Computer Problems for Calculus with Complex Numbers

1. Consider the principal branch of $\ln z$. For each of the plots below take as your domain the square going from $-2-2i$ to $2+2i$.

   (a) Create a three-dimensional graph with z on the horizontal plane and the modulus $|f(z)|$ on the vertical axis. What does your plot show $|f|$ doing on the branch cut? What does it show $|f|$ doing as $z \to 0$? Where is the minimum value of $|f|$ on your plot?

   (b) Plot $Re(f)$ and $Im(f)$ in two side-by-side plots. What happens to each one on the branch cut? In the limit $z \to 0$?

   (c) The formula $z = e^{i\phi}$ for all $-\pi < \phi \leq \pi$ describes every point on the unit circle. Sketch the set of complex points $f(z)$ for all z along this circle. On the same plot make a sketch of $f(z)$ for all points on the larger circle $z = 2e^{i\phi}$. What shapes are your plots? (This part does not require a computer, although you may use one if you wish.)

2. Use the equation below to derive the formula $|\cos z| = (1/2)\sqrt{e^{2y}+e^{-2y}+2\left(\cos^2 x - \sin^2 x\right)}$.

$$\cos(x+iy) = \frac{1}{2}\cos x\left(e^y + e^{-y}\right) + \frac{1}{2} i \sin x\left(e^{-y} - e^y\right)$$

   Then create a three dimensional plot of $|\cos(x+iy)|$ as a function of x and y. List three facts that you can see about the function $\cos z$ by looking at your graph.

3. For each of the functions below, make 3D plots of $|f|$, $Re(f)$, and $Im(f)$. In each case the horizontal plane will represent z and the vertical axis will be the real quantity you are plotting. Choose your domains so that you can clearly see how the function behaves. Describe some property of each function that is most easily seen in one of the plots, and what about that plot reveals that property.

   (a) $f(z) = z^3$

   (b) $f(z) = z^*$

   (c) $f(z) = \tan z$

   (d) $f(z) = \ln(\cos z)$

4. Find an improper integral where the standard definition and the Cauchy principal value differ, and evaluate both of them. (If you've already found one in one of the previous problems, you may use that one. If not, figure one out.) Then use a computer to evaluate the integral. Which answer does the computer give?

5. "Picard's theorem" says that in any neighborhood of an essential singularity a function $f(z)$ must take on every possible complex value, with at most one exception.

   (a) Find the Laurent series for $f(z) = e^{1/z^2}$ about the origin and use it to prove that f has an essential singularity at the origin.

   (b) Plot $f(\rho e^{i\phi})$ for fixed ρ and $0 < \phi < 2\pi$. Do this for several decreasing values of ρ so you can see how f behaves around the origin as you get closer to it. Describe your results.

   (c) Find the one value f never takes as you get close to the origin.

6. Find the region the function $f(z) = e^z$ maps the quarter circle of radius 2 in the first quadrant, centered on the origin, to.

7. **Exploration: The Joukowski Airfoil**

The function $z + R^2/z$ is real on a circle of radius R centered on the origin. An equivalent way of saying that is that the mapping $z(f) = f + R^2/f$ maps this circle to part of the x axis. In 1908 Nikolai Joukowski applied this mapping to circles with other centers and found that they produced interesting shapes. You'll work with one such example here.

(a) Apply the mapping $z = f + 1/f$ to a circle that passes through the point $f = -1$ but is centered on $f = .1 + .2i$. Plot the resulting shape. This shape is called an "airfoil," and is often used to model airplane wings.[1]

(b) Invert the mapping to find $f(z)$. You should get two solutions. For now you'll just hold onto both of them.

For each of the two inverse functions $f(z)$ you found, have the computer define functions $u(x, y)$ and $v(x, y)$ and use them to define a function $\psi(x, y)$. *Don't copy this function down. You don't even have to print it on your screen. It's pretty ugly.*

(c) Make two plots showing the contours of $\psi(x, y)$ around the airfoil, one for each stream function you defined. You should find that each of them fits around the airfoil perfectly in some parts of the plot and not in others.

(d) Define a new function equal to the appropriate stream function in each part of the domain. Plot the contours of this piecewise function around the airfoil to see the flow. You should be able to see the stagnation point.

The two problems below are a set; the first should be done without a computer and the second is a computer-based follow up.

8. The multi-valued function $f(z) = \sqrt{1 + z^2}$ can be made single-valued with two branch cuts. The starting point is to rewrite $f(z)$ as $\sqrt{z - i}\sqrt{z + i}$.

(a) First consider $\sqrt{z - i}$. Where is the branch cut for this function (using the principal phase of $z - i$)?

(b) Where is the branch cut for $\sqrt{z + i}$?

(c) Draw a z plane with a small circle around $z = i$. Sketch the curve this circle maps to on the f plane. What happens to $f(z)$ as the z circle crosses a branch cut?

9. *[This problem depends on Problem 8.]* In this problem you will make plots of mappings of the function $f(z) = \sqrt{1 + z^2}$ using the branch that has two branch cuts, as discussed in Problem 8.

(a) Have a computer make a z plane with the circle $|z| = 2$ and an f plane with the curves this circle maps to.

(b) What are the values of f just before and after crossing each branch cut in this mapping?

(c) Repeat Parts (a)–(b) for the circle $|z - i| = 1$. The curve will only cross one branch cut this time.

[1]See e.g. *Theoretical Aerodynamics, 4th ed.* by L. M. Milne-Thomson, Macmillan and Company, 1966.