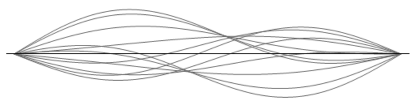
Discovery Exercise for Normal Modes

A guitar string extends from $x = 0$ to $x = \pi$. It is fixed at both ends, but free to vibrate in between.



The position of the string $y(x, t)$ is subject to the equation:

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{9}\frac{\partial^2 y}{\partial t^2} \tag{1}$$

The phrase "fixed at both ends" in the problem statement gives us the boundary conditions:

$$y(0, t) = y(\pi, t) = 0 \tag{2}$$

1. One solution to this problem is $y = 5\sin(2x)\cos(6t)$. (Later you will learn to find such solutions for yourself; right now we are focusing on understanding the equation and its solutions.)

   (a) Confirm that this solution solves the differential equation by plugging $y(x, t)$ into both sides of Equation 1 and showing that you get the same answer.

   (b) Confirm that this solution meets the boundary conditions, Equation 2.

   (c) Draw graphs of the shape of the string between $x = 0$ and $x = \pi$ at times $t = 0$, $t = \pi/12$, $t = \pi/6$, $t = \pi/4$, and $t = \pi/3$. Then describe the resulting motion in words.

   (d) Which of the three numbers in this solution—the 5, the 2, and the 6—is an arbitrary constant? That is, if you change that number to any other constant, the function will still solve the differential equation and meet the boundary conditions.

2. Another solution to this equation is $y = -(1/2)\sin(10x)\cos(kt)$, if you choose the correct value of k.
   (a) Plug this function into both sides of Equation 1 and solve for k.

   (b) Does the resulting function also meet the boundary conditions?

   (c) What is the period of this function in space (i.e. the distance between adjacent peaks, or the wavelength)?

   (d) What is the period of this function in time (i.e. how long do you have to wait before the string returns to its initial position)?

   See Check Yourself #70 at felderbooks.com/checkyourself

3. For the value of k that you calculated in 2, is the function $y = 5\sin(2x)\cos(6t) - (1/2)\sin(10x)\cos(kt)$ a valid solution? (Make sure to check whether it meets both the differential equation and the boundary conditions!)

4. Next consider the solution $y = A\sin(px)\cos(kt)$.
   (a) For what values of k will this function solve Equation 1? Your answer will depend on p.

   (b) For what values of p will this solution match the boundary conditions?

5. Write the solution to this differential equation in the most general form you can.