1.9 Additional Problems

1.179 Parts (a)–(e) below give five different scenarios for the mass of a spherical snowball $M(t)$. Match each one with the appropriate differential equation.

(a) Rolling down a hill, it gains 5 g/s.	I. $dM/dt = 2M$
(b) Every second, 5 g melt.	II. $dM/dt = M - 5$
(c) Rolling, the mass doubles every second.	III. $dM/dt = -5$
(d) Rolling, the mass triples every second.	IV. $dM/dt = 5$
(e) Every second the snowball picks up its own mass, but 5 g melt.	V. $dM/dt = M$

1.180 A thermostat is set to keep the house temperature at a constant 68°. If the temperature rises too high, the air conditioner brings it down; if the temperature falls too low, the heater brings it up. Using u for the room temperature, which of the following differential equations best represents this situation? Assume every k is a positive constant with the appropriate units.

(a) $du/dt = 68k$
(b) $du/dt = 68kt$
(c) $du/dt = 68ku$
(d) $du/dt = k(u - 68)$
(e) $du/dt = k(68 - u)$
(f) $du/dt = k(u + 68)$
(g) $du/dt = 68k_1(u + k_2 t)/(u - k_2 t)$

For Problems 1.181–1.186,

(a) Confirm that the given solution solves the Differential Equation (DE) for *any* value of the arbitrary constant(s).
(b) Find the specific solution that matches the given condition.

1.181 DE: $r'(\theta) + 3r(\theta) = \cos\theta$
Solution: $r = (\sin\theta + 3\cos\theta)/10 + Ce^{-3\theta}$
Condition: $r(0) = 1$

1.182 DE: $dx/dt = (tx - x^2)/t^2$
Solution: $x = t/(\ln t + C)$
Condition: $x(1) = e$

1.183 DE: $\dfrac{dy}{dx} = \dfrac{(x+y)\ln(x+y) - (x+2)}{x+2}$
Solution: $y = e^{C(x+2)} - x$
Condition: $y(0) = 10$

1.184 DE: $\dfrac{dy}{dx} = \dfrac{(\ln x - 1)(y^2+1)}{xy\ln x}$
Solution: $y = \sqrt{\dfrac{Cx^2}{(\ln x)^2} - 1}$
Condition: $y(e^2) = 3$

1.185 DE: $V''(x) - 4V(x) = 6x - 4x^3$
Solution: $V(x) = x^3 + C_1 e^{2x} + C_2 e^{-2x}$
Conditions: $V(0) = 0,\ V(\ln 2) = 1$

1.186 DE: $s''(t) - 9s + 8 = \sin t$
Solution: $s(t) = 8/9 - (\sin t)/10 + C_1 e^{3t} + C_2 e^{-3t}$
Conditions: $s(0) = 0,\ s'(0) = 0$

For Problems 1.187–1.192,
a) Find the general solution to the given differential equation. You may solve by simple inspection, by separation of variables, or by guess-and-check with an unknown constant.
b) Find the specific solution corresponding to the given condition(s).

1.187 $dy/dt = ky,\ y(0) = p$

1.188 $d^2y/dt^2 = -16y,\ y(0) = 0,\ y'(0) = 2$

1.189 $du/dx = u + 3,\ u(0) = 0$

1.190 $du/dx = (2x+1)u,\ u(0) = 1$

1.191 $d^2f/dx^2 - 5df/dx + 6f = 0,\ f(0) = 0,\ f(1) = 1$

1.192 $d^2f/dx^2 - 3df/dx + 2f = 6e^{3x}$,
$f(0) = 0,\ f'(0) = 0$

1.193 With sufficient food and no predators, the population of tribbles multiplies by 121 every day. However, the population is kept in check by a ravenous horde of vermicious knids, who eat 12,000 tribbles a day.

(a) Write a differential equation for the number of tribbles as a function of time.
(b) What is the equilibrium solution to your equation? Is it a stable or unstable equilibrium? How can you tell?
(c) Solve your differential equation with the initial condition $P(0) = 125$.

1.194 Solve the differential equation $d^2y/dt^2 = 3y$ with initial conditions $y(0) = 0$, $y'(0) = 4\sqrt{3}$.

1.195 $dy/dx = e^{x+y}$.

(a) Solve.

(b) Find the solution that contains the point $y(0) = -2$.

(c) Demonstrate that your solution satisfies both the differential equation and the initial condition.

1.196 The function $y = f(x)$ is the solution to the differential equation $dy/dx = (y-2)^2/(x-2)$ that goes through the point $(3, 0)$.

(a) Find the slope of this curve at the point $(3, 0)$.

(b) Find a formula for the *concavity* of this curve as a function of x and y. (Remember that concavity is the derivative with respect to x of dy/dx. After you use the quotient rule, your formula for concavity will have x, y, and dy/dx in it. Then you can replace dy/dx with $(y-2)^2/(x-2)$ to get a function of x and y.)

(c) Find the concavity of this curve at the point $(3, 0)$.

(d) Based on your answers to parts (a) and (c), draw a quick sketch of the curve around the point $(3, 0)$.

(e) Find the function $y = f(x)$.

1.197 The function $V(t)$ follows the differential equation $dV/dt = 2V^2 + V^3 - V^4$. Predict the long-term behavior of V. Your answer will consist of several different statements of the form "If V starts in this range, then it will head toward…"

1.198 Consider the differential equation $dy/dt = y^2$.

(a) Begin by drawing a slope field at all integer points in $t \in [0, 5]$, $y \in [-2, 2]$.

(b) Based on looking at your slope field, what is $\lim_{t\to\infty} y(t)$ if $y(0) = 1$? If $y(0) = -1$?

(c) Now solve the differential equation using separation of variables. Your solution should have an arbitrary constant C.

(d) Find two specific solutions for the two initial conditions $y(0) = 1$ and $y(0) = -1$.

(e) Find $\lim_{t\to\infty} y(t)$ for your two specific functions.

(f) In one case, you should find that the limit you predicted based on your slope field does not match the limit you predicted based on your solution. Explain why one (or both) of your predictions was wrong.

1.199 Consider the differential equation $dy/dx = y - x^2$.

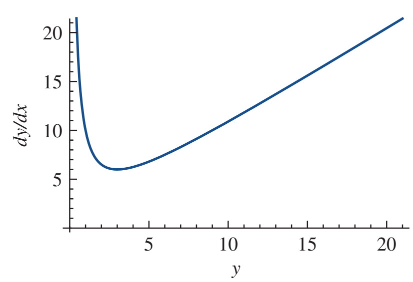
(a) Have a computer make a slope field for this equation. You may need to do some trial and error to find good ranges for x and y. You should have at least 80 points on your slope field.

(b) Have the computer plot the function $y = 2 + 2x + x^2$ on the same plot as the slope field you just found. You should be able to convince yourself that this function is following the slopes indicated on your slope field, and is thus a solution to the differential equation. (You can also check this analytically.)

(c) Predict how the function will behave if it has an initial condition that places it above the solution you just plotted. How will it behave if it has an initial condition below the one you plotted? Explain your predictions by referring to the behavior of the slope field above and below that solution.

(d) Have the computer numerically solve this differential equation with initial conditions $y(0) = 1$ and $y(0) = 3$. Make a final plot showing the slope field, the particular solution you plotted before, and these two numerical solutions. Does their behavior match your predictions? Explain.

1.200 The drawing below shows dy/dx as a function of y. $\lim_{y\to 0^+}(dy/dx) = \infty$ and $\lim_{y\to\infty}(dy/dx) = \infty$, and the absolute minimum occurs where $y = 3$ and $dy/dx = 6$. The function is undefined for $y \le 0$.



Sketch the function $y(x)$ that goes through the point $x = 0$, $y = 3$.

1.201 A dish contains an amount of bacteria $B(t)$. On average each bacteria cell produces one new offspring each day. At the same time an enzyme in the dish kills 1000 bacteria cells per day. Write a differential equation for $B(t)$ and solve it, assuming the dish starts with 1200 bacteria cells. (In writing your equation you should assume time is measured in days and B is measured in number of cells.) How many bacteria cells does the dish contain after ten days?

1.202 In 1930, psychologist Louis L. Thurstone proposed[9] that learning follows the differential equation $dp/dt = k[p(1-p)]^{3/2}$ where p represents mastery of a task or topic on a scale of 0 to 1 and k is a positive constant.

- **(a)** What are the equilibrium solutions, and what do they represent about learning?
- **(b)** At what p-value does a student learn the fastest?
- **(c)** Sketch the general shape of a learning curve according to Thurstone's model.

This equation can be solved analytically. We encourage you to have a computer solve it for you, which will give you a good appreciation for why sometimes an analytic solution to an ODE is not the best way to understand it.

1.203 We have worked several times with drag forces proportional to velocity. In some cases it is more accurate to model a drag force as proportional to velocity *squared*: $F = -kv^2$ where k is a positive constant.

- **(a)** Explain why this law, as written, only makes sense for positive v-values.
- **(b)** Using Newton's Second Law, write a first-order differential equation for the velocity of an object experiencing this drag force and no other forces.
- **(c)** Draw a slope field at all integer points in $t \in [0,5]$, $v \in [0,2]$.
- **(d)** Based on looking at your slope field, what is $\lim_{t\to\infty} v(t)$ if $v(0) = 0$? If $v(0) = 1$?
- **(e)** Solve the differential equation using the initial condition $v(0) = v_0$.
- **(f)** How long will it take the object to reach 1/10 of its original speed? Your answer will depend on k, m, and v_0.
- **(g)** Find the body's *position* function $x(t)$, assuming $x(0) = 0$.
- **(h)** Draw a quick sketch of $x(t)$ for $t \geq 0$.

1.204 Black Holes and Hawking Radiation In 1974 Stephen Hawking predicted that, due to quantum effects, black holes should lose mass. This process, called "evaporation," follows the equation $dM/dt = -\hbar c^4/(15{,}360\pi G^2 M^2)$ where M is the mass of the black hole and all other quantities are constants.

- **(a)** We can make this equation more tractable by grouping the constants together. Let $k = \hbar c^4/(15{,}360\pi G^2)$ and rewrite the differential equation in a simpler form.
- **(b)** Looking at your differential equation, does it predict that black holes will grow or shrink? Does it predict that large black holes will change more slowly or more quickly, than small ones? Based on these facts, write a brief description of the life of a black hole.
- **(c)** Solve the differential equation with the initial condition $M = M_0$.
- **(d)** Look up values of $\hbar$ (Planck's constant divided by 2π), c (the speed of light), and G (the gravitational constant). Make sure they are all in standard SI units! Put them together to find the value of your constant k.
- **(e)** The most commonly observed type of black hole is a "stellar black hole," which has a mass comparable to that of the sun: 2×10^{31} kg. How long would such a black hole take to evaporate entirely?
- **(f)** The universe is roughly 1.4×10^{10} years old. Express the lifetime of a stellar black hole as a multiple of the current age of the universe.
- **(g)** It's possible that particle physics experiments will produce microscopic black holes with mass comparable to a proton: about 2×10^{-27} kg. As of this writing, there has been no evidence of such black holes being produced, but some people worry that if they were they would destroy the Earth. How long would such a black hole last? Is this something we should be worried about?[10]

[9] Thurstone, L.L. The Learning Function. *Journal of General Psychology,* 1930, 3, 469–493.

[10] Aside from the short lifetime of such black holes, there's a simpler argument for why they couldn't destroy Earth. Cosmic rays constantly strike the upper atmosphere with energies far greater than we can produce in a lab. If these experiments could produce anything that would destroy Earth it would have happened billions of years ago.

1.205 Rocket Science
A rocket engine produces thrust by burning and expelling fuel. Newton's Second Law $F = ma$ holds as always, but m is not a constant. Assume the rocket begins with a total mass of m_0 and burns off mass at a steady rate of k (measured in mass per unit time), producing a constant force of F.

(a) Write a function for the rocket's mass $m(t)$. (This is a high school algebra problem; don't try to make it complicated.)

(b) Write a second-order differential equation for the rocket's position $x(t)$.

(c) Because your equation gives d^2x/dt^2 as a function of t only (no x-dependence), you can integrate twice to find the general solution. Your final answer should have two independent arbitrary constants.

(d) Assuming the rocket begins its journey at rest at $x = 0$, solve for your arbitrary constants and find the function $x(t)$. Your answer will, of course, contain the constants F, m_0, and k.

(e) If the rocket begins with a mass of 5000 kg and burns mass at a rate of 60 kg/s, exerting a constant force of 5,000,000 N, how fast is it going after 50 s? How far has it traveled?

1.206 The Barometric Formula
A region is filled with fluid (gas or liquid). This fluid is everywhere subjected to the downward force of gravity, balanced by the "pressure gradient" (lower altitudes are at higher pressure). This equilibrium is represented by the equation $dP/dz = -\rho g$, where P is the pressure, z is height, ρ is density, and $g = 9.8$ m/s^2 is the acceleration due to gravity. (Deriving this equation is a physical problem unrelated to what we want to convey in this chapter, so we're just giving it to you.)

(a) For a liquid, density generally stays pretty constant. Treating ρ as a constant, find $P(z)$. Your answer should of course include an arbitrary constant.

(b) For a gas, the ideal gas law gives the density as $\rho = mP/(RT)$, where m is the molar mass and R is the ideal gas constant. Assuming constant temperature, find $P(z)$. This result is known as the "barometric formula."

1.207 *[This problem depends on Problem 1.206.]* In Problem 1.206(b) you derived the barometric formula for pressure variation with depth by assuming a constant temperature. For the lower portion of the Earth's atmosphere, known as the troposphere, a more realistic model is a linear variation of temperature with altitude: $T = T_0 - kz$. Calculate $P(z)$ under this assumption.

1.208 Newton's Law of Heating and Cooling
A small object (such as a hot cup of coffee or a cold soda) at temperature Q is placed inside a room with ambient temperature Q_r. According to "Newton's Law of Heating and Cooling," the rate of change of the object's temperature is proportional to the *difference* in temperature between the object and the room.

(a) Newton's Law can be expressed as $dQ/dt = k(Q - Q_r)$ or as $dQ/dt = k(Q_r - Q)$. Both are mathematically valid, but we want to choose the one that will lead to a positive k-value. (That way we know that e^{kt} is blowing up and e^{-kt} is approaching zero.) Which of these equations will give us a positive k-value? Will it work for both the hot coffee and the cold soda? How can you tell?

(b) Solve Newton's Law by separating variables. Note that your function $Q(t)$ will have three constants in it: Q_r (the temperature of the room), k (the constant of proportionality), and C (an arbitrary constant).

(c) What is $\lim_{t\to\infty} Q(t)$?

(d) A 90°C cup of coffee placed in a 70°C room reaches 80°C in 10 min. How long will it take to reach 71°C?

1.209 Radioactive Decay
A radioactive sample consists of a large collection of unstable atoms. In any given day (or year or century), every one of those atoms has a certain chance of decaying, independent of all the other atoms. Therefore, the more atoms you have, the more atoms decay: $dM/dt = -kM$ where k is a positive constant.

(a) Draw a slope field for this differential equation on all integer points in $0 \le t \le 4$, $0 \le M \le 4$ using $k = 1/2$. Then draw two sample curves through your slope field.

(b) Based on your slope field, what is $\lim_{t\to\infty} M(t)$?

(c) Solve the differential equation with the initial condition $M(0) = M_0$. (Do *not* set $k = 1/2$ as you did in Part (a); leave it as an unknown constant that varies from one substance to another.)

Does the resulting function match the curves you drew in Part (a)?

(d) The "half-life" $t_{1/2}$ of a radioactive substance is the amount of time elapsed before $M = M_0/2$. Write a formula for the half-life as a function of k.

(e) If you measure that 10% of a radioactive sample has decayed in one day, how long will it be until 90% decays?

1.210 *[This problem depends on Problem 1.209.]*

(a) Uranium-235 is used in atomic bombs. How long does it take a sample of ^{235}U to decay to a tenth of its original mass?

(b) Carbon-14 is used in dating ancient organic samples. How long does it take a sample of ^{14}C to decay to a tenth of its original mass?

1.211 Rate of a Chemical Reaction. When iron and sulfur are heated together, they can combine to create iron sulfide. Each time an Fe molecule collides with an S molecule, there is a chance that they will form an FeS molecule: **Fe+S→FeS**.

Let f equal the number of g-moles of iron, s equal the number of g-moles of sulfur, and p equal the number of g-moles of iron sulfide produced. The system begins with $f = f_0$, $s = s_0$, and $p = 0$.

(a) Explain why, at any given time, $f = f_0 - p$.

(b) Suppose at a given moment $df/dt = -5$. What is ds/dt? What is dp/dt?

(c) Because the reaction depends upon random collisions between Fe and S molecules, the rate of reaction dp/dt is proportional to the *product fs*. Write a differential equation for $p(t)$. Note that your differential equation will include the variable p and the constants f_0 and s_0, but cannot include the variable quantities f and s.

(d) Solve your differential equation assuming that $f_0 = s_0$.

(e) Solve your differential equation assuming that $f_0 \neq s_0$. (*You can solve this on a computer or do it by hand using partial fractions.*)

(f) Find $\lim_{t\to\infty} p(t)$. The answer will depend on whether $s_0 > f_0$, $s_0 = f_0$, or $s_0 < f_0$.

1.212 Spread of a Disease
The *logistic equation* is used to predict, among other things, the spread of disease through a fixed population. For instance, a disease might follow the equation $dP/dt = 2P(1 - P)$ where P is the fraction of the population that is infected.

(a) For very low values of P, $1 - P$ is approximately 1, so the differential equation is approximately $dP/dt = 2P$. What kind of growth is predicted by this equation? What does it suggest about the mechanism of disease spread?

(b) For very high values of P, $1 - P$ approaches zero, so the growth slows down. Why does this make sense in terms of how the disease spreads?

(c) Draw a slope field for this equation for $P = 0, 1/8, 1/4, 3/8, 1/2, 5/8, 3/4, 7/8, 1$. Ignore negative t-values.

(d) Based on looking at your slope field, list the equilibrium solutions and classify them as stable or unstable.

(e) Trace a sample curve through your slope field, starting at $P(0) = 1/8$. Your curve should extend far enough to determine $\lim_{t\to\infty} P(t)$.

1.213 *[This problem depends on Problem 1.212.]*

(a) Solve the differential equation in Problem 1.212. (*You can solve this on a computer or do it by hand using partial fractions.*)

(b) Find the specific solution that corresponds to the initial condition $P(0) = 1/8$.

(c) Find $\lim_{t\to\infty} P(t)$. Make sure it matches your prediction in Problem 1.212.

(d) Graph your solution on a computer. Make sure the graph matches the graph you drew based on your slope field.