CHAPTER 13

Calculus with Complex Numbers (Online)

13.6 Integrating Along Branch Cuts and Through Poles

13.6.1 Explanation: Integrating Along Branch Cuts and Through Poles

We have seen that it is important to know if a closed contour encloses a pole. But what if the contour goes directly through a pole? Or what if a contour travels along a branch cut of the function being integrated?

There is a fundamental problem with evaluating an integral where the function itself is ill-defined. For instance, along the positive real axis you can consider ϕ to be 0 or 2π (or infinitely many other possibilities). So $\int_1^2 \phi\, dz$ could be 0 or 2π and so on. The "right answer" is just a matter of definition.

So we might declare all such integrals to be undefined, but there are compelling reasons to work with such tricky cases. For instance, we have seen that complex integrals along the real axis are used to evaluate real integrals. But the real axis includes the branch cuts for common functions such as $\sqrt{x}$ and $\ln x$. If we invalidate all integrals along branch cuts we lose a valuable approach to many important problems.

Such problems have an analogue in the real-valued world. You may recall that an integral such as $\int_0^1 \ln x\, dx$ is called an "improper integral" because the integrand has a vertical asymptote at $x = 0$. We approach such an integral as a limit.

$$\int_0^1 \ln x\, dx = \lim_{a \to 0^+} \int_a^1 \ln x\, dx$$

In words, we evaluate the integral in a region where the function is always well defined, and then take a limit as that region approaches the asymptote.

Analogously, when a contour integral goes directly through a pole or along a branch cut, we begin by drawing a contour that comes *near* that pole or branch cut. Along such a contour the function, and therefore the integral, is uniquely defined. Then we take a limit as the contour approaches the singularity.

The following example demonstrates most of what you need to know about this technique. (A few more issues are brought out in Problem 13.84.) But we hope you will notice that almost everything you need are the things you *already* know about contour integrals, just being put together in a new way.

At one key point we will have to use the following theorem, which will also prove useful in many of the problems in this section. (We state it here without proof but it makes a lot of sense if you think about it. You can also prove it reasonably easily with a Riemann sum.)

The modulus of a contour integral must be less than or equal to the maximum modulus of the integrand on the contour times the length of the contour.

$$\left|\int_C f(z)\,dz\right| \le |f(z)|_{\max} \times arclength(C) \tag{13.6.1}$$

We will be using that theorem to prove that certain integrals must be zero by showing that the product (modulus times arclength) is zero.

Example: Integrating Along a Branch Cut

We begin with the integral of a real function along the real number line.

$$R = \int_0^{\infty} \frac{1}{\sqrt{x}(x+1)}\,dx \tag{13.6.2}$$

The problem has nothing to do with complex numbers, but we have seen that the easiest way to approach such problems often begins with a contour integral.

$$\int_0^{\infty} \frac{1}{\sqrt{z}(z+1)}\,dz \tag{13.6.3}$$

Are these two integrals the same? This question is subtler than you might think, because in Equation 13.6.2 everything is real and $\sqrt{x}$ is by definition a positive number. In Equation 13.6.3 we have to decide what branch of $\sqrt{z}$ we are using.

Remember that $\sqrt{z}$ has phase $\phi_z/2$. We usually use the principal branch of ϕ, which goes from $-\pi$ to π. In that case $\sqrt{z}$ has a branch cut on the negative real axis, while on the positive real axis $\phi = 0$ and $\sqrt{z}$ is positive. For this problem, however, we will choose to put the branch cut on the positive real axis. Thus when z is slightly above the positive real axis $\phi \approx 0$ and $\sqrt{z}$ is (approximately) a positive real number, but when z is slightly below the positive real axis $\phi \approx 2\pi$ and $\sqrt{z}$ is (approximately) a negative real number.

With this choice Equation 13.6.3 is not strictly defined, but if you shift the contour slightly up in the complex plane then it equals R, the solution to Equation 13.6.2. If you shift the contour slightly down the solution to Equation 13.6.3 equals $-R$.

It might not seem like any of this is getting us closer to evaluating Equation 13.6.2, but at this point we have all the ingredients we need. To see how this works, consider integrating $f(z)$ around the closed contour C shown in Figure 13.11. The contour avoids both the pole at $z = 0$ and the branch cut along the real axis. Contour C consists of four distinct pieces: a horizontal line just above the positive real axis, a horizontal line just below the positive real axis, a small circle around the origin (whose radius we will call ρ_1), and a large circle around the origin (radius ρ_2). The integral around the entire contour C must equal the sum of these individual integrals. In the limit as $\rho_1 \to 0$ and $\rho_2 \to \infty$ the two horizontal lines go from $x = 0$ to $x \to \infty$.

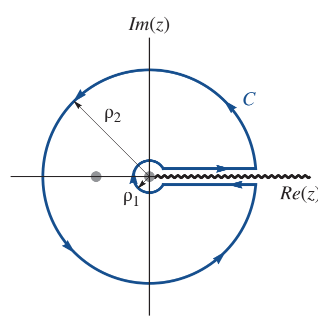


FIGURE 13.11 The contour C is made up of four parts, two horizontal lines just above and below the positive real axes, and two (almost) circles of radius ρ_1 and ρ_2. The gray dots indicate poles.

- **The entire contour.** The contour C is closed, and the integral around any closed contour depends on the enclosed poles. The pole at $z = 0$ is *not* enclosed by this contour! But for $\rho_1 < 1$ and $\rho_2 > 1$ the pole at $z = -1$ is enclosed. We can rewrite the function as this.

$$f(z) = \frac{1/\sqrt{z}}{z - (-1)}$$

That is the correct form for Cauchy's integral formula with $g(z) = 1/\sqrt{z}$ and $z_0 = -1$. The integral is therefore:

$$2\pi i g(z_0) = \frac{2\pi i}{\sqrt{-1}} = 2\pi$$

- **The small circle around the origin.** We are interested in the integral around this loop as $\rho_1 \to 0$. We will show that this integral must approach zero by invoking Equation 13.6.1. To find the modulus of $f(z)$ we multiply it by its complex conjugate and then take the square root.

$$f(z)f(z)^* = \frac{1}{\sqrt{z}(z+1)}\frac{1}{\sqrt{z^*}(z^*+1)} = \frac{1}{\sqrt{zz^*}(zz^* + z + z^* + 1)}$$

The quantity $z + z^*$ is $2Re(z)$. On a circle of radius ρ_1 the quantity zz^* is always ρ_1^2.

$$f(z)f(z)^* = \frac{1}{\rho_1(\rho_1^2 + 2Re(z) + 1)}$$

We don't need calculus to maximize this quantity: we just have to minimize the denominator, which occurs on the negative real axis where $Re(z) = -\rho_1$.

$$f(z)f(z)^*_{\max} = \frac{1}{\rho_1(\rho_1^2 - 2\rho_1 + 1)} = \frac{1}{\rho_1(1-\rho_1)^2}$$

$$|f(z)|_{\max} = \sqrt{f(z)f(z)^*_{\max}} = \frac{1}{\sqrt{\rho_1}(1-\rho_1)}$$

(Within the world of positive real numbers, the square root always means a positive number. Because $\rho_1 < 1$ we therefore wrote $(1 - \rho_1)$ for the square root of $(1 - \rho_1)^2$.) Now we multiply $|f(z)|_{\max}$ by the arclength of the contour, which is $2\pi\rho_1$, to put an upper limit on the integral.

$$\left|\oint f(z)dz\right| \le \frac{2\pi\sqrt{\rho_1}}{1-\rho_1}$$

In the limit as $\rho_1 \to 0$ this approaches zero. We conclude that the integral itself must approach zero.

- **The outer circle.** The math is almost the same as for the inner circle with ρ_1 replaced with ρ_2. Because the values of ρ_2 that interest us are greater than 1, we replace $\sqrt{(\rho_2 - 1)^2}$ with $\rho_2 - 1$. Then we take the limit $\rho_2 \to \infty$, and we once again get 0.

$$\left|\oint f(z)dz\right| \le \lim_{\rho_2 \to \infty} \frac{2\pi\sqrt{\rho_2}}{\rho_2 - 1} = 0$$

- **The horizontal lines.** In the limit where the horizontal lines approach the positive real axis the integral along the upper line becomes R, the real-valued integral that we are trying to find. Along the bottom horizontal line, as $\phi \to 2\pi$, all the values of $\sqrt{z}$ are the negative versions of those same real numbers. But we are integrating that line in the negative direction, which brings in another negative sign. Therefore the bottom line contributes exactly the same R as the top line.

We now have all the pieces to evaluate the integral we started with.

entire loop =	small (ρ_1) circle	+	big (ρ_2) circle	+	top line	+	bottom line
$2\pi =$	0	+	0	+	R	+	R

We finally conclude that $R = \pi$, and thus solve the problem we began with.

The example above showcases the general approach to such problems, along with most of the tricks that you will need along the way. Only one complication remains: what if there had been a pole on the positive x-axis, right in the middle of the region of integration? Our contour C would then have to circle around that pole. You will work such an example in Problem 13.84.

13.6.2 Problems: Integrating Along Branch Cuts and Through Poles

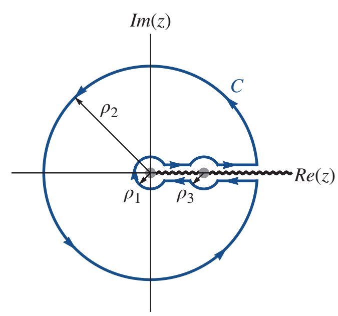
13.84 Walk-Through: Integration Through a Pole. In this problem you're going to evaluate the integral

$$R = \int_0^\infty \frac{1}{x^{1/3}(x-1)}dx$$

This looks similar to the problem we solved in the Explanation (Section 13.6.1) but there are two differences. The first is that we've used a cube root instead of a square root, which will give you a bit more practice thinking about branch cuts. The more important difference is that this integrand has a pole on the *positive* real axis, right on the region of integration. You can define the value of this improper integral as:

$$R = \lim_{\rho_1 \to 0^+, \rho_2 \to \infty, \rho_3 \to 0^+} \left[\int_{\rho_1}^{1-\rho_3} \frac{1}{x^{1/3}(x-1)}dx + \int_{1+\rho_3}^{\rho_2} \frac{1}{x^{1/3}(x-1)}dx \right]$$

Take the branch cut of $z^{1/3}$ to be along the positive real axis. To solve this problem, you're going to define $f(z) = 1/[z^{1/3}(z-1)]$ and integrate it on the closed contour C shown below. The small circle around the origin has radius ρ_1, the large circle has radius ρ_2, and the circle around the pole has radius ρ_3.



(a) Evaluate the contour integral $\oint_C f(z)dz$ along the entire contour C by considering the poles of $f(z)$.

(b) As in the Explanation, use Equation 13.6.1 to show that the integrals around the inner and outer loops vanish in the limits $\rho_1 \to 0$ and $\rho_2 \to \infty$.

(c) Now consider the loop of radius ρ_3 around $z = 1$. We can represent this entire loop as $z = 1 + \rho_3 e^{i\theta}$. (We didn't call it ϕ to avoid confusion with the phase of z.) As $\rho_3 \to 0^+$ the $z^{1/3}$ in the denominator of our fraction simply approaches $1^{1/3}$ (although even there you have to be careful to use the correct branch for each half-loop). But the $z - 1$ in the denominator approaches zero, and as you know zeroes in the denominator of a limit require extra care.

i. For the top half of the loop, the phase ("where-you-are-in-the-complex-plane" ϕ, not "where-you-are-in-our-little-loop" θ) is zero so $z^{1/3} = 1$. Rewrite $f(z)$, replacing $z^{1/3}$ with 1 and $z - 1$ with $\rho_3 e^{i\theta}$, and then integrate as θ goes from 0 to π. Then let $\rho_3 \to 0$ in your answer. *Hints*: Don't forget to calculate dz in terms of $d\theta$, and pay attention to the direction of integration shown in the picture above.

ii. For the bottom half of the loop, $\phi = 2\pi$. Rewrite $f(z)$, replacing $z^{1/3}$ with $e^{2\pi i/3}$ and $z - 1$ with $\rho_3 e^{i\theta}$, then integrate (the limits will be different for this half), and finally take the limit.

(d) As $\rho_1 \to 0$ and $\rho_2 \to \infty$ the upper horizontal contour equals R, the real integral you are trying to find. How is the lower horizontal contour related to R? *Hint*: The answer is neither R nor $-R$.

(e) Set the integral around the entire loop equal to the sum of all the contours that make it up and solve the resulting equation for R, the real integral we set out to find. Make sure your answer is a real number.

13.85 *[This problem depends on Problem 13.84.]* We plugged the integral from Problem 13.84 into a computer and got the (quite correct) answer that it diverges. Find a way to get your computer to evaluate this integral without using any complex numbers and get the answer from Problem 13.84, which is the Cauchy principal value.

Evaluate the integrals in Problems 13.86–13.89 by following the basic template outlined in the Explanation (Section 13.6.1). If the region of integration goes directly through a pole you will need to bend your contour around that pole as demonstrated in Problem 13.84. At the end you should write your answers in real form.

13.86 $R = \int_0^\infty dx/[x^{2/3}(x-1)]$.

13.87 $R = \int_0^\infty dx/[x^{1/4}(x+3)]$.

13.88 $R = \int_0^\infty dx/[\sqrt{x}(x^2-4)]$.

13.89 $R = \int_0^\infty dx/[x^k(x-x_0)]$ where $0 < k < 1$, $x_0 < 0$. (You'll do the case where $x_0 > 0$ in Section 13.11 Problem 13.153.) *Hint*: When you are putting your final answer in real form you may find it helpful to replace x_0 with $|x_0|e^{i\pi}$, but don't do this until you get to the end.

13.90 Try evaluating $\int_0^\infty \sqrt{x}/(x+1)dx$ using the techniques of this section. Explain why it doesn't work.

13.91 In this problem we introduce a trick for contour integrals involving logarithms. Our first use of this trick will be an integral that you can easily evaluate without complex numbers (which is a good way to check your answer); in Problem 13.92 you will apply the same trick to a harder problem. The question for this problem is $\int_0^{x_0} \ln x \, dx$ where $x_0 > 0$. Use R to represent the real-valued integral we are looking for.

(a) Start with the complex function $\ln z$, defining the branch cut along the positive real axis. The integral of $\ln z$ just above the positive real axis going forward from 0 to x_0 is R, the integral we want. How is the integral of $\ln z$ that goes backward from x_0 to 0 just *below* the positive real axis related to R? Given that answer, explain why directly applying the technique we have used in this section will *not* help us find the integral we want.

(b) Here is an apparently unrelated question: find the real and imaginary parts of the function $f(z) = (\ln z)^2$. (This is the function that we will define as $f(z)$ for the rest of this problem.)

(c) Now consider the integral of $f(z)$ along the two horizontal paths you used in Part (a). Write an expression for the sum of those two integrals. Your answer should be in terms of R, the original integral we wanted to find.

(d) Finish drawing a closed contour using those two horizontal lines and avoiding both the branch cut and the origin. This contour will look just like many of the ones we have used in this section, with the notable exception that the larger loop is *not* going to infinity. Find the integral of $f(z)$ around this entire closed loop.

(e) Argue that the integral of $f(z)$ around the inner loop is zero in the limit where the radius approaches zero. The following limit (which you can prove using l'Hôpital's rule) may prove helpful: $\lim\limits_{x \to 0^+} x(\ln x)^2 = 0$.

(f) Along the outer circle $z = x_0 e^{i\phi}$. Evaluate the integral of $f(z)$ along the outer circle by converting to an integral over ϕ. (Earlier you broke $f(z)$ into its real and imaginary parts, which should have given you a total of three terms. You can do those three integrals separately and sum the results. Two of them will require integration by parts.)

(g) Use all the pieces you have found to solve for the original integral.

13.92 Let $R = \int_0^\infty [(\ln x)/[(x+1)(x+2)]dx$. To solve this you'll begin by defining $f(z) = (\ln z)^2/[(x+1)(x+2)]$, with the branch cut of $\ln z$ to be along the positive real axis.

(a) Draw a contour like the ones we've been using in this section, going above and below the branch cut and closing the contour with a small circle around the origin and a large circle that approaches infinity. Evaluate $\int f(z)\, dz$ around this entire contour.

(b) Find the sum of the integrals above and below the branch cut. Remember that the second one goes backwards and that $\ln z$ is different on these two contours. Your answer should be given in terms of R. *Hint*: at one point in these calculations you should find that you need to integrate $dx/[(x+1)(x+2)]$. You can do that by rewriting it as $dx/(x+1) - dx/(x+2)$. You could find that using partial fractions, but you can even more easily check that these formulas are equal.

(c) Prove that the integrals around the small and large loops both go to zero. The following limit (which you can prove using l'Hôpital's rule) may prove helpful: $\lim_{x\to 0^+} x(\ln x)^2 = 0$. (You will also need to use l'Hôpital's rule for another, more straightforward limit.)

(d) Put all of your answers together to find R.