Motivating Exercise for Introduction to Differential Equations Flow of a Compressible Fluid

Consider a pipe filled with some kind of gas. Under certain simplifying assumptions,[1] you can use the "Hagen-Poiseuille equation" and the ideal gas law to show that the pressure along such a pipe will obey the following equation.

$$\frac{dP}{dx} = -\frac{k}{P} \tag{1.1}$$

where...

- The independent variable x specifies a location in the pipe (measured for instance in meters from one end, and increasing in the direction of the gas flow).
- The positive constant k is based on the pipe's radius and the gas's viscosity, temperature, mass flow rate, and molar mass.
- The dependent variable P is the pressure of the gas.

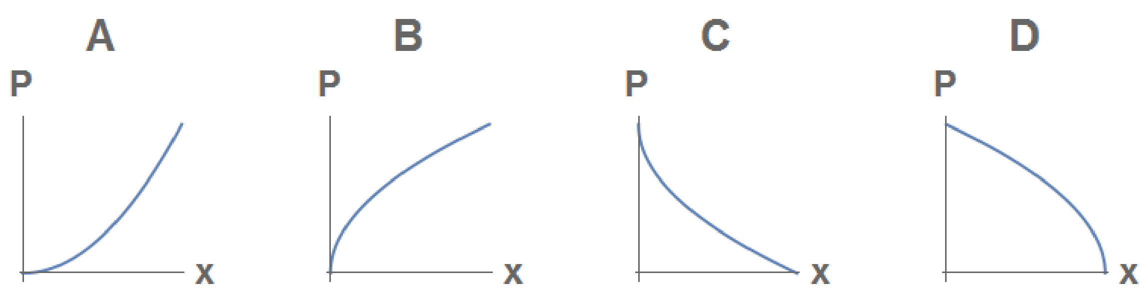
Equation 1.1 is called a "differential equation." (Also "ordinary differential equation" or sometimes "ODE.") Note that it does not tell us P (the pressure), and it does not tell us dP/dt (how the pressure changes over time). It tells us dP/dx, how the pressure varies from one location in the pipe to another.

For the first two questions below, explain briefly how Equation 1.1 tells us the answer.

1. As you move along the pipe in the direction of the gas flow (so x is increasing) is the pressure increasing or decreasing?

2. If the pressure is very high, does the pressure change very rapidly or very slowly?

3. Based on those two answers, which of the following is a plausible plot of $P(x)$?



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[1] a horizontal circular pipe of constant radius, an ideal gas, and gas flow that is "steady state" (not changing over time) and "laminar" (no swirls and eddies)

At this point you may have no idea how you would go about finding the actual function, so we're going to give you one. Note that our solution below introduces a new constant P_0.

$$P(x) = \sqrt{P_0^2 - 2kx} \tag{1.2}$$

4. The "boundary condition" means the pressure at the beginning of the pipe, where $x = 0$. What boundary condition does Equation 1.2 set?

5. Calculate dP/dx using Equation 1.2.

6. Calculate $-k/P$ using Equation 1.2.

If dP/dx and $-k/P$ came out the same, you have just demonstrated that this $P(x)$ function satisfies Equation 1.1. (If they didn't, you made a mistake somewhere.) You can also verify that the overall shape of this function matches the shape you chose. But we haven't shown you how to derive Equation 1.1 on your own, or how to find a solution like Equation 1.2. Those are both skills that you will learn as you study differential equations.