Discovery Exercise for the Quantum Harmonic Oscillator and Ladder Operators

The derivative d/dx is called an "operator," meaning it takes as input a function and produces as output another function. We will abbreviate that derivative as D. The operator x just multiplies any function by x. So for example, the operator $D - x$ acting on the function $f(x) = x^2$ produces the function $(D - x)f = 2x - x^3$. For this exercise we define the operators $\hat{a}_R = D - x$ and $\hat{a}_L = D + x$.

1. Calculate $\hat{a}_L \sin x$. (This should be very simple; we just want to make sure you're clear on the notation.)

   See Check Yourself #88 at felderbooks.com/checkyourself

2. Calculate $\hat{a}_L \hat{a}_R x^2$. (Read this as "act with $\hat{a}_R$ on x^2, then act with $\hat{a}_L$ on the result.")

The easiest way to do algebra with operators is to see what they do to an arbitrary function. For example, the operator Dx acting on a function f gives $(d/dx)(xf) = x(df/dx) + f$. We can then take out the f and write $Dx = xD + 1$, where the 1 represents the operator "multiply the function by 1."

3. Calculate $\hat{a}_L \hat{a}_R$. *Hint*: a second derivative is D^2 in operator notation.

   See Check Yourself #89 at felderbooks.com/checkyourself

4. The "commutator" of two operators $\hat{A}$ and $\hat{B}$ is defined as $\left[\hat{A}, \hat{B}\right] = \hat{A}\hat{B} - \hat{B}\hat{A}$. Calculate the commutator $[\hat{a}_L, \hat{a}_R]$.