Discovery Exercise for Euler's Formula II—Modeling Oscillations

The integral

$$\int \sin(2x)e^x dx \tag{1}$$

conventionally requires integration by parts (twice) and an extra trick. In this exercise you will solve it in an easier way. Begin by considering a different integral.

$$\int e^{2ix}e^x dx \tag{2}$$

1. Use the properties of exponentials to combine the two exponentials in Equation 2 into one exponential. Do *not* expand the complex exponential with Euler's formula.

2. Evaluate the integral to find $\int e^{2ix}e^x dx$. Don't forget to include an arbitrary constant C.

   See Check Yourself #16 at felderbooks.com/checkyourself

3. Rewrite your solution in the standard form $a + bi$ with real a and b. This will require using Euler's formula. Then, when you have a complex number in a denominator you can simplify by multiplying the top and bottom of the fraction by the complex conjugate of the denominator. To split your arbitrary constant you can just write it as $C = A + iB$. Your answer should be in the form

$$\int e^{2ix}e^x dx = (\text{Some real function of } x) + i\,(\text{Some other real function of } x) \tag{3}$$

4. Use Euler's formula to rewrite $\int e^{2ix}e^x dx$ as the integral of a real function plus i times the integral of another real function $\left(\int a(x)dx + i\int b(x)dx\right)$. For this part you are *not* using the results you've derived so far; just split the integrand into real and imaginary parts.

5. Recall that every complex equation is two real equations in disguise: one that says the real part of the left-hand-side equals the real part of the right-hand-side and one that says the same for the imaginary parts. Using your results from Parts 3 and 4 rewrite Equation 3 as two real equations.

It should now be clear why we set out to solve a complex integral when what we really wanted was the solution to a real integral. One of the two real equations you just wrote down should be in the form $\int \sin(2x)e^x dx =$ (*some real function of* x). In other words, by solving the relatively simple integral $\int e^{2ix}e^x dx$ you were able to find the solution to the more difficult integral $\int \sin(2x)e^x dx$.

6. Take the derivative of the solution you found for $\int \sin(2x)e^x dx$ and verify that it does give you $\sin(2x)e^x$.

In this exercise you solved a purely mathematical problem with no physical context, but in doing so you've learned a technique for solving many physical problems: define a sinusoidal quantity as the real or imaginary part of a complex quantity, and you replace a math problem involving trigonometry with an easier math problem involving exponentials.