Discovery Exercise for The Method of Fourier Transforms

The temperature in a bar obeys the heat equation:

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2} \tag{1}$$

Now consider the temperature $u(x,t)$ of an *infinitely long* bar.

1. If you use the method of eigenfunction expansions to solve this problem for a finite bar you begin by expanding the unknown solution $u(x,t)$ in a Fourier series. Explain why you cannot do the same thing in this case.

You can take an approach that is similar to the method of eigenfunction expansion, but in this case you will use a Fourier *transform* instead of a Fourier series. You begin by taking a Fourier transform of both sides of Equation 1. Using $\mathcal{F}$ to designate a Fourier transform with respect to x, this gives:

$$\mathcal{F}\left[\frac{\partial u}{\partial t}\right] = \mathcal{F}\left[\alpha \frac{\partial^2 u}{\partial x^2}\right] \tag{2}$$

2. The u you are looking for is a function of x and t. When you solve Equation 2 you will find a new function $\mathcal{F}[u]$. What will that be a function of? (It's not a function of u. That's the original function you're taking the Fourier transform of.)

3. One property of Fourier transforms is "linearity" which tells us that, in general, $\mathcal{F}[af+bg] = a\mathcal{F}(f) + b\mathcal{F}(g)$. Another property of Fourier transforms is that $\mathcal{F}\left[\partial^2 f/\partial x^2\right] = -p^2\mathcal{F}[f]$. Apply these properties (in order) to the right side of Equation 2.

4. Another property of Fourier transforms is that, if the Fourier transform is with respect to x and the derivative is with respect to t, you can move the derivative in and out of the transform: $\mathcal{F}\left[\partial f/\partial t\right] = \frac{\partial}{\partial t}\mathcal{F}[f]$. Apply this property to the left side of Equation 2.

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5. Solve this first-order differential equation to find $\mathcal{F}[u]$ as a function of p and t. Your solution will involve an arbitrary function $g(p)$.

6. Use the formula for an inverse Fourier transform to write the general solution $u(x,t)$ as an integral. (Do not evaluate the integral.) Your answer should depend on x and t. (Even though p appears in the answer, it only appears inside a definite integral, so the answer is *not* a function of p.)

7. What additional information would you need to solve for the arbitrary function $g(p)$ and thus get a particular solution to this PDE?