Motivating Exercise for Calculus with Complex Numbers: Laplace's Equation

In a region with no heat sources or sinks the steady state temperature T obeys Laplace's equation: $\partial^2 T/\partial x^2 + \partial^2 T/\partial y^2 = 0$. For some boundary conditions this is relatively easy to solve by inspection. For example, consider the semi-infinite slab shown in Figure 1. The left edge at $x = 0$ is held at $T = 1$ (in some units). The right edge at $x = \pi/2$ is held at $T = 0$. The bottom at $y = 0$ is insulated, which means $\partial T/\partial y = 0$ along that edge.[1]

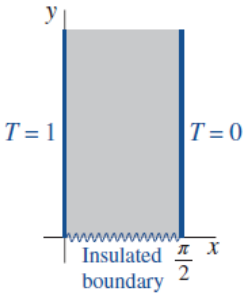

Figure 1

1. The symmetry of these boundary conditions means that the solution $T(x, y)$ must be independent of y. Using that fact, rewrite Laplace's equation as an ODE for $T(x)$ and solve it with the boundary conditions $T(0) = 1$ and $T(\pi/2) = 0$.

See Check Yourself #90 at felderbooks.com/checkyourself

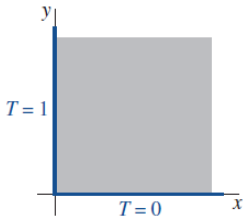

Figure 2

Next consider Laplace's equation in the entire first quadrant, with boundary conditions $T = 0$ on the positive x axis and $T = 1$ on the positive y axis. For these boundary conditions, it's more convenient to write Laplace's equation in polar coordinates.

$$\frac{\partial^2 T}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial T}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 T}{\partial \phi^2} = 0$$

2. Once again, from symmetry, we can conjecture that T only depends on one of the two polar coordinates. Which one, and why?

3. Using that fact, rewrite Laplace's equation as an ODE and solve it with the correct boundary conditions.

See Check Yourself #91 at felderbooks.com/checkyourself

[1] For the problems in this exercise to have unique solutions we have to assume the implicit boundary condition that T remains finite as $y \to \infty$.

Those problems may have required dusting off a few skills with partial derivatives and ODEs, but hopefully nothing about them seemed new or unsolvable. Now try this one on for size. Consider the same semi-infinite slab we started with, but this time assume the bottom and right edges are at $T = 0$ and the left edge is at $T = 1$. You're welcome to try to solve this, but unless you've studied complex analysis you're going to find it a lot harder than the others.

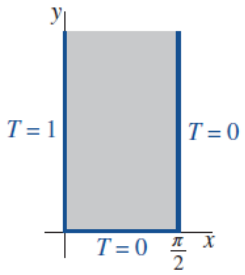

Figure 3

4. Even though you can't solve Laplace's equation with these boundary conditions, you can still predict a fair amount about the behavior from what you know. What is the temperature range inside the slab? Where is it hottest, and where coldest? What other qualitative predictions can you make?

We'll go ahead and solve the equation for you: $T(x, y) = (2/\pi)\tan^{-1}\left[\cot(x)(e^y - e^{-y})/(e^y + e^{-y})\right]$. Sorry to give that away when you were just on the verge of guessing it!

Part of the study of "complex analysis" is learning how to use "analytic functions" to solve Laplace's equation in simple regions. Another part is using complex functions to "map" one region to another. Once you have learned both these skills you can put them together to show that our third problem, which looks superficially like the slab problem we started with, is actually mathematically equivalent to the problem in Figure 2. Once you have the solution to the first quadrant problem, 5-10 minutes of easy algebra can convert it to this hideous-looking solution to the second slab problem.