Discovery Exercise for Linear Approximations

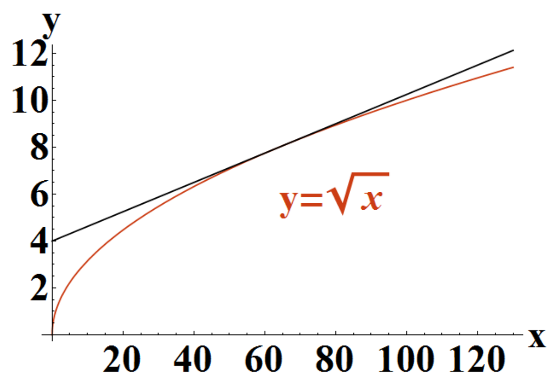

Figure 1: The function $y = \sqrt{x}$ (solid) and the tangent line to that curve at the point $x = 64$ (dashed)

1. Suppose the curve $y = \sqrt{x}$ models some real-world function, and we use the tangent line to approximate this function. In general, for what kinds of x-values will this approximation work best? For what kinds of x-values will it work very poorly?

A "tangent line," by definition, matches the curve at one point in two ways: it has the same y-value, and it has the same derivative.

2. At the point where $x = 64$, calculate the y-value of the curve, $f(64)$. (The line will have the same y-value at $x = 64$.)

3. At the point where $x = 64$, use a derivative to calculate the slope of the curve, $f'(64)$. (The line will have the same slope.)

4. Based on the point from Part 2 and the slope from Part 3, find the equation of the tangent line.

5. Plug $x = 69$ into both functions: the original curve $y = \sqrt{x}$, and the tangent line. If the tangent line value is used to approximate the real value, what is the percent error? Recall that the formula for percent error is $\left|\frac{\text{(real value)}-\text{(approximation)}}{\text{(real value)}}\right| \times 100$.

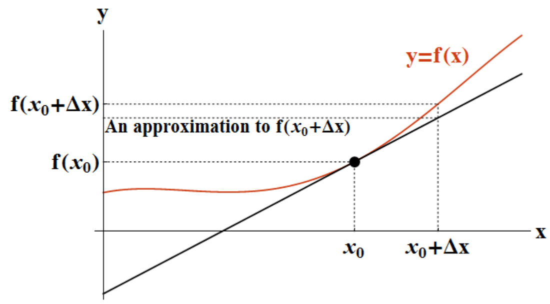

Figure 2: A function $f(x)$ and the tangent line at $x = x_0$.

6. Use the tangent line to approximate $\sqrt{100}$. What is the percent error this time?

7. Use the tangent line to approximate $\sqrt{64.5}$. What is the percent error this time?

Now we're going to repeat the same exercise, but for a generalized function $y = f(x)$ at a point $x = x_0$.

8. At the point where $x = x_0$, the y-value of the curve—and therefore the line—is $f(x_0)$. The slope of the curve—and therefore the line—is $f'(x_0)$. Find the equation of the line. That is, find a general formula for the only function $y = mx + b$ that has a slope of $f'(x_0)$ and goes through the point $(x_0, f(x_0))$.

9. Plug the point $x = (x_0 + \Delta x)$ into the tangent line equation to calculate its y-value on the line, and therefore approximate its y-value on the curve.

   See Check Yourself #8 at felderbooks.com/checkyourself

10. When we approximated $\sqrt{69}$, the linear approximation was higher than the actual value. For the curve in Figure 2, will the linear approximation be high or low?

11. *In general*, for what kinds of curves will the linear approximation come out high?