) $|\mathbf{M}| = |\mathbf{N}| = -24$

7.3(b) rotates 90° clockwise

(c) reflects about $y = x$

(d) reflects about the x-axis

(e) reflects about the y-axis

(f) reflects about the x-axis and stretches by a factor of 2 in the y-direction

(g) $|\mathbf{A}| = 1$, $|\mathbf{B}| = -1$, $|\mathbf{C}| = -2$

7.11(a) 2, with eigenvalues 1 and -1

(b) 3, with eigenvalues 1, 1, and -1

(c) 2, with eigenvalues 1 and -1

(d) 3, all with eigenvalue 1

(e) 2, both with eigenvalue 1

(f) 1, with eigenvalue -1

7.13(c) Matrix $\mathbf{M}$ stretches vectors by a factor of 6 in the $\hat{i} + 2\hat{j}$ direction.

7.15 counterclockwise rotation of 30°

7.17 stretch by a factor of 6 along $6\hat{i} - 5\hat{j}$, and 3 along $\hat{i} - \hat{j}$

7.21 $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$

7.23(a) $6\hat{i} - 15\hat{j}$

(b) yes, 3.

(c) $6\hat{i} - 5\hat{k}$

(d) no

7.27(a) It's a cube of side length 1 with two opposite corners at the origin and $(1, 1, 1)$.

(b) $\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

(c) $\begin{pmatrix} 0 & -1 & -1 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 2 & 2 & 2 & 2 & 2 & 0 & 0 & 2 & 2 & 0 & 0 \end{pmatrix}$

7.31(a) same orientation

(b) **i** same orientation

ii opposite orientation

iii opposite orientation

(c) $|\mathbf{T}_1| = 1$, $|\mathbf{T}_2| = 4$, $|\mathbf{T}_3| = -1$, $|\mathbf{T}_4| = -3$

7.41(a) $\begin{pmatrix} 5/4 & \sqrt{3}/4 \\ \sqrt{3}/4 & 7/4 \end{pmatrix}$

(b) $\begin{pmatrix} 1/\sqrt{3} \\ 1 \end{pmatrix}$ with eigenvalue 2, $\begin{pmatrix} -1 \\ 1/\sqrt{3} \end{pmatrix}$ with eigenvalue 1.

7.43 $\dfrac{1}{4}\begin{pmatrix} 13+5\sqrt{3} & -7+3\sqrt{3} \\ -3+3\sqrt{3} & 7-5\sqrt{3} \end{pmatrix}$

7.45(a) $\begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix}$

(b) $\begin{pmatrix} 7/2 & -1/2 \\ -1/2 & 7/2 \end{pmatrix}$

7.49 $\begin{pmatrix} 1/2+\cos(\pi/9) & -1/2+\cos(\pi/9) & -\sin(\pi/9)/\sqrt{2} \\ -1/2+\cos(\pi/9) & 1/2+\cos(\pi/9) & -\sin(\pi/9)/\sqrt{2} \\ \sqrt{2}\sin(\pi/9) & \sqrt{2}\sin(\pi/9) & \cos(\pi/9) \end{pmatrix}$

7.51(a) $\vec{F}_g = -mg\hat{j}$, $\vec{F}_{fr} = |F_{fr}|\cos\theta\hat{i} + |F_{fr}|\sin\theta\hat{j}$,
$\vec{F}_N = -|F_N|\sin\theta\hat{i} + |F_N|\cos\theta\hat{j}$, $\vec{F}_T = -|F_T|\cos 45°\hat{i} + |F_T|\sin 45°\hat{j}$

(b) $\vec{F}_g = -mg\sin\theta\hat{i}' - mg\cos\theta\hat{j}'$, $\vec{F}_{fr} = |F_{fr}|\hat{i}'$, $\vec{F}_N = |F_N|\hat{j}'$, $\vec{F}_T = -|F_T|\cos(45° + \theta)\hat{i}' + |F_T|\sin(45° + \theta)\hat{j}'$

(c) $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

7.53(a) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(b) The identity matrix **I**

7.55(a) $\begin{pmatrix} 0 & 2ay \\ 0 & 0 \end{pmatrix}$

(b) $\mathbf{L}' = \dfrac{ay}{2}\begin{pmatrix} \sqrt{3} & 3 \\ -1 & -\sqrt{3} \end{pmatrix}$

(c) $\mathbf{L}' = \dfrac{a}{4}\begin{pmatrix} \sqrt{3}x' + 3y' & 3x' + 3\sqrt{3}y' \\ -x' - \sqrt{3}y' & -\sqrt{3}x' - 3y' \end{pmatrix}$

(d) $\vec{v}\,' = (a/8)\left(x'^2 + 2\sqrt{3}x'y' + 3y'^2\right)(\sqrt{3}\hat{i}' - \hat{j}')$

7.57(a) $\mathbf{C} = \begin{pmatrix} \gamma & -\gamma v/c^2 & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

(b) $\mathbf{C}^{-1} = \begin{pmatrix} \gamma & \gamma v/c^2 & 0 & 0 \\ \gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

7.59(a) $\mathbf{M} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

(b) $\mathbf{M}' = \begin{pmatrix} 2-1/\sqrt{2} & (\sqrt{2}-1)/c & 1/(c\sqrt{2}) & 0 \\ (1-\sqrt{2})c & \sqrt{2}-1 & -1 & 0 \\ c/\sqrt{2} & 1 & 1/\sqrt{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

7.61(a) $\vec{r}_A = (t,0,0,0)$, $\vec{r}_B = (t,L,0,0)$

(b) γL

(c) $\Delta t' = \gamma\tau$, $\Delta x' = \gamma v\tau$

7.71 No

7.77(a) Hermitian but not unitary

(b) Unitary but not Hermitian

(c) Not Hermitian or unitary

7.79(a) $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

(c)
i Yes
ii No.
iii No.
iv No.
v No.
vi Yes.

7.81(b) $(1/\sqrt{2})(\hat{i}+\hat{j})$

(c) $(1/\sqrt{11})(\hat{i}-\hat{j}-3\hat{k})$

(d) $(1/\sqrt{22})(-3\hat{i}+3\hat{j}-2\hat{k})$

(f) $\vec{e}_1 = (1/\sqrt{11})(\hat{i}+\hat{j}-3\hat{k})$, $\vec{e}_2 = (1/\sqrt{990})(14\hat{i}+25\hat{j}+13\hat{k})$

7.85(a) $\begin{pmatrix} 2 & 3 & 1 & 1 & 2 \\ -2 & -1 & 2 & 4 & -21 \\ 8 & 20 & 7 & 4 & 10 \\ 2 & 15 & 10 & 11 & -34 \end{pmatrix}$

(b) $\begin{pmatrix} 2 & 3 & 1 & 1 & 2 \\ 0 & 2 & 3 & 5 & -19 \\ 8 & 20 & 7 & 4 & 10 \\ 2 & 15 & 10 & 11 & -34 \end{pmatrix}$

(c) $\begin{pmatrix} 2 & 3 & 1 & 1 & 2 \\ 0 & 2 & 3 & 5 & -19 \\ 0 & 8 & 3 & 0 & 2 \\ 2 & 15 & 10 & 11 & -34 \end{pmatrix}$

(d) $\begin{pmatrix} 2 & 3 & 1 & 1 & 2 \\ 0 & 2 & 3 & 5 & -19 \\ 0 & 8 & 3 & 0 & 2 \\ 0 & 12 & 9 & 10 & -36 \end{pmatrix}$

(e) $\begin{pmatrix} 2 & 3 & 1 & 1 & 2 \\ 0 & 2 & 3 & 5 & -19 \\ 0 & 0 & -9 & -20 & 78 \\ 0 & 0 & -9 & -20 & 78 \end{pmatrix}$

(f) $\begin{pmatrix} 2 & 3 & 1 & 1 & 2 \\ 0 & 2 & 3 & 5 & -19 \\ 0 & 0 & -9 & -20 & 78 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

(h) Rank 3, linearly dependent equations, you could solve for three variables in terms of the fourth.

7.87 Rank 2, $x = 5/2$, $y = 1/2$

7.89 Rank 2, linearly dependent, you could solve for two variables in terms of the third.

7.91 Rank 3, Inconsistent

7.93 Rank 4, $x = -3$, $y = 4$, $z = 0$, $t = 8$

7.95 Rank 3, linearly dependent, we can solve for three variables in terms of the fourth.

7.97 Rank 4, $a = -3$, $b = 4$, $c = 1$, $d = 2$

7.99(b) $-3C + 2D + 4F = 2A$, $-2C + E + 2F = A$, $-5C + 4D + 6F = 2B$, $-7C + 4E + 6F = 2B$

(f) measure 3, calculate 3

7.103 $f = 2$

7.105 $f = 4$

7.107(a) 2

(b) 7 basic, 2 nonbasic

(c) 3

(d) m basic, 3 nonbasic

7.111 $f = 2$

7.113 $f = 1$

7.115 $f = 2$

7.117 $f = 2$

7.119 $f = -5/2$

7.121(a) $S_1 = 5$, $S_2 = 6$, $S_3 = 1$ and $f = 3$.

(b) $S_1 = 203$, $S_2 = 303$, $S_3 = 1$ and $f = 102$

(c) f increases without bound.

(d) S_3 becomes negative, which is not allowed.

7.125 $f = 3/2$

7.127 $f = 4$

7.129 $f = 12$

7.131 $f = 18$

7.137(c) $\begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix}$

(f) $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

(h) Rotate 90° clockwise

(i) $-\hat{j}$, $-(1/2)\hat{i} - \hat{j}$, $-(1/2)\hat{i} + \hat{j}$

(j) $\begin{pmatrix} 0 & -1/2 \\ -1 & 0 \end{pmatrix}$

(k) $\begin{pmatrix} 0 & -1 \\ -2 & 0 \end{pmatrix}$

7.139(a) $\begin{pmatrix} -1 & -2 \\ 2 & 1 \end{pmatrix}$

(b) $\begin{pmatrix} -3 & -1 \\ -1 & -3 \end{pmatrix}$

(c) $\mathbf{B} = \mathbf{CA}$

(d) $\mathbf{C} = \mathbf{BA}^{-1}$

(e) $\mathbf{C} = \dfrac{1}{3}\begin{pmatrix} -1 & -5 \\ 5 & 1 \end{pmatrix}$

7.141(c) $\begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \\ 1 & 1 & 1 \end{pmatrix}$

(d) adds X to all x coordinates, Y to y coordinates and Z to z coordinates.

(e) yes

7.143(a) $\dfrac{1}{2}\begin{pmatrix} \lambda_1 + \lambda_2 & \lambda_1 - \lambda_2 \\ \lambda_1 - \lambda_2 & \lambda_1 + \lambda_2 \end{pmatrix}$

(b) yes

(c) 2

7.147(a) $\mathbf{I} = \begin{pmatrix} 2/3 & -1/4 & -1/4 \\ -1/4 & 2/3 & -1/4 \\ -1/4 & -1/4 & 2/3 \end{pmatrix} ML^2$

(b) $\left[((2/3)a - (1/4)b)\,\hat{i} + (-(1/4)a + (2/3)b)\,\hat{j} + (-(1/4)a - (1/4)b)\,\hat{k}\right] ML^2$

Chapter 8

8.1(a) 1

(b) 2

(c) $\lim_{x\to\infty} \sigma(x) = \infty$

(d) $\lim_{x\to-\infty} \sigma(x) = \infty$

(e) -1

(f) $\lim_{y\to\infty} \sigma(x) = 0$

(g) $\lim_{y\to-\infty} \sigma(x) = \infty$

(j) $(16/3)(1-1/e)$

8.5(a) ii.

(c) force in the $-x$ direction

8.7(a) North

8.9(a) Northeast

(b) Southwest

8.13(a) planes

(b) planes

(c) Cylinders centered around the origin

(d) Spheres centered on the origin

(e) Ellipsoids centered on the origin, shorter in the z-direction

8.23(a) $2\pi r$

(b) $n/2\pi r$

8.29(a) negative

(b) positive

8.31 Speed up to the left

8.37(b) $x = \pi/2, 3\pi/2, 5\pi/2, 7\pi/2$

(c) $x = 3\pi/2$ and $7\pi/2$

(d) $x = \pi/2$, and $5\pi/2$

8.51(a) left

(b) left

(d) positive charge left, negative charge right

8.53(a) $\vec{g} = \left(-GM/(x^2+y^2+z^2)^{3/2}\right)(x\hat{i} + y\hat{j} + z\hat{k})$

8.55 0

8.57 $\vec{0}$

8.59(a) In the direction of $3\hat{i} - 2\hat{j}$ or about 34° below the positive x-axis

(b) $y = 3x + 1$

(c) neither up nor down

8.61(a) $\vec{\nabla}P(x, y, z) = -\hat{k}$

8.65(b) $V = k/\sqrt{x^2+y^2} - k/\sqrt{(x-5)^2+y^2}$

(c) $\vec{E} = \left[kx/(x^2+y^2)^{3/2} - k(x-5)/((x-5)^2+y^2)^{3/2}\right]\hat{i} + \left[ky/(x^2+y^2)^{3/2} - ky/((x-5)^2+y^2)^{3/2}\right]\hat{j}$

8.69(a) $-1/3$

(b) $-4/3$

(c) $-(1/3)y^3 - x^3 y$

(e) different by a constant

8.71 $F = -x^2 - \sin y$

8.73 $F = -\ln|x|$

8.75 $F = -\ln\left|3x+y^2\right|$

8.83(a) $2/(2x+3y) + 2x/z + (x+1)y$

(b) $5/3$

(d) $((x+1)z + 2xy/z^2)\hat{i} - yz\hat{j} + (2y/z - 3/(2x+3y))\hat{k}$

(e) $2\hat{i} - 2\hat{j}$

8.85 divergence=0; curl is out of the page

8.87 divergence is negative; curl=0

8.89 divergence is positive; curl=0

8.93 $\vec{\nabla}\cdot\vec{L} = 1$, $\vec{\nabla}\times\vec{L} = \vec{0}$

8.95 $\vec{\nabla}\cdot\vec{f} = 4$, $\vec{\nabla}\times\vec{f} = (3\pi - 48)\hat{i} + 3\hat{j} - 2\hat{k}$

8.97 $\vec{\nabla}\cdot\vec{h} = 1/4$, $\vec{\nabla}\times\vec{h} = -404.75\,\hat{j} + (3/16)\hat{k}$

8.99(b) 8

(c) -8

(d) 0

8.101(a) $\vec{E} = kQ(x/(x^2+y^2+z^2)^{3/2}\hat{i} + y/(x^2+y^2+z^2)^{3/2}\hat{j} + z/(x^2+y^2+z^2)^{3/2}\hat{k})$

(c) 0

(e) 0

8.103 Laplacian is a scalar

8.109 positive: $|x| > |y|$; negative: $|x| < |y|$; 0: $x = \pm y$

8.111(a) $df_x/dx = 0$

(b) $dg_x/dx = 2$

(c) $dh_x/dx = -1/x^2$

8.115(a) $4\pi kq$

(b) $(3kq)/R^3$

(c) ∞

8.117(b) $<-\sqrt{3}/2, 1/2>$

(c) $<-\sqrt{3}/2, 1/2>$

8.119 $\vec{\nabla}\cdot\vec{f}(1,0) = 0$

8.121 $\vec{\nabla}\cdot\vec{f}(1,0) = 2$

8.123 $\vec{\nabla}f = (2+3\phi)\hat{\rho} + 3\hat{\phi}$, $\nabla^2 f = (2+3\phi)/\rho$

8.125 $\vec{\nabla}f = 2\rho\hat{\rho} + 2z\hat{k}$, $\nabla^2 f = 6$

8.127 $\vec{\nabla} f = (\theta+\phi)\hat{r} + \hat{\theta} + (1/\sin\theta)\hat{\phi}$, $\nabla^2 f = 2(\theta+\phi)/r + \cos\theta/(r\sin\theta)$

8.129 $\vec{\nabla} f = (\tan\theta/r)\hat{\theta} + \cos\phi/(r\sin\theta\sin\phi)\hat{\phi}$,
$\nabla^2 f = -1/(r^2\sin^2\theta\sin^2\phi) + 1/r^2 + 1/(r^2\cos^2\theta)$

8.131 $\vec{\nabla}\cdot\vec{f} = 1/\rho$, $\vec{\nabla}\times\vec{f} = 2\hat{k}$

8.133 $\vec{\nabla}\cdot\vec{f} = z^2/\rho + 2\rho\sin\phi\cos\phi$, $\vec{\nabla}\times\vec{f} = (2z-2\rho)\hat{\phi} + 3\rho\sin^2\phi\;\hat{k}$

8.135 $\vec{\nabla}\cdot\vec{f} = 0$, $\vec{\nabla}\times\vec{f} = (\cot\theta)/r\;\hat{r} - (1/r)\hat{\theta}$

8.137 $\vec{\nabla}\cdot\vec{f} = 3\cos\theta + \cos(2\theta)\csc\theta$, $\vec{\nabla}\times\vec{f} = \cos(2\theta)\csc\theta\hat{r} - 2\cos\theta\;\hat{\theta} + (2\cos\theta+\sin\theta)\hat{\phi}$

8.141 div$= 0$; curl$= \vec{0}$.

8.143(a) $f_z(\rho,\phi,z+\Delta z)\times\Delta\rho(\rho\Delta\phi)$

(b) $-f_z(\rho,\phi,z)\times\Delta\rho(\rho\Delta\phi)$

(c) $\partial f_z/\partial z$

(d) $(1/\rho)(\partial f_\phi/\partial\phi)$

8.147 0

8.149 $3L^3$

8.151 $(1/3)\pi^4(1/e - e)$

8.153 $(8/3)\pi R^3$

8.155 $(12/5)\pi R^5$

8.157(a) $kD_0R^3(4\pi)$

8.159 2π

8.161 $-1/2$

8.163 $2\pi R\ln R$

8.165 0

8.167 $1/2$

8.173(d) $V(\rho) = -[\lambda/(2\pi\varepsilon_0)]\ln\rho$, $V(1) = 0$

8.175 (a),(c),(d)

8.177 conservative

8.179 not conservative

8.181 not conservative

8.183 conservative

8.187(a) $kq/\left(x^2+y^2+z^2\right)^{3/2}\left(x\hat{i}+y\hat{j}+z\hat{k}\right)$

(b) $(kq/r^2)\hat{r}$

8.189(a) $\oint\vec{B}\cdot d\vec{s} = \mu_0 I$

8.191 $\vec{\nabla}\cdot\vec{f} = 2(3x-y+z)$; $\vec{\nabla}\times\vec{f} = 2(y-z)\hat{i} - 6x\hat{k}$

8.193 $\vec{\nabla}\cdot\vec{f} = \sin\phi - \rho^2/z^2$; $\vec{\nabla}\times\vec{f} = -(2\rho/z)\hat{\phi} + \cos\phi\;\hat{z}$

8.195 $\vec{\nabla}\cdot\vec{f} = 3\sin\theta - \sin\phi/\sin\theta$; $\vec{\nabla}\times\vec{f} = (\cos\phi\cos\theta/\sin\theta)\;\hat{r} - 2\cos\phi\;\hat{\theta} - \cos\theta\;\hat{\phi}$

8.197 no solution

8.199 no solution

8.201 $F = y + ze^x$

8.203 $(4/3)\pi R^3 k$

8.205(a) 0

(b) 0

8.207(b) square in xy-plane

(c) 0

8.209 0

8.211 $-2/3$

8.213 $4\pi/3$

8.215(a) $3H\rho(t) = -d\rho/dt$

(b) 3 Billion years ago

(c) $\rho(t) = C/t^2$

(d) 4.1 Billion years ago

Chapter 9

9.1 $f(x) \approx 6\sin(2x) + \sin(4\pi x)$

9.3(a) -2

(b) 6

(c) $1/3$

(d) $-1/3$

(e) odd

(f) 8

(g) 4

(h) neither

9.5(c) neither

9.15 $\displaystyle \frac{1}{2} + \sum_{odd\ n} \frac{2}{n\pi}\sin(nx)$

9.17 $\displaystyle \pi^2 + \sum_{n=1}^{\infty} \frac{(-1)^n 12}{n^2}\cos(nx) + \sum_{n=1}^{\infty} \frac{(-1)^n 2}{n}\sin(nx)$

9.19 $\displaystyle \frac{e^{\pi} - e^{-\pi}}{\pi}\left[\frac{1}{2} + \sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^2}(\cos(nx) - n\sin(nx))\right]$

9.23(a) 4π, 2π, $4\pi/3$, π

(b) 4π

9.27(a) $\displaystyle \frac{1}{4} + \frac{\sqrt{2}}{\pi}\cos(x) + \frac{1}{\pi}\cos(2x) + \frac{\sqrt{2}}{3\pi}\cos(3x) - \frac{\sqrt{2}}{5\pi}\cos(5x) - \frac{1}{3\pi}\cos(6x) - \frac{\sqrt{2}}{7\pi}\cos(7x)$

(b) $a + bS$ where $a = 1/4$ and $b = 1/\pi$

(c) $\pi/4$

9.29(a) odd

9.33(a) π

(b) π

(c) $1/2$ for both of them

9.35 $(1/\pi)\int_{-\pi}^{\pi} f(x)\cos(kx)dx$

9.37(a) 2π

(b) π

(c) 10π

(d) $2\pi/3$

(e) π

9.39(a) $\pi/3$

(b) $(2/3)\pi$

(c) period=$6/n$, $p = \pi n/3$ where n is any positive integer

(d) 6

9.41(a) period=6

(b) period=6

(c) period=3

(d) same periods

9.47 $\frac{e^5-1}{5}+\sum_{n=1}^{\infty}\frac{10(e^5-1)}{25+4n^2\pi^2}\cos\left(\frac{2n\pi}{5}x\right)+\sum_{n=1}^{\infty}\frac{4n\pi(1-e^5)}{25+4n^2\pi^2}\sin\left(\frac{2n\pi}{5}x\right)$

9.49 $\frac{3}{2}-\sum_{odd\ n}\frac{2}{n\pi}\sin\left(\frac{n\pi}{3}x\right)$

9.51 $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}4}{n\pi}\sin\left(\frac{n\pi}{2}x\right)$

9.53 Sine series: $\sum_{odd\ n}\frac{12}{n\pi}\sin(n\pi x)$, Cosine series: 3

9.59 $\sum_{n=1}^{\infty}\frac{27\sin(2n\pi/3)}{n^2\pi^2}\sin\left(\frac{n\pi}{65}x\right)$

9.61 $\sum_{odd\ n}-\frac{2i}{n\pi}e^{inx}$

9.63 $\frac{1}{2}+\sum_{odd\ n}\frac{2}{\pi^2n^2}e^{inx}$

9.65 $\frac{5}{2}+\sum_{n=-\infty}^{\infty}\frac{5i}{2n\pi}e^{i(2n\pi/5)x}$

9.67 $\frac{i}{3}+\sum_{n=-\infty}^{\infty}\frac{(-1)^n(2+n\pi)i}{n^2\pi^2}e^{in\pi x}$

9.71(b) c_n is imaginary and $c_{-n}=c_n^*$.

9.73(a) $2A\cos(n\pi x/L)-2B\sin(n\pi x/L)$

(b) $2\sqrt{A^2+B^2}$

9.75(a) $c_n=(-1)^n i/n\ (n\neq 0)$, $c_0=0$

(b) $\sum_{n=-\infty}^{\infty}|c_n|^2=2\sum_{n=1}^{\infty}(1/n^2)$

(c) $\pi^2/3$

(d) $\pi^2/6$

9.77 $\pi^4/90$

9.89 $t_0/[2\pi(1+it_0\omega)]$

9.93(b) 1

(c) e^{ix}

9.99(a) $\hat{f}_0=-1, \hat{f}_1==8-3i, \hat{f}_2=-7$

(b) $8+3i$

(c) $\hat{f}_{-2}=-7, \hat{f}_3=8+3i$

(d) $\hat{f}_{-1}$ and $\hat{f}_1$ together represent the frequency $\pi/4$. $\hat{f}_0$ represents the constant term, and $\hat{f}_2$ the frequency $\pi/2$.

9.101 $(18, 10+6i, 2, 10-6i, 18)$, $p_{min}=5\pi/6$, $p_{max}=5\pi/3$

9.103 $(2, 2+2i, 14, 2-2i, 2)$, $p_{min}=\pi/2$, $p_{max}=\pi$

9.105 $(8, -4+4\sqrt{3}\,i, 8+4\sqrt{3}\,i, -4, 8-4\sqrt{3}\,i, -4-4\sqrt{3}\,i, 8)$, $p_{min}=\pi/12$, $p_{max}=\pi/4$

9.117 $f(x,y)=(5/2)+\sum_{n=-\infty}^{\infty}(2i/n\pi)e^{in\pi y}+\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}(2/mn\pi^2)e^{im\pi x}e^{in\pi y}$ for odd m and n only

9.119 $$f(x) = \frac{2}{3} + \sum_{m=-\infty}^{\infty} \frac{2i}{3m\pi} e^{2m\pi ix} + \sum_{n=-\infty}^{\infty} \frac{1+in\pi}{n^2\pi^2} e^{n\pi iy} + \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \frac{i-n\pi}{n^2 m\pi^3} e^{2m\pi ix} e^{n\pi iy} \quad m, n \neq 0$$

9.121 $$\sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^{m+n}(e^{4\pi} - e^{-4\pi})(e^{2\pi} - e^{-2\pi})}{4\pi^2(4-im)(2+in)} e^{m\pi ix/2} e^{n\pi iy/2}$$

9.123(a) $z = (2H/W)y$, $z = (2H/L)x$, $z = 2H(1 - y/W)$, $z = 2H(1 - x/L)$

(b) odd

9.129 $\sin(2x) + \cos(3x)$, $(i/2)e^{-2ix} - (i/2)e^{2ix} + (1/2)e^{3ix} + (1/2)e^{-3ix}$

9.133 $$\frac{1}{2}(1-\cos 1) + \sum_{n=1}^{\infty} \frac{(\cos 2) - 1}{n^2\pi^2 - 1}\cos(2n\pi x) + \sum_{n=1}^{\infty}\left(\frac{8n}{\pi(4n^2-1)} - \frac{(\sin 2)n\pi}{n^2\pi^2-1}\right)\sin(2n\pi x),$$
$$\sum_{n=-\infty}^{\infty}\left(\frac{4in}{\pi(1-4n^2)} + \frac{\cos 2 + in\pi\sin 2 - 1}{2(n^2\pi^2-1)}\right)e^{2in\pi x}$$

9.137(b) $3 + \sum_{n=1}^{\infty}(6/n\pi)\sin(2n\pi x/3)$

(c) odd

(d) $3 - \sum_{n=-\infty}^{\infty}(3i/n\pi)e^{2in\pi x/3}$ $(n \neq 0)$

9.139(a) 2π

(b) 8π

(c) odd

(d) even

(e) $y(x,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} 16/(nm\pi^2)\sin(n\pi/2)\sin(mx)\cos(nt/4)$, odd m only

9.143 $$A\sin(2t) + B\cos(2t) + \frac{1}{4} + \sum_{\text{odd } n} \frac{4}{n\pi(4-n^2\pi^2)}\sin(n\pi t)$$

Chapter 10

10.1(a) $(d^2y/dx^2) + 3(dy/dx) + 2y = 0$

(b) $y_c = C_1 e^{-2x} + C_2 e^{-x}$

(c) $2Ax^2 + (6A + 2B)x + (2A + 3B + 2C) = 6x^2$

(e) $y_p = 3x^2 - 9x + 21/2$

(f) $y(x) = 3x^2 - 9x + 21/2 + C_1 e^{-2x} + C_2 e^{-x}$

10.3(a) $Ny = -1/x^3$

(e) $y = \sqrt{x + C}$

10.5(d) $y(x) = -5 \ln x/x + C_1 x + C_2/x$

10.7 $y(x) = -5 + C_1 e^{x/2} + C_2 e^{-4x}$

10.9 $y(x) = -(1/3)xe^{-4x} + C_1 e^{x/2} + C_2 e^{-4x}$

10.11 $y(x) = 2e^{2x} + C_1 \sin(3x) + C_2 \cos(3x)$

10.13 $y(x) = (p/k)x^2 - 2(p/k^2)x + 2(p/k^3) + Ce^{-kx}$

10.15 $y(x) = 5x - 2 + C_1 e^{-x} \cos(2x) + C_2 e^{-x} \sin(2x)$

10.17 $y(x) = (ad/b^2) - (c/b) - (d/b)x + C_1 e^{(1/2)[-a+\sqrt{a^2-4b}]x} + C_2 e^{(1/2)[-a-\sqrt{a^2-4b}]x}$

10.19 $y(x) = -3 + C_1 x^2 + C_2 x^{-4/3}$

10.21 $y(x) = (27/10)e^{-x} - (17/325)\cos(4x) - (306/325)\sin(4x) + C_1 e^{-6x} + C_2 e^{-3x}$

10.25 linear

10.27 $x = -2e^{-t} \sin t$

10.29 $x = Ae^{p_1 t} + Be^{p_2 t}$ where $p = \dfrac{-b \pm \sqrt{b^2 - 4mk}}{2m}$

10.31(a) $I(t) = -0.059\cos(360t) + 0.072\sin(360t) + e^{-50t}\left[0.059\cos(312\ t) - 0.073\sin(312\ t)\right]$

10.35(a) $dy/dx = -2y/(x + y)$

(b) $-2mx/(x + mx) = m$

(c) $m = 0$ or $m = -3$

(d) $y = -3x$

10.37(a) right

10.51 $x'(t) = -f(x, y)$, $y'(t) = -g(x, y)$

10.63(a) $dx/dy - (\sin x)y = \sin x$

(b) $-\sin x$

(c) $e^{\cos x}$

(d) $e^{\cos x}(dy/dx) - e^{\cos x}(\sin x)y = e^{\cos x}(\sin x)$

(f) $e^{\cos x}y(x) = -e^{\cos x} + C$

(g) $y(x) = Ce^{-\cos x} - 1$

10.65 $y(x) = (1/6)e^{3x} + Ce^{-3x}$

10.67 $y(x) = (e^x + C)/\sqrt{x}$

10.69 $y(x) = (x - e^{-x})/(e^x + 1) + C/(e^x + 1)$

10.71 $y = Ce^{-(2/3)x^3-3x} + 2$

10.73(b) $I(t) = V_0(5\sin(\omega t) - \omega\cos(\omega t))/(50 + 2\omega^2) + Ce^{-5t}$

10.77(a) $dM/dt = A + RM$

(b) $M(t) = -20,000 + Ce^{-e^{-t}/20}$

(c) $M(10) = 20,000\left[e^{(1-e^{-10})/20} - 1\right]$

10.83 $xy = C$

10.85 $3x^2 + 3x^2y + 10xy + 7y^2 = C$

10.87 $-y/(x+y) = C$

10.89(a) $dL_0\sqrt{1 - v^2/c^2}$

(b) $-vL_0/(c^2\sqrt{1-v^2/c^2})\ dv$

(c) $(\sqrt{1-v^2/c^2})\ dL_0 - (vL_0/(c^2\sqrt{1-v^2/c^2}))\ dv = 0$

(d) $L_0\sqrt{1-v^2/c^2} = C$

10.91 $V(x,y) = -e^{-x^2-y^2} - x + C$

10.93 D_1 is exact, with solution $xyzt + xe^z + t^2 = C$.

10.95 $y^2/2 - (3y+2)/x = C$

10.97 $y^2e^{x+2y} = C$

10.99 $(x^2+y)\left[\ln(x^2+y) - 1\right] = C$

10.103 $N^{5/3}V^{-2/3}e^{(2/3)S/(Nk_B)} = C$

10.109 Linearly dependent

10.111 Linearly dependent

10.113 Linearly independent

10.115(a) $-4e^x \sin x$

(b) 0

(c) $-4e^{\pi/2}$

(e) no

10.117(a) $W'(x) = y_1y_2'' - y_1''y_2$

(b) $W'(x) = -a_1(y_1y_2' - y_1'y_2)$

(c) $W'(x) = -a_1W$

(d) $W(x) = Ce^{-\int a_1(x)dx}$

10.119(a) $f'(0) = 3g'(0)$

(c) $f''(0) = 3g''(0)$

10.121(a) $du/dt = 2t + 2$

(b) $dx/dt = (2t+2)(dx/du)$

(d) $4u(d^2x/du^2) + 6u(dx/du) + 2ux = 0$

(e) $x(u) = Ae^{-u} + Be^{-u/2}$

(f) $x(t) = Ae^{-(t+1)^2} + Be^{-(t+1)^2/2}$

10.123 $y(x) = -x \pm \sqrt{2x + C}$

10.125 $y(x) = x^{-4}/(C - \ln x)$

10.127 $x(t) = \sqrt{A\cos(7t) + B\sin(7t)}$

10.129 $f(x) = A\cos x + B\cos x\ln(\cos x)$

10.131 $y(x) = x + 2/(3 + Ce^{6x})$

10.133 $y(x) = -(1/6)\ln(-6x + C) + x/3$

10.135 $x(t) = Ae^{-e^{2t}/2} + Be^{-e^{2t}}$

10.137 $x(t) = Ae^{\cos(2t)} + Be^{3\cos(2t)}$

10.139 $y(x) = x^2 - \sin^{-1}(-x + C)$

10.141(a) $k^2u^2(d^2x/du^2) + k^2u(dx/du) - 3ku(dx/du) + 36u^{6/k}x = 0$

(b) 3

(c) $x = A\cos(2u) + B\sin(2u)$

(d) $x(t) = A\cos(2e^{3t}) + B\sin(2e^{3t})$

10.143 $f(\theta) = AP_n(\cos\theta) + BQ_n(\cos\theta)$

10.145(a) $d^2\phi/dt^2 + (3/2)t^{-1}(d\phi/dt) + \lambda\phi^3 = 0$

(c) $k = 3/4$

(d) $u''(t) + (15/16)t^{-2}u + \lambda t^{-3/2}u^3 = 0$

10.147 $s = \dfrac{3}{2} \quad \rightarrow \quad a^{-3/2}\dfrac{d^2u}{dt^2} - \dfrac{3}{2}a^{-5/2}u\dfrac{d^2a}{dt^2} - \dfrac{3}{4}ua^{-7/2}\left(\dfrac{da}{dt}\right)^2 + a^{3/2}\dfrac{dV}{du} = 0$

10.151 $y(x) = [5/(18x^4 + Cx^9)]^{1/3}$

10.153 $y(x) = (x/3 + C/\sqrt{x})^2$

10.155 $y(x) = x/(C - \ln x)$

10.157 $y(x) = \pm x\sqrt{-\ln(-2\ln x + C)}$

10.159 $y(x) = \pm x\sqrt{\sin^{-1}(C + 2\ln x)}$

10.161 $y(x) = \pm 2/\sqrt{1 + Ce^{(8/3)x^3}}$

10.163 $y(t) = -\sin^{-1}(C - t) - t$

10.165 $x(t) = A\sin(1/x) + B\cos(1/x)$

10.167 $x(t) = A\sin t + B\cos t + Ct + D$

10.169(a) $dx/dt = -vx/\sqrt{x^2 + y^2}$

(b) $dx/dt = -vy/\sqrt{x^2 + y^2}$

(c) $dy/dx = y/x$

(d) $y = Cx$

(e) $dy/dx = \left(vy - s\sqrt{x^2 + y^2}\right)/(vx)$

(f) $dy/dx = \left(vy/x - s\sqrt{1 + (y/x)^2}\right)/v$

(g) $y = \left(x^{1-s/v} - L^{-2s/v}x^{1+s/v}\right)/2$

10.171(a) $v(x) = \sqrt{(2/3)gx + Cx^{-2}}$

(b) $v(x) = \sqrt{(2/3)gx}$

(c) $a(x, v) = \sqrt{g/6}\, x^{-1/2}v$

(d) $a = g/3$

10.173(a) $dy/dx = x(du/dx) + u$

(b) $x(du/dx) + u = f(u)$

(c) $du/(f(u) - u) = dx/x$

(d) $\int(du/(f(u) - u)) = \ln x + C$

10.177(b) $a_1(x) = 1/x^3 + 1/x,\ a_0(x) = -1/x^4 - 1/x^2$

(c) $u = -e^{1/(2x^2)}$

(d) $y_{c2}(x) = xe^{1/(2x^2)}$

(f) $y(x) = Ax + Bxe^{1/(2x^2)}$

10.181 $y(x) = Ax + B(x-1)e^{1/x}$

10.183(a) $y_{c1} = e^x$

(b) $3u' + 3u'' + u''' = 0$

(c) $u = \dfrac{e^{kx}}{k}$ where $k = \dfrac{-3 \pm \sqrt{-3}}{2}$

(d) $y = Ae^x + Be^{p_1 x} + Ce^{p_2 x}$ where $p_{1,2} = \dfrac{-1 \pm \sqrt{-3}}{2}$

10.185 $y(x) = -e^{5x}(1 + \ln x) + C_1 e^{5x} + C_2 x e^{5x}$

10.187 $y(x) = e^{4x}/14 + C_1 e^{x/2} + C_2 e^{2x}$

10.189 $y(x) = -x^2 - 1 + Ae^x + Bx$

10.191 $y(x) = x\cos x - x^2 \sin x + Ax + Bx \sin x$

10.195(a) $d^2x/dt^2 + 7(dx/dt) + 12x = F_e$

(b) $x_{1c} = e^{-3t}$ and $x_{2c} = e^{-4t}$

(c) $x_p(t) = e^{-3t} \int e^{3t} F_e(t)dt - e^{-4t} \int e^{4t} F_e(t)dt$

(d) **(i)** $x_p(t) = 5/12$

(ii) $x_p(t) = (8/63)e^{t/2}$

10.197(a) $b_{c1} = \sin(v n\pi t/L)$ and $b_{c2} = \cos(v n\pi t/L)$

(b) $\sin\left(\dfrac{vn\pi}{L}t\right) \displaystyle\int -\frac{4vL}{n^2\pi^2} \cos\left(\frac{vn\pi}{L}t\right) a(t)\, dt + \cos\left(\frac{vn\pi}{L}t\right) \int \frac{4vL}{n^2\pi^2} \sin\left(\frac{vn\pi}{L}t\right) a(t)\, dt$

(c) $b_{yn}(t) = -4kL^2/(n^3\pi^3)$

(d) $b_{yn}(t) = -t(4L^2/n^3\pi^3)$

10.209 $H(t-1) - H(t-4)$

10.211 $\sin(t)H(t-\pi)$

10.213 force over time

10.215 k/s

10.217 $1/s^2$

10.219 $(e^{-2s} - e^{-5s})/s$

10.221 $-e^{-s}(1/s + 1/s^2) + 1/s^2$

10.223 $e^{-3s} + e^{-4s}$

10.225 $(e^{-s} - e^{-4s})/s$

10.231(a) 1

(b) 0

(c) 4

(d) $4e^2$

10.233 $mv\delta(t)$

10.235(b) $t[H(t) - H(t-1)] + (2-t)[H(t-1) - H(t-2)]$

(c) $\left(e^{-2s} - 2e^{-s} + 1\right)/s^2$

10.237(a) $f(-50) = f(-5) = f(-1) = 0$; $f(1) = f(10) = f(100) = 1$

(c) Heaviside function

10.239 seconds^{-1}; thing·seconds

10.241(a) $1/(s-1)$

(c) $1/(s+1)$

(d) $1/(s^2-1)$

10.243(a) e^{-5s}

(b) $X(s) = (e^{-5s} + 6s + 23)/(3s^2 + 10s - 8)$

10.259 $x(t) = 13\cos t - 3\cos\left(\sqrt{11}\,t\right) - 3\sin t + 3\sqrt{11}\sin\left(\sqrt{11}\,t\right)$,
$y(t) = 39\cos t + \cos\left(\sqrt{11}\,t\right) - 9\sin t - \sqrt{11}\sin\left(\sqrt{11}\,t\right)$

10.261 e^{7t}

10.263 $\cos(5t)$

10.265(a) e^{-s}

(e) $x(t) = \left(1/\sqrt{2}\right)\sin\left(\sqrt{2}\,t\right)H(t)$

10.271 $I(s) = 9s(1 - e^{-2\pi s})/[(1+s^2)(2s^2 + 7s + 6)]$

10.273(a) $-kx_2$

(b) $x_2 - x_1$

(c) Left mass: $k(x_2 - x_1)$. Right mass: $-k(x_2 - x_1)$.

(d) $mx_1'' = kx_2 - 2kx_1$, $mx_2'' = kx_1 - 2kx_2$

(e) $X_1 = 20s/(3s^4 + 40s^2 + 100)$, $X_2 = 2s(3s^2 + 20)/(3s^4 + 40s^2 + 100)$

10.275(a) $X(s) = F(s)/(s^2 + a_1 s + a_0)$

(b) $G(s) = 1/(s^2 + a_1 s + a_0)$

(c) $x''(t) + 2x'(t) + 3x(t) = f(t)$; $G(t) = 1/(s^2 + 2s + 3)$

10.277(a) $\mathcal{L}\left[tf(t)\right] = -F'(s)$

(b) $\mathcal{L}[ty'(t)] = -sY'(s) - Y(s)$

(c) $\mathcal{L}[ty''(t)] = -s^2Y'(s) - 2sY(s) + y(0)$

(d) $Y'(s)\left(-s^2 + s\right) + Y(s)(-s + 1 + n) = 0$

10.279(a) $f(x) = \int_0^\infty f(p)\delta(p - x)\,dp$

(b) $G''(x) + 2G'(x) - 15G(x) = \delta(x - p)$

(c) $G(x,p) = \begin{cases} Ae^{-5x} + Be^{3x} & 0 \le x < p \\ Ce^{-5x} + De^{3x} & x > p \end{cases}$

(d) $A + B = 0$

(e) $D = 0$

(f) $Ae^{-5p} + Be^{3p} = Ce^{-5p}$

(g) $-5Ae^{-5p} + 3Be^{3p} + 1 = -5Ce^{-5p}$

(h) $G(x,p) = \dfrac{1}{8}\begin{cases} (e^{-5x} - e^{3x})e^{-3p} & 0 \le x < p \\ e^{-5x}(e^{-3p} - e^{5p}) & x > p \end{cases}$

(i) $y(x) = (1/8)\left\{e^{-5x}\int_0^x (e^{-3p} - e^{5p})f(p)dp + (e^{-5x} - e^{3x})\int_x^\infty e^{-3p}f(p)dp\right\}$

(j) $y(x) = (1/16)(e^{-5x} - e^{-x})$

10.281 $y(x) = (1/6)x^3 - (50/3)x$

10.283 $y(x) = -(1/8)\left(e^{-5x}\int_{-\infty}^x e^{5p}f(p)dp + e^{3x}\int_x^\infty e^{-3p}f(p)dp\right)$

10.285 $G(x,p) = \begin{cases} -(\cos p)\sin x & x < p \\ -(\sin p)\cos x & x > p \end{cases}$

10.287 $y(x) = \cos x\ln(\cos x) + (x - \pi/2)\sin x$

10.289(b) $x(t) = -(1/4) + (1/36)e^{4t} + (2/9)e^{-t/2}$

10.291(b) $y_1 = A\sin x$

(c) $y_3 = D\cos x$

(d) $y_2 = 1/(2\epsilon) + B\sin x + C\cos x$

(f) $A = -\sqrt{3}/(2\epsilon)\sin\epsilon$, $D = -1/(2\epsilon)\sin\epsilon$

(g) $\Delta y = -(1/2)(\sin^2\epsilon/\epsilon)$, $\Delta y' = (\sqrt{3}/2)(\sin^2\epsilon/\epsilon) + \sin\epsilon\cos\epsilon/\epsilon$

(h) $\lim_{\epsilon\to 0}\Delta y = 0$, $\lim_{\epsilon\to 0}\Delta y' = 1$

10.293(a) $x''(t) + (b/m)x'(t) = F_{ext}(t)/m$

(b) $G(t,p) = (m/b)\left[1 - e^{(b/m)(p-t)}\right]$ for $t > p$

(c) For $t < t_1$, $x(t) = 0$. For $t_1 < t < t_2$, $x(t) = -(km/b^2)(t-t_1) + k/(2b)(t-t_1)^2 + km^2/(b^3)\left(1 - e^{-(b/m)(t-t_1)}\right)$. For $t > t_2$, $x(t) = -km/(b^2)(t_2-t_1)e^{(b/m)(t_2-t)} + k/(2b)(t_2-t_1)^2 + km^2/(b^3)\left(e^{(b/m)(t_2-t)} - -e^{-(b/m)(t-t_1)}\right)$

10.295 $f(t) = (1/4) + 2e^{-t^4}$

10.297 $f(t) = (1/50)\left(e^{5t} + 9\sin(5t) + 49\cos(5t)\right)$

10.299 $f^3 t + (1/3)t^3 + f^2 = C$

10.301 $f = (\sin^2 t + C)/(2\cos t)$

10.303 $e^{f^2+t} + \left(f^2+t\right)^2 - t^2 = C$

10.305 $fe^t - (1/4)t^4 + (1/3)f^3 = C$

10.307 The Laplace transform of the solution is $F(s) = (2 + e^{-s})/(s^2 + 4s + 3)$.

10.309 $f(t) = -2 + C_1 t^{-2} + C_2 t^2$

10.311 $(f/t) + \ln(f/t) = \ln t + C$

10.313 $f(t) = (1/3)\cos^4 t + \sin^2 t - (1/3)\sin^4 t + A\cos t + B\sin t$

10.315 The Laplace transform of the solution is $s/(s+1)^2$

10.317 $y(x) = \ln(Ae^x + Be^{-x} + 1)$

10.319 $y(x) = -(1/C_1)\ln\left(1 - C_1 x^2\right) + C_2$

10.321 $y(x) = \pm x\sqrt{2\ln x + C}$

10.323(a) $x(t) = Ae^{-3t} + Bte^{-3t}$

(c) $x(t) = Ae^{-3t} + Bte^{-3t} + e^{-3t}\left(t\int e^{3t}F_e(t)dt - \int te^{3t}F_e(t)dt\right)$

(d) $x(t) = Ae^{-3t} + Bte^{-3t} + t^7 e^{-3t}/42$

10.325(a) $x(t) = (4/15)e^{-t} - (1/15)e^{-4t} - (1/5)\cos(2t)$

10.327(b) $\left(e^{-2s} - e^{-(2+a)s}\right)/as$

(c) e^{-2s}

(d) $\delta(t-2)$

10.331(b) $-x/y$

(c) $dy/dx = y/x$

(d) $y(x) = Cx$

(e) $y(x) = Cx^4$

10.333(a) $y(x) = 1/(8kx^2) + C$

(b) $m_{eq} = 4y/x$

(c) $y(x) = \pm\sqrt{C - x^2/4}$

Chapter 11

11.1 (b), (c)

11.3 (c), (d)

11.5 (a), (b)

11.7 (c)

11.9(d) Yes

(e) Yes

11.11(c) Yes

(d) No

11.23 $z(3, y) = 0$ is the only homogeneous condition

11.25 homogeneous

11.29 $\partial u/\partial t = \alpha \left(\partial^2 u/\partial x^2\right) + kx$

11.31(a) $\partial u/\partial t = \alpha \nabla^2 u$

(b) $\nabla^2 u_0 = 0$

11.33 $\partial^2 y/\partial t^2 = v^2 \left(\partial^2 y/\partial x^2\right) - k^2 \left(\partial y/\partial t\right)$

11.35(a) $\partial \rho/\partial t = k \left(\partial^2 \rho/\partial x^2\right)$

11.45(a) $T_R = T \sin(\theta_R)$

(b) $T_R = T \sin \tan^{-1}(\partial y/\partial x)_R$

(c) $T_R \approx T \left(\partial y/\partial x\right)_R$

(d) $T_L \approx -T \left(\partial y/\partial x\right)_L$

(e) $T \left(\partial y/\partial x\right)_R - T \left(\partial y/\partial x\right)_L$

(f) $\lambda\ dx$

(g) $[T \left(\partial y/\partial x\right)_R - T \left(\partial y/\partial x\right)_L]/(\lambda\ dx)$

(h) $v = \sqrt{T/\lambda}$

11.51(a) $y(x, t) = \sin(5x)\cos(5vt)$

(c) $y(x, t) = 3\sin(2x)\cos(2vt)$

(d) $y(x, t) = 6\sin(x)\cos(vt) - 5\sin(7x)\cos(7vt)$

(e) $y(x, t) = \sum_{n=1}^{\infty}(1/n^2)\sin(nx)\cos(nvt)$

11.55 $y(x, t) = \sin(2x)\cos(4t) + (1/10)\sin(10x)\cos(20t)$

11.57 $y(x, t) = \sum_{n=1}^{\infty}(2/\pi n)\left[\cos(\pi n/3) - \cos(2\pi n/3)\right]\sin(nx)\cos(2nt)$

11.63 $y(x, t) = \sin(x)\cos(vt)$

11.65 $s(x, t) = \cos(x)\cos(3t) + (1/10)\cos(10x)\cos(30t)$

11.69(a) $y(x, 0) = \sin(n\pi x/L)$ where n is any integer

(b) $y(x, t) = \sin(n\pi x/L)\cos(nv\pi t/L)$

(c) $y(x, t) = \sin(3\pi x/L)\cos(3v\pi t/L)$

(d) $y(x, t) = 2\sin(3\pi x/L)\cos(3v\pi t/L) + 5\sin(8\pi x/L)\cos(8v\pi t/L)$

(e) $y(x, t) = \sum_{n=1}^{\infty}(2/\pi n)\left[\cos(\pi n/3) - \cos(2\pi n/3)\right]\sin(n\pi x/L)\cos(nv\pi t/L)$

11.71 $y(x, t) = \sin(kx + \phi)\cos(kvt)$ where k and ϕ are any real numbers.

11.75(a) $(1/2)[\sin(n(x + vt)) + \sin(n(x - vt))]$

(b) $f(x) = g(x) = \sum_{n=0}^{\infty}(b_n/2)\sin(nx)$

11.77 $y(x,t) = F\sin(2\pi x/L)\cos(2v\pi t/L)$

11.79 $y(x,t) = H\sin(2\pi x/L)\cos(2v\pi t/L) + (cL/v\pi)\sin(\pi x/L)\sin(v\pi t/L)$

11.81 $y(x,t) = \sum_{n=1}^{\infty}\sin\left(\frac{n\pi}{L}x\right)\left[C_n\sin\left(\frac{nv\pi}{L}t\right) + D_n\cos\left(\frac{nv\pi}{L}t\right)\right]$ where
$C_n = \frac{2cL}{n^2\pi^2 v}\left[\cos\left(\frac{n\pi}{3}\right) - \cos\left(\frac{2n\pi}{3}\right)\right]$ and $D_n = \frac{4L}{\pi^2 n^2}\sin\left(\frac{n\pi}{2}\right)$

11.83(a) $D_n = (2/n\pi)\left[\cos(n\pi/3) - \cos(2n\pi/3)\right]$

(b) $y(x,t) = \sum_{n=1}^{\infty}(2/n\pi)\left[\cos(n\pi/3) - \cos(2n\pi/3)\right]\sin(n\pi x)\cos\left(\sqrt{n^2\pi^2+1}\ t\right)$

11.85 $y(x,t) = \sin(\pi x/L)\, e^{-c^2\pi^2 t/L^2}$

11.87 $y(x,t) = \sum_{n=1}^{\infty} D_n\sin(n\pi x/L)\cos\left(\sqrt{1+c^2n^2\pi^2/L^2}\ t\right)$ where
$D_n = (2/L)\int_0^L f(x)\sin(n\pi x/L)\,dx.$

11.89 $y(x,t) = \frac{1}{2}\left(\frac{L}{3\pi c}\right)^{3/2}\sin\left(\frac{3\pi}{L}x\right)\left[\sinh\left(\sqrt{\frac{3\pi c}{L}}\ t\right) - \sin\left(\sqrt{\frac{3\pi c}{L}}\ t\right)\right]$

11.91 $u(x,t) = u_0\sin(2\pi x/L)\, e^{-4\alpha\pi^2 t/L^2}$

11.93 $u(x,t) = \sum_{n=1}^{\infty}(2u_0/n\pi)(\cos(n\pi/3) - \cos(2n\pi/3))\sin(n\pi x/L)e^{-\alpha n^2\pi^2 t/L^2}$

11.97(a) $y(x,t) = 0$

(c) $X''(x) = PX(x),\ T''(t) = v^2PT(t)$

11.99(a) $-(\hbar^2/2m)(\psi''/\psi) + V(x) = i\hbar(T'/T) = E$

(b) $T = Ae^{-Eit/\hbar}$

(c) $E = n^2\pi^2\hbar^2/\left(2mL^2\right)$ for $n = 1, 2, 3\ldots$

(d) $\sum_{n=1}^{\infty} A_n\sin\left(\frac{n\pi}{L}x\right)$

(e) $\sum_{n=1}^{\infty} A_n\sin\left(\frac{n\pi}{L}x\right)e^{-n^2\pi^2\hbar it/(2mL^2)}$

(f) $\sum_{n=1}^{\infty}\frac{2\psi_0}{n\pi}\left[\cos\left(\frac{n\pi}{4}\right) - \cos\left(\frac{3n\pi}{4}\right)\right]\sin\left(\frac{n\pi}{L}x\right)e^{-n^2\pi^2\hbar it/(2mL^2)}$

11.103(a) $V(x,y,z) = [\sinh(\sqrt{5}\pi z/L)/\sinh(\sqrt{5}\pi)]\left[\sin(\pi x/L)\sin(2\pi y/L)\right.$
$\left.+\sin(2\pi x/L)\sin(\pi y/L)\right]$

11.107(b) $F_{nm} = (4/mn\pi^2)[\cos(2m\pi/3) - \cos(m\pi/3)][\cos(2n\pi/3) - \cos(n\pi/3)]$

11.109 $u(x,y,t) = \sin(2\pi x/L)\sin(3\pi y/H)e^{-\pi^2\left[4(a/L)^2+9(b/H)^2\right]t}$

11.111 $u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\sin(m\pi x/L)\sin(n\pi y/H)\sin(\sqrt{(m\pi/L)^2+(n\pi/H)^2+1}\ t)$
where $A_{mn} = (4/LH)[\int_0^H\int_0^L g(x,y)\sin(m\pi x/L)\sin(n\pi y/H)\ dx\ dy]/$
$\sqrt{(m\pi/L)^2+(n\pi/H)^2+1}$

11.113 $z(x,y,t) = \sin(\pi x/L)\sin(\pi y/L)\cos\left(\sqrt{2}v\pi t/L\right)$

11.117 $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} C_{mn}\sin\left(\frac{m\pi}{L}x\right)\sin\left(\frac{n\pi}{L}y\right)\sin\left(\frac{\pi}{L}\sqrt{v_x^2m^2+v_y^2n^2}\ t\right)$

$$C_{mn} = \frac{4}{\pi L\sqrt{v_x^2m^2+v_y^2n^2}}\int_0^L\int_0^L g(x,y)\sin\left(\frac{m\pi}{L}x\right)\sin\left(\frac{n\pi}{L}y\right)\ dx\ dy$$

11.119 $y = AJ_2(x^2) + BY_2(x^2)$

11.121 $y = AJ_0(e^x) + BY_0(e^x)$

11.129 $z(\rho, t) = \displaystyle\sum_{n=1}^{\infty} \frac{2c}{a\alpha_{0,n} v J_1^2(\alpha_{0,n})} \left(\int_0^a \sin\left(\frac{3\pi}{a}\rho\right) J_0\left(\frac{\alpha_{0,n}}{a}\rho\right) \rho \, d\rho \right) J_0\left(\frac{\alpha_{0,n}}{a}\rho\right) \cos\left(\frac{\alpha_{0,n}}{a} vt\right)$

11.131 $z(\rho, \phi, t) = \displaystyle\sum_{n=1}^{\infty} \left[\frac{2}{a^2 J_2^2(\alpha_{1,n})} \int_0^a (a-\rho) J_1\left(\frac{\alpha_{1,n}}{a}\rho\right) \rho \, d\rho \right] J_1\left(\frac{\alpha_{0,n}}{a}\rho\right) \cos\left(\frac{\alpha_{0,n}}{a} vt\right) \sin(\phi)$

11.135(a) $T'(t)/T(t) = x^{3/2}(X''(x)/X(x)) + x^{1/2}(X'(x)/X(x))$

(b) $X(x) = AJ_0\left(4kx^{1/4}\right) + BY_0\left(4kx^{1/4}\right)$

(c) $X(x) = AJ_0\left(4kx^{1/4}\right)$

(d) $k = \alpha_{0,n}/4$

(e) $T(t) = Ce^{-\alpha_{0,n}^2 t/16}$

(f) $y(x, t) = \sum_{n=1}^{\infty} A_n J_0(\alpha_{0,n} x^{1/4}) e^{-\alpha_{0,n}^2 t/16}$

(g) $A_n = 1/(2J_1^2(\alpha_{0,n})) \int_0^1 \sin(\pi x) J_0(\alpha_{0,n} x^{1/4}) x^{-1/2} \, dx$

11.137 $y(x, t) = \sum_{n=1}^{\infty} A_n J_1\left(\alpha_{1,n} x/3\right) e^{-\alpha_{1,n}^2 t/9}$ where $A_n = (2/9 J_2^2(\alpha_{1,n})) \int_1^2 J_1\left(\alpha_{1,n} x/3\right) x dx$

11.139 $z(x, y, t) = J_{2\pi}(\alpha_{2\pi,3} x) \sin(2\pi y) e^{-\alpha_{2\pi,2}^2 t}$

11.141(a) $T(t) = A\sin(kt) + B\cos(kt)$

(b) $c = 2k/\sqrt{g}$.

(d) $u(y, t) = \displaystyle\sum_{n=1}^{\infty} J_0\left(\sqrt{\frac{y}{L}}\alpha_{0,n}\right) \left[A_n \sin\left(\frac{\alpha_{0,n}\sqrt{g}}{2\sqrt{L}} t\right) + B_n \cos\left(\frac{\alpha_{0,n}\sqrt{g}}{2\sqrt{L}} t\right) \right]$

where $A_n = \dfrac{2}{\alpha_{0,n}\sqrt{g}\sqrt{L} J_1^2(\alpha_{0,n})} \displaystyle\int_0^L h(y) J_0\left(\frac{\alpha_{0,n}}{\sqrt{L}}\sqrt{y}\right) dy$ and

$B_n = \dfrac{1}{L J_1^2(\alpha_{0,n})} \displaystyle\int_0^L f(y) J_0\left(\frac{\alpha_{0,n}}{\sqrt{L}}\sqrt{y}\right) dy$

11.143 $t = a^2 \ln 2 / \left(D\alpha_{2,3}^2\right)$

11.145 $A_n = \dfrac{2c}{a\alpha_{0,n}^2 J_1(\alpha_{0,n})}$ and $B_n = \dfrac{2}{a^2 J_1^2(\alpha_{0,n})} \displaystyle\int_0^a (a-\rho) J_0\left(\frac{\alpha_{0,n}}{a}\rho\right) \rho d\rho$

11.149(a) $(1-x^2)(X''(x)/X(x)) - 2x(X'(x)/X(x)) = T'(t)/T(t)$

(b) $X(x) = AP_l(x)$

(c) $T(t) = Ce^{-l(l+1)t}$

(d) $y(x, t) = \sum_{l=0}^{\infty} A_l P_l(x) e^{-l(l+1)t}$

(e) $A_l = [(2l+1)/2] \int_{-1}^1 x P_l(x) dx$

(f) $y(x, t) = xe^{-2t}$

11.151 $y(x, t) = \sum_{l=0}^{\infty} A_l P_l(x/3) e^{-l(l+1)t}$ where $A_l = [(2l+1)/2] \int_{-1}^1 P_l(u) \sin(\pi u) du$

11.153 $4x\cos(\sqrt{2}\, t)$

11.155 $y(x, \phi, t) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} C_{lm} e^{-l(l+1)t} Y_l^m(\cos^{-1} x, \phi)$
where $C_{lm} = -\int_0^{2\pi} \int_{-1}^1 \left(1-x^2\right) \cos(\phi) \left[Y_l^m(\cos^{-1} x, \phi)\right]^* dx \, d\phi$

11.157(a) $R(r) = Ar^l + Br^{-l-1}$

11.159(a) $V(r, \theta, \phi) = V_0$

(b) $\vec{0}$

11.161(c) $u(r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} A_{lm} Y_l^m(\theta,\phi) j_l(\lambda r)$ where $\lambda = \alpha_{l+1/2}/a$

11.163(h) $V(r,\theta) = \sum_{l=0}^{\infty} C_l (r/a)^l P_l(\cos\theta)$ where $C_l = (2l+1)\int_0^1 cP_l(u)du$ (odd l only)

11.165(a) $f_1 = [3/\sin(7\pi/\sqrt{2})]\sin(7x/\sqrt{2})\sin(7y)$

(b) $f_2 = [4/\sin(5\sqrt{2}\,\pi)]\sin(5x)\sin(5\sqrt{2}\,y)$

11.167 $u_0 \dfrac{e^{\mu(y-L)} - e^{-\mu(y-L)}}{e^{-\mu L} - e^{\mu L}} \sin\left(\dfrac{2\pi}{L}x\right) + \sum_{n=1}^{\infty} \dfrac{4u_0}{n\pi}\dfrac{e^{\nu x} - e^{-\nu x}}{e^{\nu L} - e^{-\nu L}} \sin\left(\dfrac{n\pi}{L}y\right)$ for odd n only, $\mu = \sqrt{\gamma^2 + 4\pi^2/L^2}$ and $\nu = \sqrt{\gamma^2 + n^2\pi^2/L^2}$

11.169 $\sin(\pi x)e^{(y-3)/2}\dfrac{e^{(1/2)\sqrt{4\pi^2+1}\;y} - e^{-(1/2)\sqrt{4\pi^2+1}\;y}}{e^{(3/2)\sqrt{4\pi^2+1}} - e^{-(3/2)\sqrt{4\pi^2+1}}}$
$+3\sin(2\pi x)e^{y/2}\dfrac{e^{(1/2)\sqrt{1+16\pi^2}\;y} - e^{3\sqrt{16\pi^2+1}}e^{-(1/2)\sqrt{1+16\pi^2}\;y}}{1 - e^{3\sqrt{16\pi^2+1}}}$

11.171(a) $(e^x - e^{-x})/(e - e^{-1})$

(b) $\sum_{n=1}^{\infty} A_n \sin(n\pi x)e^{-(1+n^2\pi^2)t}$

(c) $(e^x - e^{-x})/(e - e^{-1}) + \sum_{n=1}^{\infty} A_n \sin(n\pi x)e^{-(1+n^2\pi^2)t}$

(d) $\dfrac{e^x - e^{-x}}{e - e^{-1}} + \sum_{n=1}^{\infty}\dfrac{(-1)^{n+1}2}{n\pi(1+n^2\pi^2)}\sin(n\pi x)e^{-(1+n^2\pi^2)t}$

(e) $(e^x - e^{-x})/(e - e^{-1})$

11.173 $\alpha\sin\theta + \alpha\sin(4\theta)e^{-15t}$

11.175(b) $u(x,t) = 100x + \sum_{n=1}^{\infty}(800/n^3\pi^3)e^{-\alpha n^2\pi^2 t}\sin(n\pi x)$, odd n only

(c) 3996 s

11.179 $\sum_{n=1}^{\infty}\dfrac{4\kappa v^2L^2}{n\pi(v^2n^2\pi^2 - \omega^2L^2)}\left[\dfrac{L\omega}{n\pi v}\sin\left(\dfrac{n\pi v}{L}t\right) - \sin(\omega t)\right]\sin\left(\dfrac{n\pi}{L}x\right)$, odd n only

11.181 $\dfrac{\kappa v^2L^2}{p^2\pi^2v^2 + \omega^2L^2}\left[-e^{-\omega t} - \dfrac{\omega L}{p\pi v}\sin\left(\dfrac{p\pi v}{L}t\right) + \cos\left(\dfrac{p\pi v}{L}t\right)\right]\sin\left(\dfrac{p\pi}{L}x\right)$

11.183 $\sum_{n=1}^{\infty}\left[A_n\sin\left(\dfrac{n\pi v}{L}t\right) + B_n\cos\left(\dfrac{n\pi v}{L}t\right) - \dfrac{4\kappa v^2L^2}{n\pi\left(\omega^2L^2 + v^2n^2\pi^2\right)}e^{-\omega t}\right]\sin\left(\dfrac{n\pi}{L}x\right)$ for odd n only,
where $A_n = -\dfrac{4\kappa vL^3\omega}{n^2\pi^2\left(\omega^2L^2 + v^2n^2\pi^2\right)}$, $B_n = \dfrac{4L}{n\pi}\left[\dfrac{\kappa v^2L}{\omega^2L^2 + v^2n^2\pi^2} + \dfrac{(-1)^{(n-1)/2}}{n\pi}\right]$

11.185(b) $(2/n\pi)\left[\cos(n\pi/3) - \cos(2n\pi/3)\right] - (-1)^n 2/n^5$

11.187 $y(x,t) = L^2/(L^2+\pi^2)\left[1 - \cos(\sqrt{1+\pi^2/L^2}\;t)\right]\sin(\pi x/L)$

11.189 $\sum_{n=1}^{\infty}\left[\left(b_{fn} - \dfrac{[1-(-1)^n]2L^2}{n\pi(n^2\pi^2 - L^2)}\right)e^{-n^2\pi^2t/L^2} + \dfrac{[1-(-1)^n]2e^{-t}L^2}{n\pi(n^2\pi^2 - L^2)}\right]\sin\left(\dfrac{n\pi}{L}x\right)$ where $b_{gn} = (2/L)\int_0^L g(x)\sin(n\pi x/L)dx$

11.191 $\sum_{n=1}^{\infty} 4\kappa/(n\pi)\left(1 - e^{L^2t/(n^2\pi^2)}\right)\sin(n\pi x/L)$

11.193 $\sum_{n=1}^{\infty}\dfrac{4L^2}{n^3\pi^3}\left(H\dfrac{e^{n\pi y/L} - e^{-n\pi y/L}}{e^{n\pi H/L} - e^{-n\pi H/L}} - y\right)\sin\left(\dfrac{n\pi}{L}x\right)$ (odd n only)

11.195 $\sum_{n=1}^{\infty}\dfrac{4\kappa L^2}{L^2 + n^2\pi^2}\left(\dfrac{e^{\sqrt{1+n^2\pi^2/L^2}\;y} + e^{\sqrt{1+n^2\pi^2/L^2}\;(H-y)}}{1 + e^{\sqrt{1+n^2\pi^2/L^2}\;H}} - 1\right)\sin\left(\dfrac{n\pi}{L}x\right)$, odd n only

11.199(b) $f(x) = \sum_{n=1}^{\infty}(-1)^{n-1}(2/n\pi)\sin(n\pi x)$

11.201(b) $a''_{sn}(t) + (n^2\pi^2 c_s^2/L^2)a_{sn}(t) = -c_s^2 a_{qn}$

(c) $a_{sn}(t) = A_n \sin(n\pi c_s t/L) + B_n \cos(n\pi c_s t/L) - (a_{qn}L^2/n^2\pi^2)$, $a_{s0}(t) = -(1/2)c_s^2 a_{q0} t^2 + Ct + D$

(d) $s(x,t) = -(1/2)c_s^2 a_{q0} t^2 + \sum_{n=1}^{\infty}(a_{qn}L^2/n^2\pi^2)\left[\cos(n\pi c_s t/L) - 1\right]\cos(n\pi x/L)$

11.203(c) $L^2/(14\pi^2\varepsilon_0)\sin(\pi x/L)\sin(2\pi y/L)\sin(3\pi z/L)$

11.205(b) $R_n(\rho) = J_0\left(\alpha_{0,n}\rho/a\right)$

(c) $$\frac{d^2A_{zn}}{dt^2} + \frac{v^2\alpha_{0n}^2}{a^2}A_{zn} = \frac{2}{a^2 J_1^2(\alpha_{0,n})}\int_0^{a/2}\kappa\cos(\omega t)J_0\left(\frac{\alpha_{0,n}}{a}\rho\right)\rho\, d\rho$$

(d) $(d^2A_{zn}/dt^2) + (v^2\alpha_{0n}^2/a^2)A_{zn} = \gamma_n\kappa\cos(\omega t)$

(e) $A_{zn} = \gamma_n\kappa a^2/(v^2\alpha_{0n}^2 - a^2\omega^2)\left[\cos(\omega t) - \cos(v\alpha_{0n}t/a)\right]$

(f) $z(\rho,t) = \sum_{n=1}^{\infty}\gamma_n\kappa a^2/(v^2\alpha_{0n}^2 - a^2\omega^2)\left[\cos(\omega t) - \cos(v\alpha_{0n}t/a)\right]J_0(\alpha_{0n}\rho/a)$

11.209 $\hat{y} = \delta(p)\left(1 - e^{-t}\right) + \hat{f}e^{-c^2p^2t}$

11.211 $y(x,t) = \kappa(1 - e^{-t})$

11.213 $$\hat{y} = \frac{1}{2\sqrt{\pi}(c^2p^2 + 1 - \omega^2)}e^{-(p/2)^2}\left[\cos(\omega t) - \cos\left(\sqrt{c^2p^2+1}\ t\right)\right] + \frac{2e^{-ip/2} - e^{-ip} - 1}{2\pi p^2\sqrt{c^2p^2+1}}\sin\left(\sqrt{c^2p^2+1}\ t\right)$$

11.215 $(Qi/\pi p^3)\left(1 - \cos p\right)\left[1 - \cos(vpt)\right]$

11.217 $(Qv^2t^2\delta(p)/2) + (Gde^{-d^2p^2/4}/2vp\sqrt{\pi})\sin(vpt)$

11.219 $\hat{y} = \sin p\left(1 - e^{-(p^2 - ip)t}\right)/\left[\pi p(p^2 - ip)\right]$

11.221(b) $C(p) = -e^{-(p/2)^2}/[2\sqrt{\pi}\left(p^2 + 2\pi^2\varepsilon_0\right)]$

11.223(a) $s^2U + s\sin(\pi x) - \partial^2U/\partial x^2 = s\sin(\pi x)/(1+s^2)$

(b) $U(0,s) = 0,\ U(1,s) = 1/s^2$

(c) $$U(x,s) = -\frac{s^3}{(s^2+1)(s^2+\pi^2)}\sin(\pi x) + \frac{e^{sx} - e^{-sx}}{s^2(e^s - e^{-s})}$$

11.225 $$Y(x,s) = \frac{1}{s^2(1+s^2)}\left[1 - \frac{e^{(s/c)\ x} + e^{s/c}e^{-(s/c)\ x}}{\left(1 + e^{s/c}\right)}\right]$$

11.227 $Y(x,s) = (1/s^2)\sinh[\sqrt{s}(1-x)]/\sinh(\sqrt{s})$

11.229 $$Y(x,s) = \frac{x}{s^3} + \frac{e^{x\sqrt{s/c}} - e^{-x\sqrt{s/c}}}{s^3\left(e^{-\sqrt{s/c}} - e^{\sqrt{s/c}}\right)} = \frac{x}{s^3} - \frac{1}{s^3}\frac{\sinh(x\sqrt{s/c})}{\sinh(\sqrt{s/c})}$$

11.231(b) $(\partial^2Y/\partial x^2) = (s^2/v^2)Y$

(e) $Y(x,s) = [y_0\omega/(\omega^2+s^2)]e^{-(s/v)x}$

11.233(a) $z(x,0) = 2\sin(\pi x/6)$

(c) $\sin(5\pi x/6)\, e^{(-25\pi^2+20)t/16}$

(e) $\sin(\pi x/6)\, e^{(-\pi^2+20)t/16} + \sin(5\pi x/6)\, e^{(-25\pi^2+20)t/16}$

11.235(a) $\sum_{n=1}^{\infty}(2s/n\pi)[\cos(n\pi/3) - \cos(2n\pi/3)]\sin(nx)$

(b) $\sum_{n=1}^{\infty}(2s/n^2v\pi)[\cos(n\pi/3) - \cos(2n\pi/3)]\sin(nx)\sin(nvt)$

11.241 $y(x,t) = \frac{3}{2}x - \frac{1}{2}x^2 + \sum_{n=1}^{\infty}\left[-\frac{18(1-(-1)^n)}{n^3\pi^3} + \frac{2}{n\pi}\left(\cos\left(\frac{n\pi}{3}\right) - \cos\left(\frac{2n\pi}{3}\right)\right)\right]$
$\sin\left(\frac{n\pi}{3}x\right)e^{-\frac{n^2\pi^2}{9}t}$

11.243 $y(x,t) = \frac{\alpha^2 x_0}{2\sqrt{\pi}\beta}\int_{-\infty}^{\infty}\frac{1}{c^2p^2-\omega^2}e^{-p^2x_0^2/(4\beta^2)}\left(\cos(\omega t) - \cos\left(cpt\right)\right)e^{ipx}\,dp$

11.245 $y(x,t) = \sum_{n=1}^{\infty} -108(1+2(-1)^n)/(\pi^3 n^3)\sin(n\pi x/3)e^{-n^2\pi^2 t}$

11.247 The solution $u(x,t)$ is the inverse Laplace transform of
$\frac{1}{s(1+s)}\left[1 + e^{(x-1)/2}\left(\frac{e^{1/2} - e^{(1/2)\sqrt{1+4s}}}{e^{\sqrt{1+4s}} - 1}e^{(1/2)\sqrt{1+4s}x} + \frac{e^{1/2} - e^{-(1/2)\sqrt{1+4s}}}{e^{-\sqrt{1+4s}} - 1}e^{-(1/2)\sqrt{1+4s}x}\right)\right]$

11.249 $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} 16V/[mn\pi^2\sin\left(\pi\sqrt{n^2+m^2}\right)]\sin(m\pi x/L)\sin\left(n\pi y/L\right)\sin((\pi/L)\sqrt{m^2+n^2}\,z\right)$
for odd m and n

11.251 $y(x,t) = 1/(81\pi^4)\left(e^{-9\pi^2 t} + 9\pi^2 t - 1\right)\sin(3\pi x)$

11.253 $y(x,t) = \sum_{n=1}^{\infty}\left(B_n/A_n + C_n e^{-A_n t}\right)\sin(n\pi x/3)$ where $A_n = \alpha^2 n^2\pi^2/9 + \beta^2$,
$B_n = 6/(n^2\pi^2)\sin(n\pi/3) + (6-2n\pi)/(n^2\pi^2)\sin(2n\pi/3) + 2/(n\pi)\cos(2n\pi/3)$,
$C_n = -18/(\pi^3 n^3)\left[n\pi\sin(n\pi) + 2\cos(n\pi) - 2\right] - B_n/A_n$

11.255 $x - \sum_{n=1}^{\infty} 8/(n^3\pi^3)\sin(n\pi x)e^{-n^2\pi^2 t}$ for odd n only

11.257 $\hat{y}(p) = e^{-(p/2)^2}/[2\sqrt{\pi}p^2(v^2-1)]\left[e^{ipt} - (i/v)\sin(vpt) - \cos(vpt)\right]$

11.259(a) $\partial^2 V/\partial r^2 + \partial^2 V/\partial\phi^2 = e^{-r}\sin\phi$

(b) $r \geq 0$

(d) $\hat{V}_s = -2p\sin\phi/\left[\pi(1+p^2)^2\right] + A(p)e^{p\phi} + B(p)e^{-p\phi}$

(e) $A(p) = B(p) = 0$

Chapter 12

12.1 $\sum_{n=1}^{57} n$

12.3 $\prod_{n=1}^{100} n$

12.5 $\sum_{n=0}^{28}(-1)^n \sin[(2n+1)x]$

12.7 $\sum_{n=0}^{28}(4n^2+2n+1)$

12.9(a) $a_n = a_{n-1} + 3$

(b) $a_n = 14 + 3(n-1)$, $a_{100} = 311$

(c) $\sum_{n=1}^{\infty}[14+3(n-1)]$

12.11(b) $f(n) = \ln(n-3)$

(c) $a = 3$, $b = 8$

12.13 $(n+1)c_{n+1}$

12.15 $1 + 3x + \sum_{n=2}^{\infty} 6x^n$

12.17 $\sum_{n=0}^{\infty}\left(n^2+3n+4\right)x^n$

12.19(a) $\sum_{n=0}^{\infty} c_n(n+3)x^{n-2} - \sum_{n=0}^{\infty} c_n(n+3)x^n$

(b) $\sum_{n=-2}^{\infty} c_{n+2}(n+5)x^n - \sum_{n=0}^{\infty} c_n(n+3)x^n + \sum_{n=-1}^{\infty} c_{n+1}(n-6)x^n + \sum_{n=0}^{\infty} c_n x^n$

(c) $3c_0x^{-2} + 4c_1x^{-1} - 7c_0x^{-1} + \sum_{n=0}^{\infty}\left[c_{n+2}(n+5) + c_{n+1}(n-6) - c_n(n+2)\right]x^n = 8x^{-1}$

(d) $c_0 = 0$

(e) $c_1 = 2$

(f) $c_{n+2} = \left[c_n(n+2) - c_{n+1}(n-6)\right]/(n+5)$

(g) $c_2 = 12/5$

(h) $c_3 = 3$, $c_4 = 108/35$

12.21(a) even

(b) odd

(c) ∞, ∞

(d) $\infty, -\infty$

(e) 1, 0

(g) 1, −1

12.23(b) $A = (x_0 + v_0/\omega)/2$, $B = (x_0 - v_0/\omega)/2$

(d) $C = v_0/\omega$, $D = 0$

(e) $C = 0$, $D = x_0$

(f) $C = v_0/\omega$, $D = x_0$

12.25 $\left(\sqrt{\pi}/2\right)\left(\text{erf}\,7 - \text{erf}\,5\right)$

12.27(a) 0

(b) always increasing

(c) always concave down

12.29 $\text{erf}\,x = \left(2/\sqrt{\pi}\right)x - 2x^3/\left(3\sqrt{\pi}\right) + \ldots$

12.31(a) $\text{erf}\left[x/(s\sqrt{2})\right]$

(b) 1

(d) $16/\sqrt{2}$

12.33 2

12.35 139

12.37 $\Gamma(x+21)/\Gamma(x+4)$

12.43(a) $S = k_B\left[\left(q + N - \frac{1}{2}\right)\ln(q+N-1) - \frac{1}{2}\ln(2\pi) - \left(q + \frac{1}{2}\right)\ln q - \left(N - \frac{1}{2}\right)\ln(N-1)\right]$

(b) $\dfrac{dS}{dq} = k_B\left(\dfrac{1-N}{2q(q+N-1)} + \ln(q+N-1) - \ln q\right)$

(c) $\dfrac{dS}{dq} = k_B\left(\dfrac{-N}{2q(q+N)} + \ln(q+N) - \ln q\right)$

(d) **(i)** $\dfrac{dS}{dq} = k_B\left[\dfrac{-Nx^2}{2(1+Nx)} + \ln(1+Nx)\right]$

(ii) $dS/dq \approx k_B N x$

(iii) $dS/dq \approx k_B N/q$

(e) $dS/dU = k_B N/(q\epsilon)$

(f) $T = U/(k_B N)$

12.49(g) 1

12.53(a) $c_2 = (1/2 - c_0)/2$, $c_3 = (-1/4 - c_1)/6$, $c_4 = (1/8 + c_1 - c_2)/12$

(b) $y(x) = x + (1/4)x^2 - (5/24)x^3 + (7/96)x^4$

12.55 $y = -2 - 2x - x^2$

12.57 $y(x) = 2 + x - x^2 - x^3/3 + x^4/4$

12.59 $x(t) = 1 + t^2/2 + t^4/16$

12.61 $y(x) = 24 + 24x + 12x^2 - 4x^3 - x^4$

12.65 $H_{\text{even}}(x) = c_0\left[1 - \dfrac{2k}{2!}x^2 + \dfrac{2^2k(k-2)}{4!}x^4 - \dfrac{2^3k(k-2)(k-4)}{6!}x^6 + \ldots\right]$

12.69 $(3/5)P_1(x) + (2/5)P_3(x)$

12.71 $(9/5)P_1(x) - (84/55)P_3(x)$

12.73 $(1/2)P_0(x) - (5/8)P_2(x)$

12.75 $(\sin 1)P_0(x) + 5(3\cos 1 - 2\sin 1)P_2(x)$

12.81(b) $\Theta = AP_l(\cos\theta) + BQ_l(\cos\theta)$ where $l(l+1) = k$

12.83(a) $A_n = \dfrac{2n+1}{2R^n}\displaystyle\int_0^\pi \left(\cos^2\theta\right) P_n(\cos\theta)\sin\theta\, d\theta$

(b) $A_n = \dfrac{2n+1}{2R^n}\displaystyle\int_{-1}^1 u^2 P_n(u)\, du$

(c) $A_0 = 1/3$, $A_2 = 2/(3R^2)$, and $A_1 = A_3 = 0$

12.85(b) $\displaystyle\sum_{j=0}^{l/2}(-1)^{l/2-j}\frac{l!}{(l/2-j)!(l/2+j)!}x^{l+2j}$

(c) $(-1)^{l/2}\dfrac{l!}{[(l/2)!]^2}x^l, (-1)^{l/2-1}\dfrac{l!}{(l/2-1)!(l/2+1)!}x^{l+2}, (-1)^{l/2}\dfrac{l!}{(l/2-2)!(l/2+2)!}x^{l+4}$

(d) $(-1)^{l/2}\left[\dfrac{l!}{(l/2)!}\right]^2 + (-1)^{l/2-1}\dfrac{l!}{(l/2-1)!(l/2+1)!}\dfrac{(l+2)!}{2!}x^2$
$+(-1)^{l/2}\dfrac{l!}{(l/2-2)!(l/2+2)!}\dfrac{(l+4)!}{4!}x^4$

(e) $(-1)^{l/2}\left[\dfrac{l!}{(l/2)!}\right]^2\left[(-1)\dfrac{l(l+1)}{2!}x^2 + \dfrac{l(l+1)(l-2)(l+3)}{4!}x^4\right]$

12.87(c) $(-1)^l\dfrac{1}{2^{2l}(l!)^2}\displaystyle\int_{-1}^1 (x^2-1)^l\frac{d^{2l}}{dx^{2l}}\left[(x^2-1)^l\right]dx$

(d) $(2l)!$

12.89(a) $y_{1/2}$

(b) $3 - 6x + x^2$

12.91 $c_0 + d_0x^{3/2}\left(1 - 3x/20 + 3x^2/224\right)$

12.93 $c_0x^{1/5}\left(1 - 5x^2/6\right) + d_0x^{-1/5}\left(1 - 5x^2/4\right)$

12.95(d) $y(x) = Ax^{2/3} + Bx^{-2}$

12.97(e) $c_0(1/x - 1/2)$

12.99 $y = Ax^i + Bx^{-i}$

12.101(b) $\displaystyle\sum_{n=0}^\infty c_n(r+n-1)(r+n)x^{r+n} + \sum_{n=0}^\infty c_n(r+n)x^{r+n} + \sum_{n=0}^\infty c_nx^{r+n+2} - \sum_{n=0}^\infty p^2c_nx^{r+n} = 0$

(c) $c_0(r^2-p^2)x^r + c_1\left[(r+1)^2 - p^2\right]x^{r+1} + \displaystyle\sum_{n=2}^\infty\left(c_n\left[(r+n)^2-p^2\right] + c_{n-2}\right)x^{r+n} = 0$

(d) $r = \pm p$

(e) $c_1 = 0$

(f) $c_n = c_{n-2}/\left[p^2 - (r+n)^2\right]$

12.103(a) $c_{2m} = c_{2m-2}/\left[4m(p-m)\right]$

(b) $c_6 = c_0/\left[4^3(3!)(p-1)(p-2)(p-3)\right]$

(c) $c_{2m} = c_0/\left[4^m(m!)(p-1)(p-2)(p-3)\dots(p-m)\right]$

(d) $c_{2m} = c_0(-1)^m\Gamma(-p+1)/\left[4^m(m!)\Gamma(p-m+1)\right]$

12.107(b) $H_p^{(1)} = \left[1+i\cot(\pi p)\right]J_p(x) - i\csc(\pi p)J_{-p}(x)$, $H_p^{(2)} = \left[1-i\cot(\pi p)\right]J_p(x) + i\csc(\pi p)J_{-p}(x)$

12.111 $y = AJ_0[(1-i)x/\sqrt{2}] + BY_0[(1-i)x/\sqrt{2}]$

12.113 $y(x) = \sqrt{x}\left[AJ_{n+1/2}(x) + BY_{n+1/2}(x)\right]$

12.115(a) $-1 < x < 1$

(b) $(1-x^2)y''(x) - 2xy'(x) + \lambda y(x) = 0$

(c) $AP_l(x) + BQ_l(x)$ where $\lambda = l(l+1)$

(e) $\lambda = l(l+1)$ for non-negative integer l, $y(x) = P_l(x)$

(f) $\pm 1/\sqrt{3}$

12.117(a) x

(b) $xf(x)J_0(\alpha_{0,m}x) = \sum_{n=1}^{\infty} c_n xJ_0(\alpha_{0,n}x)J_0(\alpha_{0,m}x)dx$

(c) $\int_0^1 xf(x)J_0(\alpha_{0,m}x)dx = (1/2)c_m J_1^2(\alpha_{0,m})$

(d) $c_m = 2\left(\int_0^1 xf(x)J_0(\alpha_{0,m}x)dx\right)/J_1^2(\alpha_{0,m})$

12.119 $c_m = 2\left(\int_0^1 xf(x)J_1(\alpha_{1,m}x)dx\right)/J_2^2(\alpha_{1,m})$

12.121 $c_m = \left(\int_{-\infty}^{\infty} e^{-x^2}f(x)H_m(x)dx\right)/\left(\sqrt{\pi}\,2^m m!\right)$

12.123 $\pi/2 - \sum_{n=-\infty}^{\infty} 2e^{inx}/(n^2\pi)$ (odd n only)

12.133(a) $y_m''(x) + \lambda_m y_m(x) = 0$

(b) $y_n''(x) + \lambda_n y_n(x) = 0$

(c) $y_m''(x)y_n(x) - y_n''(x)y_m(x) + y_m(x)y_n(x)\left(\lambda_m - \lambda_n\right) = 0$

(d) $\left[y_n(x)y_m'(x) - y_m(x)y_n'(x)\right]_0^L + \left(\lambda_m - \lambda_n\right)\int_0^L y_m(x)y_n(x)dx = 0$

12.141(a) 1

12.147(a) $\hat{B}\hat{A}\psi_m + \lambda_m\psi_m = 0$

(b) $\lambda_n = \lambda_m - 5$

(c) $\hat{B}\psi_0 = 0$

(d) $\lambda = 5, 10, 15, \dots$

12.151 $\sqrt[4]{m\omega/\hbar\pi}$

12.153(a) $\hbar\hat{L}_+, -\hbar\hat{L}_-$

(b) $c_m + \hbar$ and $c_m - \hbar$

12.155 $y = c_0 + c_1 t - c_0 t^2/2 + (1-c_0-c_1)t^3/6 + (c_0 - 2c_1)t^4/24 + \dots$

12.157 $y(t) = c_0 + c_1 t - (c_0 + c_1)t^2/2 + (c_0 + 1)t^3/6 + (c_1 - 1)t^4/24 + \dots$

12.159 $y(t) = c_0 t^{-1/2} + d_0 - (11/16)c_0 t^{3/2} - (1/5)d_0 t^2 + \dots$

12.161(e) $L_0(x) = 1$

(f) $L_1(x) = 1 - x$

(g) $L_2(x) = x^2/2 - 2x + 1$, $L_3(x) = -x^3/6 + 3x^2/2 - 3x + 1$

12.163(f) $r = r_0 + v_0 t - \dfrac{GM}{2r_0^2} t^2 + \dfrac{GMv_0}{3r_0^3} t^3 + \dots$

Chapter 13

13.1(a) $u = x^2 - y^2$, $v = 2xy$

(b) $|f| = |z|^2$, $\phi_f = 2\phi_z$

13.5 $(1/2)\sin x\,(e^{-y} + e^{y}) - i(1/2)\cos x\,(e^{-y} - e^{y})$

13.11(a) $\phi_{\sqrt{z}} = (1/2)\phi_z$, $|\sqrt{z}| = \sqrt{|z|}$

(b) $2e^{i\pi/4}$, $2e^{-3i\pi/4}$.

(c) $\pm\pi$

13.13 Zeros: $z = -1$ and $z = 2i$. Poles: $z = 3$ and $z = -5 + 2i$.

13.15(a) $z = -1 - i$

(b) $z = -4i$

13.17 nz_0^{n-1}

13.21 analytic

13.23 analytic

13.27 $V = V_0 \ln\left(\sqrt{x^2 + y^2}/R\right)/(\ln 3)$

13.29 $V(x, y) = (1/2)(V_2 - V_1)y + (5/2)V_1 - (3/2)V_2$

13.33(a) vertical lines

(b) The boundary conditions would only depend on x.

13.35(a) $\sqrt{\dfrac{\sqrt{x^2 + y^2} + x}{2}} + i\sqrt{\dfrac{\sqrt{x^2 + y^2} - x}{2}}$

(b) parabola

13.37(b) First-order pole at $z = 0$, second-order pole at $z = -1$

(c) $z = -1$

(d) $-2e^{-1}$

(e) $-4\pi i e^{-1}$

(f) 0

(g) $2\pi i(1 - 2/e)$

13.39 0

13.41 $2\pi i$

13.43 0

13.45 $(2/3)\pi\ln(3i)$

13.47 0

13.49 $(\pi i/120)(e^{-1} + e)$

13.53(a) $\displaystyle\sum_{n=1}^{2\pi/(\Delta\phi)} \left(e^{i\Delta\phi} - 1\right)$

(b) $2\pi i$

13.55(a) R^{m-n}

(b) πR

13.57(b) 0

13.59 They both give 0.

13.63 $\pi/3$

13.65 $-5\pi/6$

13.67 π/e^5

13.69 $\pi/(4e^4)$

13.71(a) $\int_0^{2\pi} dx/\left[5-2i(e^{ix}-e^{-ix})\right]$

(b) $\int_0^{2\pi} du/(2u^2+5iu-2)$

(c) $(2u+i)(u+2i)$

(d) unit circle

(e) $2\pi/3$

13.73 $\pi/6$

13.75 $2\pi/\sqrt{3}$

13.77 $f(t)=e^t$

13.79 $f(t)=1$

13.81 $f(t)=e^{-3t}(1-3t+t^2)$

13.87 $\sqrt{2}\,\pi/\sqrt[4]{3}$

13.89 $\pi/\left[\left|x_0\right|^k \sin(\pi k)\right]$

13.91(b) $(\ln|z|)^2-\phi^2+2\ln|z|\phi i$

(c) $4\pi^2 x_0-4\pi iR$

(d) 0

(f) $4\pi x_0\left(-\pi+i\ln x_0-i\right)$

(g) $x_0\ln x_0-x_0$

13.93(b) $1+1/z+1/(2z^2)+1/(6z^3)+\ldots+1/(n!z^n)+\ldots$

(c) Essential singularity

(d) 1

(e) $2\pi i$

13.95(a) 1

(b) $\sqrt{2}$

(c) π

13.97(a) $1/\sqrt{2}\left[(z-\pi/4)^{-1}+1-(1/2)(z-\pi/4)+\ldots\right]$

(b) $1/\sqrt{2}$

(c) $\sqrt{2}\,\pi i$

13.99 $f(z)=-z^{-1}-z-z^3+\ldots,\ R=1$

13.101(a) $a=-1,\ b=1$

(b) $f(z)=-z^{-1}-1+z-z^2+\ldots$

(d) $f(z)=\sum_{n=2}^{\infty} z^{-n}$

13.103 $g^{(n-1)}(z_0)/(n-1)!$

13.111 the disk enclosed by the unit circle.

13.113 quarter-circle between $e^{-i\pi/4}$ and $e^{i\pi/4}$

13.115 fourth quadrant between the circles of radius $1/2$ and 1

13.117 circle of radius $7/26$ centered on $(29/26)+i(7/13)$

13.121(a) $|f|^2 = (x^2+y^2)^2 + 2x^3 + 2xy^2 + x^2 + y^2$

(c) $T = 0$ on the inner circle and $T = 1$ on the outer.

(d) $\left(\ln\sqrt{u^2+v^2}\right)/(\ln 2)$

(e) $u(x,y) = x^2 + x - y^2$, $v(x,y) = 2xy + y$

(f) $\left(\ln\sqrt{(x^2+y^2)^2 + 2x^3 + 2xy^2 + x^2 + y^2}\right)/(\ln 2)$

(h) **(i)** $\ln(36/25)/\ln(2) \approx 0.53$

(ii) $(36/25, 0)$

(iii) $\ln(36/25)/\ln(2) \approx 0.53$

(iv) La-de-da

13.123(a) $V(u,v) = \left(\ln\sqrt{u^2+v^2}\right)/(\ln 2)$

(b) $z = \sqrt{(1/2)e^{i\phi}+1}$ and $z = \sqrt{e^{i\phi}+1}$ for $0 \le \phi \le 2\pi$

(c) $f(z) = z^2 - 1$

(d) $u(x,y) = x^2 - y^2 - 1$ and $v(x,y) = 2xy$

(e) Find the potential function in the region bounded by $z = \sqrt{(1/2)e^{i\phi}+1}$ (held at $V = -1$) and $z = \sqrt{e^{i\phi}+1}$ (held at $V = 0$) for $0 \le \phi \le 2\pi$ by using the mapping $f(z) = z^2 - 1$.

(f) $V(x,y) = \left(\ln\sqrt{(x^2+y^2)^2 + 2(y^2-x^2)+1}\right)/(\ln 2)$

13.125 $T(x,y) = \dfrac{1}{\ln 2}\ln\left(2\sqrt{\dfrac{1-6x}{x^2+y^2}+9}\right)$

13.127 $T(x,y) = \dfrac{2}{\pi}\tan^{-1}\left(\dfrac{(x^2+y^2)^{1/4}\cos\left(\left[\tan^{-1}(y/x)\right]/2\right)}{(x^2+y^2)^{1/4}\sin\left(\left[\tan^{-1}(y/x)\right]/2\right)-1}\right)$

13.129(b) Radius $1/3$ centered at $z = -2/3$

(c) $Re(f(z)) = -1/4$

(d) $V = A\ln\left(B\sqrt{\dfrac{4x+1}{x^2+y^2}+4}\right)$ where $A = \dfrac{V_L - V_C}{\ln 2}$ and $B = e^{\left(\frac{V_C \ln 2}{V_L - V_C}\right)}$

13.131 $T = (2/\pi)\tan^{-1}(\cot x \tanh y)$

13.133(a) $V = v/\pi$

13.135(a) $\psi = \ln(x^2+y^2)$

(b) $\psi = y[1 - 1/(x^2+y^2)] + \ln(x^2+y^2)$

13.137(a) $\psi(x,y) = \left(3x^2y - y^3\right)\left(1 - R^2/(x^2+y^2)^3\right)$

(b) $\vec{v} = \left[3(x^2-y^2) - \dfrac{3R^2}{(x^2+y^2)^4}(x^4 - 6x^2y^2 + y^4)\right]\hat{i} - 6xy\left[1 + 2R^2\dfrac{x^2-y^2}{(x^2+y^2)^4}\right]\hat{j}$

13.141(b) $\chi(z) = \psi(z) + \chi(z_0)$

13.143(a) $\psi = k\tan^{-1}(y/x)$

(b) $\vec{v} = \dfrac{kx}{x^2+y^2}\hat{i} + \dfrac{ky}{x^2+y^2}\hat{j}$

(c) $k = 800\ \text{cm}^2/\text{s}$

13.145(a) Maps the upper half-plane to itself

(b) first quadrant

(c) segment from $(0,0)$ to $\left(\sqrt{2},0\right)$.

(d) $(x^2+y^2)^2 = 2(x^2-y^2)$

13.149(a) $z = x+i$ where $x \le 0$

(b) $z = x-i$ where $x \le 0$

13.151(f) The branch cut is on the imaginary axis between $z=-i$ and $z=i$.

13.153 $\pi \sin(2\pi k)/\left[x_0^k - x_0^k \cos(2\pi k)\right]$

13.155(a) $T(u,v) = (2T_0/\pi)\tan^{-1}(v/u)$

(c) $T(x,y) = \dfrac{2T_0}{\pi}\tan^{-1}\left(\dfrac{\tan^{-1}(y/x)}{\ln\sqrt{x^2+y^2}}\right)$

13.157(a) $u(x,y) = (1/2)\sin x(e^y+e^{-y})$, $v(x,y) = (1/2)\cos x(e^y-e^{-y})$

(c) $f(x+5i) = 74.21\sin x + 74.20i\cos x$ where $0 \le x \le \pi/2$

(a) $T(u,v) = (2/\pi)\tan^{-1}(v/u)$

(b) $T(x,y) = \dfrac{2}{\pi}\tan^{-1}\left(\dfrac{\cos x(e^y-e^{-y})}{\sin x(e^y+e^{-y})}\right)$