Discovery Exercise for Functions of Complex Numbers

Recall that any complex number can be written in the "polar" form $z = |z|e^{i\phi}$, where the modulus $|z|$ is the distance from the origin to the point in the complex plane, and the phase ϕ is the polar angle. For example, the number $2i$ can be written as $2e^{i\pi/2}$.

1. Find a number that can correctly fill in this blank: $e^{\underline{?}} = i$. In other words, find one possible value for $\ln i$.

2. Using the fact that the phases ϕ and $\phi + 2\pi$ represent the same direction in the complex plane, write all of the possible values of $\ln i$. There are infinitely many of them, but you should be able to write them all in a simple form using n to stand for an arbitrary integer.

3. Find the real and imaginary parts of $\ln z$, where $z = |z|e^{i\phi}$ for real $|z|$ and ϕ. The rules of logs will be helpful here, but to find *all* the answers you will also need to think about the work you did above.

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You've just shown that $\ln z$ is a "multiple-valued function," meaning that for any complex number z the function $\ln z$ yields more than one answer (infinitely many in this case). To define a single-valued version of the log function, you need to restrict the values of the phase ϕ to one value for each direction in the complex plane. The most common choice for how to do that is $-\pi < \phi \leq \pi$. The resulting values are called the "principal values" of the logarithm.

4. What is the principal value of $\ln(-i)$?

5. If you move around the complex plane taking the principal value of $\ln z$ at each point, it will be continuous everywhere except when you cross one ray (half a line). What is that ray?

6. You could define a single-valued logarithm function by restricting the phase to $0 < \phi \leq 2\pi$. If you do that, where will the discontinuity in the function occur?