Discovery Exercise for Sturm-Liouville Theory and Series Expansions

Consider a particle moving freely inside a sphere of radius a. According to quantum mechanics the particle is described by a "wavefunction" $\psi(x, y, z)$ that obeys a partial differential equation called "Schrödinger's equation." You don't need to know any quantum mechanics to do this exercise; you can just take our word that when you separate variables in this PDE you get the following ODE, where r is distance from the center of the sphere.[1]

$$r^2 R''(r) + 2rR'(r) + \frac{2mE}{\hbar^2} r^2 R(r) = 0 \tag{1}$$

Both $\hbar$ (a physical constant of the universe) and m (the mass of the particle) are given constants in this problem. E (the energy of the particle) is *not* given—it comes from the solution. In classical mechanics the energy could be any non-negative real number; for any particular energy, you could write an equation for the motion. In quantum mechanics, as you will see, only certain energy levels are possible; for any allowable energy level, you can write an equation for the wavefunction.

1. Look up the equation for "spherical Bessel functions." Do a variable substitution of the form $u = kr$ and find the value of k that turns Equation 1 into that equation. (The spherical Bessel equation has a parameter p in it. Equation 1 should match it for one particular value of p.)

2. Write the general solution to Equation 1 in terms of spherical Bessel functions. Write your answer in terms of r, not in terms of u.

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3. One boundary condition says that $R(r)$ must be finite at $r = 0$. Look up the properties of spherical Bessel functions and use that boundary condition to set one of the arbitrary constants in your general solution equal to zero.

4. The second boundary condition says that $R(a) = 0$. Using this boundary condition, what are the possible values of E?

[1] Actually the ODE is more complicated because it involves terms related to the particle's angular momentum. Here we're considering particles with zero angular momentum.

5. This problem—which is not just Equation 1, but also includes the two boundary conditions—cannot be solved for all values of E. It can only be solved for the values of E that you listed in Part 4. Each allowable E is called an "eigenvalue" of this problem. Is there a finite number of eigenvalues, or an infinite number? Is there a lowest eigenvalue? Is there a highest eigenvalue?

6. Every eigenvalue E is associated with a particular solution to this problem, which is called its "eigenfunction." Write the first two eigenfunctions. If you are doing this exercise as homework (or in a classroom with computers), choose any positive values for m and a and graph those eigenfunctions.