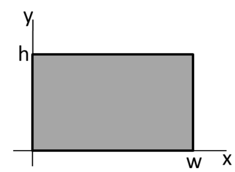
Discovery Exercise for Separation of Variables—More than Two Variables

A thin rubber sheet is stretched on a rectangular frame. The sheet is glued to the frame; however, it is free to vibrate inside the frame. (This system functions as a rectangular drum; the more common circular drum is much easier to analyze with polar coordinates.) For convenience we place our axes in the plane of the frame, with the lower-left-hand corner of the rectangle at the origin.



The motion of this sheet can be described by a function $z(x, y, t)$ which will obey the wave equation in two dimensions:

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{1}{v^2}\frac{\partial^2 z}{\partial t^2} \quad \textit{the wave equation in two dimensions} \tag{1}$$

The boundary condition is that $z = 0$ on all four edges of the rectangle. The initial conditions are $z(x, y, 0) = z_0(x, y)$, $\dot{z}(x, y, 0) = v_0(x, y)$.

1. Begin with a "guess" of the form $z(x, y, t) = X(x)Y(y)T(t)$. Substitute this expression into Equation 1.

2. Separate the variables so that the left side of the equation depends on x (not on y or t), and the right side depends on y and t (not on x). (This will require two steps.)

3. Explain why both sides of this equation must now equal a constant.

4. The function $X(x)$ must satisfy the conditions $X(0) = 0$ and $X(w) = 0$. Based on this restriction, is the separation constant positive or negative? Explain your reasoning. If positive, call it k^2; if negative, $-k^2$.

5. Solve the resulting differential equation for $X(x)$.

6. Plug in both boundary conditions on $X(x)$. The result should allow you to solve for one of the arbitrary constants and express the real number k in terms of an integer-valued n.

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7. You have another equation involving $Y(y)$ and $T(t)$ as well as k. Separate the variables in this equation so that the $Y(y)$ terms are on one side, and the $T(t)$ and k terms on the other. Both sides of this equation must equal a constant, but that constant is *not* the same as our previous constant.

8. The function $Y(y)$ must satisfy the conditions $Y(0) = 0$ and $Y(h) = 0$. Based on this restriction, is the *new* separation constant positive or negative? If positive, call it p^2; if negative, $-p^2$.

9. Solve the resulting differential equation for $Y(y)$.

10. Plug in both boundary conditions on $Y(y)$. The result should allow you to solve for one of the arbitrary constants and express the real number p in terms of an integer-valued m.

Important: p is not necessarily the same as k, and m is not necessarily the same as n. These new constants are unrelated to the old ones.

11. Solve the differential equation for $T(t)$. (The result will be expressed in terms of both k and p, or in terms of both n and m.)

12. Write the solution $X(x)Y(y)T(t)$. Combine arbitrary constants as much as possible. If your answer has any real-valued k or p constants, replace them with the integer-valued n and m constants by using the formulas you found from boundary conditions.

13. The general solution is a sum over *all possible* n and m values. For instance, there is one solution where $n = 3$ and $m = 5$, and another solution where $n = 12$ and $m = 5$, and so on. Write the double sum that represents the general solution to this equation.

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14. Write equations relating the initial conditions $z_0(x, y)$ and $v_0(x, y)$ to your arbitrary constants. These equations will involve sums. Solving them requires finding the coefficients of a double Fourier series, but for now it's enough to just write the equations.