Discovery Exercise for Derivatives and Analytic Functions

A derivative is the limit of a ratio. For real numbers, we can write this.

$$\frac{df}{dx} = \lim_{\Delta x \to 0} \left(\frac{\Delta f}{\Delta x} \right)$$

So df/dx does not just ask how much f changed; it divides that by how much x changed, which measures how far you traveled along the x axis.

The derivative of a complex function can be written the same way.

$$\frac{df}{dz} = \lim_{\Delta z \to 0} \left(\frac{\Delta f}{\Delta z} \right)$$

So df/dz divides the change in f by how much z changed. But Δz is *not* a measure of how far you traveled; it is a complex number that measures how much z changed as you traveled in whatever distance you did your traveling. To consider what all that means, we present you with the following fact.

The derivative of $\mathbf{f(z) = z^2}$ at the point $\mathbf{z = 5}$ is 10.

This tells us that if we start at $z = 5$ and move off by a reasonably small Δz the function should change by roughly 10 Δz *no matter what direction we move in.* Let's see how that pans out.

1. What is $f(5)$?

2. Suppose you move away from $z = 5$ by one unit in the positive real direction. What is the new z, and what is Δz? What is the new f, and what is Δf? Is their ratio approximately 10?

3. Suppose you move away from $z = 5$ by one unit in the positive imaginary direction. What is the new z, and what is Δz? What is the new f, and what is Δf? Is their ratio approximately 10?

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4. Suppose you move away from $z = 5$ by one unit in the negative real direction. What is the new z, and what is Δz? What is the new f, and what is Δf? Is their ratio approximately 10?

5. Suppose $\Delta z = 1 + i$. What is the new z? What is the new f, and what is Δf? Is it still approximately $10\Delta z$?

6. In every case above you should have found that Δf was approximately 10 times Δz, but never exactly. If $f'(5) = 10$ exactly (which it is), why didn't your answers come out exactly that way?