Motivating Exercise for Partial Differential Equations: The Heat Equation

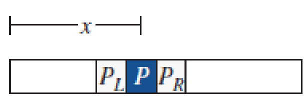
A coin at temperature u_c is placed in a room at a constant temperature u_r.[1] "Newton's Law of Heating and Cooling" states that the rate of change of the coin's temperature is proportional to the *difference* between the temperatures of the room and the coin. (In other words, the coin will cool down faster in a freezer than in a refrigerator.) We can express this law as a differential equation: $du_c/dt = k(u_r - u_c)$ where k is a positive constant.

1. What does this differential equation predict will happen if a cold coin is placed in a hot room? Explain how you could get this answer from this differential equation, even if you didn't already know the answer on physical grounds.

2. Verify that $u_c(t) = u_r + Ce^{-kt}$ is the solution to this differential equation.

Now, suppose we replace the coin with a long insulated metal bar. We assumed above that the coin had a uniform temperature u_c, changing with time but not with position. A long bar, on the other hand, can have different temperatures at the left end, the right end, and every point in between. That means that temperature is now a function of time and position along the bar: $u(x, t)$.

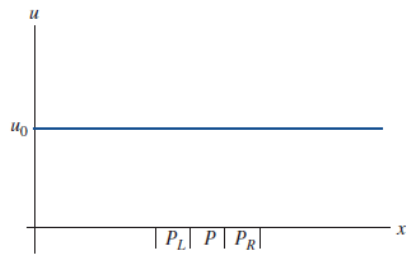
To write an equation for $u(x, t)$, consider how a small piece of the bar (call it P) at position x will behave in some simple cases.



We assume that piece P is so small that its temperature is roughly uniform. However, the pieces to the left and right of it (P_L and P_R) have their own temperatures. Piece P interacts with these adjacent pieces in the same way the coin interacted with the room: the rate of heat transfer between P and the pieces on each side depends on the temperature difference between them. We also assume that heat transfer between the different parts of the bar is fast enough that we can ignore any heat transfer between the bar and the surrounding air.

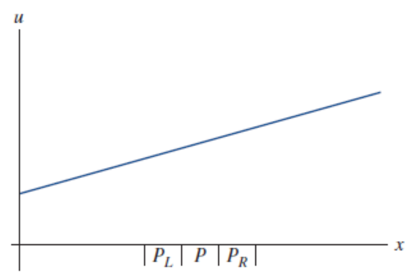
[1]Both u and T are commonly used in thermodynamics to represent temperature. Throughout this exercise we will use u.

3. First, suppose we start with the temperature of the bar uniform. We use $u(x, 0)$ to indicate the *initial temperature* of the bar—that is, the temperature at time $t = 0$—so we can express the condition "the initial temperature is a constant" by writing $u(x, 0) = u_0$.



   (a) Will P give heat to P_L, absorb heat from P_L, or neither?
   (b) Will P give heat to P_R, absorb heat from P_R, or neither?
   (c) Will the temperature of P go up, go down, or stay constant?

4. Now consider a linearly increasing initial temperature: $u(x, 0) = mx + b$, with $m > 0$.



   (a) Will P give heat to P_L, absorb heat from P_L, or neither?
   (b) Will P give heat to P_R, absorb heat from P_R, or neither?
   (c) Will the rate of heat transfer between P and P_L be faster, slower, or the same as the rate of heat transfer between P and P_R?
   (d) Will the temperature of P go up, go down, or stay constant?

5. Now consider a parabolic initial temperature: $u(x,0) = ax^2 + bx + c$. Assume $a > 0$ so the parabola is concave up, and assume that P is on the increasing side of the parabola, as shown in Figure 1.

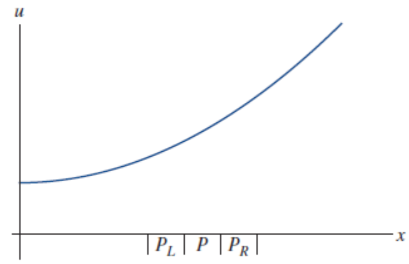


Figure 1

   (a) Will P give heat to P_L, absorb heat from P_L, or neither?
   (b) Will P give heat to P_R, absorb heat from P_R, or neither?
   (c) Will the rate of heat transfer between P and P_L be faster, slower, or the same as the rate of heat transfer between P and P_R?
   (d) Will the temperature of P go up, go down, or stay constant?

6. For each of the three cases you just examined, what were the signs (negative, positive, or zero) of each of the following quantities at point P: u, $\partial u/\partial x$, $\partial^2 u/\partial x^2$, and $\partial u/\partial t$? Just to be clear, you're giving 12 answers in all to this question. You'll get the signs of u, $\partial u/\partial x$, and $\partial^2 u/\partial x^2$ from our pictures, and $\partial u/\partial t$ from what was happening at point P in each case.

	u	$\frac{\partial u}{\partial x}$	$\frac{\partial^2 u}{\partial x^2}$	$\frac{\partial u}{\partial t}$
Part 3				
Part 4				
Part 5				

7. Which of the following differential equations would be consistent with the answers you gave to Part 6? In each one k is a real constant, so k^2 is a positive number and $-k^2$ is a negative number.

   (a) $\partial u/\partial t = k^2 u$
   (b) $\partial u/\partial t = -k^2 u$
   (c) $\partial u/\partial t = k^2(\partial u/\partial x)$
   (d) $\partial u/\partial t = -k^2(\partial u/\partial x)$
   (e) $\partial u/\partial t = k^2(\partial^2 u/\partial x^2)$
   (f) $\partial u/\partial t = -k^2(\partial^2 u/\partial x^2)$

The "heat equation" you just found is an example of a "partial differential equation," which involves partial derivatives of a function of more than one variable. In this case, it involves derivatives of $u(x,t)$ with respect to both x and t. In general, partial differential equations are harder to solve than ordinary differential equations, but there are systematic approaches that enable you to solve many linear partial differential equations such as the heat equation analytically. For nonlinear partial differential equations, the best approach is often numerical.