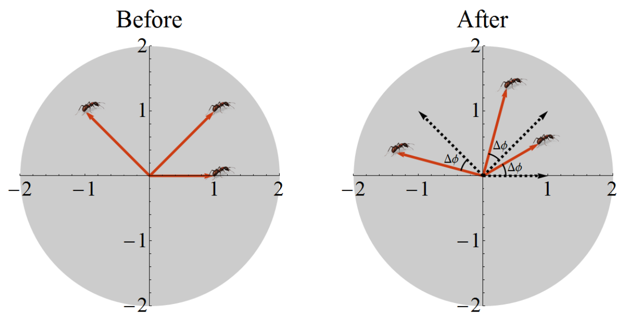
Discovery Exercise for Geometric Transformations

The picture below shows three ants on a turntable. A short time later the turntable has rotated counterclockwise by an angle $\Delta\phi$. You're going to find the new position of the ants.



1. Start by considering an ant at some arbitrary position $x_0\hat{i} + y_0\hat{j}$. Express x_0 and y_0 in terms of the polar coordinates ρ_0 and ϕ_0.

2. Similarly express x_f and y_f, the coordinates after rotation, in terms of ρ_f and ϕ_f.

3. Express ρ_f and ϕ_f in terms of ρ_0, ϕ_0, and the rotation angle $\Delta\phi$. Plug this into your answer for Part 2 to get expressions for x_f and y_f in terms of the variables ρ_0, ϕ_0, and $\Delta\phi$.

   See Check Yourself #44 at felderbooks.com/checkyourself

4. Your formulas for x_f and y_f should have expressions of the form $\sin(a + b)$ and $\cos(a + b)$. Use the sine and cosine addition formulas to expand these out.

5. Now you should be able to recognize your expressions for x_0 and y_0 from Part 1 showing up inside your expressions for x_f and y_f. Substitute x_0 and y_0 in for them. The result will be an expression for x_f and y_f that only depends on x_0, y_0, and $\Delta\phi$.

6. Write your answer in the form of a matrix equation, filling in the question marks below with formulas that include $\Delta\phi$.

$$\begin{pmatrix} x_f \\ y_f \end{pmatrix} = \begin{pmatrix} ? & ? \\ ? & ? \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

   You now have a matrix that can be multiplied by any vector $x_0\hat{i} + y_0\hat{j}$ to determine where that vector will be after the rotation.

7. In the picture above the initial positions are $\hat{\imath}$, $\hat{\imath}+\hat{\jmath}$, and $-\hat{\imath}+\hat{\jmath}$ and the rotation angle is 30°. Use your matrix to determine where the three ants end up after rotation. Make sure your answers seem to roughly match the final positions shown in the figure.

8. Without doing any calculations, predict where each of the three ants would have ended up if the turntable had been rotated counterclockwise by 90°. Check your answer by replacing $\Delta\phi$ with 90° in your matrix, and then acting with that matrix on the ants' initial positions.

What you've just derived is a "transformation matrix," a matrix that can be used to geometrically transform vectors. In this unit you'll encounter transformation matrices that stretch and reflect vectors as well as rotating them.