Motivating Exercise for Complex Numbers: The Underdamped Harmonic Oscillator

A "damped harmonic oscillator," such as an underwater mass on a spring, follows the force law $ma = -cv - kx$.

1. Remembering that $a = d^2x/dt^2$ and $v = dx/dt$, rewrite this equation as a second order differential equation for $x(t)$.

2. Consider the constants $m = 1/2$ kg, $c = 7$ kg/s, $k = 20$ N/m. Plug in a guess of the form $x(t) = e^{pt}$ and solve for p. You should find two possible values for p.

3. Using the values of p you just found, write the general solution to this differential equation for the given values of the constants. Remember that the general solution to a second order differential equation must have two arbitrary constants.

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The damping force from the water prevents the mass from oscillating at all; it decays exponentially towards equilibrium. Such an oscillator is "overdamped."

4. Now consider the same mass on the same spring, but with a weaker damping force: $c = 2$. Rewrite the differential equation with this constant changed, plug in the guess $x(t) = e^{pt}$, and solve for p. You will find, in your solutions, the square root of a negative number. Just leave it there as part of your solutions, but simplify everything as much as possible.

5. Write the general solution to this differential equation, with the square root of a negative number as a part of your answer.

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6. Choose one of the two independent solutions you found and confirm that it solves the differential equation.

A solution of this form, even if it works mathematically, does not seem to tell us anything about the motion of our mass-on-spring system. But in fact it tells us everything—once we know how to interpret it! When you learn about complex numbers, and especially about the complex exponential function, you will discover that this solution describes sinusoidal oscillation with decreasing amplitude, exactly as you would expect from such a system.