Computer Problems for Partial Differential Equations

For Problems 1–9 use a computer to numerically solve the given PDEs and graph the results. You can make plots of $f(x)$ at different times, make a 3D plot of $f(x,t)$, or do an animation of $f(x)$ evolving over time. For each one describe the late term behavior (steady state, oscillating, growing without bound), based on your computer results. Then explain, based on the given equations, why you would expect that behavior even if you didn't have a computer.

1. $\partial^2 f/\partial t^2 = \partial^2 f/\partial x^2$, $0 < x < 1$, $0 < t < 10$, $f(0,t) = 0$, $f(1,t) = e - 1$, $f(x,0) = e^x - 1$, $\dot{f}(x,0) = 0$
2. $\partial^2 f/\partial t^2 = \partial^2 f/\partial x^2$, $0 < x < 1$, $0 < t < 10$, $f(0,t) = 0$, $f(1,t) = 0$, $f(x,0) = 0$, $\dot{f}(x,0) = \sin(\pi x)$
3. $\partial f/\partial t = \partial^2 f/\partial x^2$, $0 < x < 1$, $0 < t < 10$, $f(0,t) = 0$, $f(1,t) = e - 1$, $f(x,0) = e^x - 1$
4. If you are unable to numerically solve this equation explain why.
   $\partial^2 f/\partial t^2 = -\partial^2 f/\partial x^2$, $0 < x < 1$, $0 < t < 0.5$, $f(0,t) = 0$, $f(1,t) = e - 1$, $f(x,0) = e^x - 1$, $\dot{f}(x,0) = 0$
5. $\partial f/\partial t = \partial f/\partial x$, $0 < x < 1$, $0 < t < 1$, $f(1,t) = 0$, $f(x,0) = e^{-20(x-.5)^2}$
6. $\partial f/\partial t = \partial f/\partial x$, $0 < x < 1$, $0 < t < 3$, $f(1,t) = \sin(10t)$, $f(x,0) = 0$
7. $\partial f/\partial t = -\partial f/\partial x$, $0 < x < 1$, $0 < t < 1$, $f(0,t) = 0$, $f(x,0) = e^{-20(x-.5)^2}$
8. Try solving this out to several different final times and see if you get consistent behavior. If you are unable to numerically solve this equation explain why.
   $\partial f/\partial t = -\partial f/\partial x$, $0 < x < 1$, $f(1,t) = 0$, $f(x,0) = e^{-20(x-.5)^2}$
9. $\partial f/\partial t = x\,(\partial f/\partial x)$, $0 < x < 1$, $0 < t < 1$, $f(1,t) = 0$, $f(x,0) = e^{-20(x-.5)^2}$

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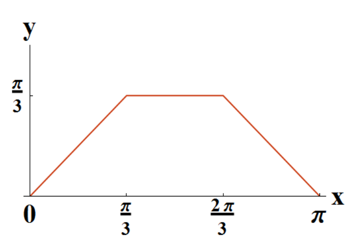
10. A string of length π is fixed at both ends, so $y(0,t) = y(\pi,t) = 0$. You pull the string up at two points and then let go, so the initial conditions are:

$$y(x,0) = \begin{cases} x & x < \pi/3 \\ \pi/3 & \pi/3 \le x \le 2\pi/3 \\ \pi - x & 2\pi/3 < x \end{cases} \qquad \text{and} \qquad \frac{\partial y}{\partial t}(x,0) = 0$$



   (a) Rewrite the initial condition as a Fourier sine series.
   (b) Write the solution $y(x,t)$ to the wave equation with these initial and boundary conditions. Your answer will be expressed as a series and will include a constant v.
   (c) Take $v = 2$ and have a computer calculate the 20^{th} partial sum of the solution you found Part (b). Plot the solution at a series of times and describe its evolution.

In Problems 11–13 a string of length π, initially at rest, has boundary conditions $y(0,t) = y(\pi,t) = 0$. For the initial shape given in the problem:

(a) Find the solution $y(x,t)$ to the wave equation, taking $v = 2$. Your answer will be in the form of a series.

(b) Make plots of the 1^{st} partial sum, the 3^{rd} partial sum, and the 20^{th} partial sum of the series solution at several times.

11. $y(x,0) = \begin{cases} 1 & \pi/3 < x < 2\pi/3 \\ 0 & \text{elsewhere} \end{cases}$

12. $y(x,0) = \pi^2/4 - (x - \pi/2)^2$

13. Make up an initial position $y(x,0)$. You may use any function that obeys the boundary conditions *except* the trivial case $y(x,0) = 0$, or any function we have already used in the problems above.

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14. A string of length 1 is fixed at both ends and obeys the wave equation with $v = 2$. For each of the initial conditions given below assume the initial velocity of the string is zero.
    (a) Have a computer numerically solve the wave equation for this string with initial condition $y(x,0) = \sin(2\pi x)$ and animate the resulting motion of the string. Solve to a late enough time to see the string oscillate at least twice, using trial and error if necessary. Describe the resulting motion.
    (b) Have a computer numerically solve for and animate the motion of the string for initial condition $y(x,0) = \sin(20\pi x)$, using the same final time you used in part (a). How is this motion different from what you found in part (a)?
    (c) Consider the initial condition $y(x,0) = \sin(2\pi x) + (.1)\sin(20\pi x)$. What would you expect the motion of the string to look like in this case? Solve the wave equation with this initial condition numerically and animate the results. Did the results match your prediction?
    (d) Finally, make an animation of the solution to the wave equation for the case $y(x,0) = .2\sin(2\pi x)\left[7 + 6x - 100x^2 + 100x^3 + \cos(36x) - e^{-36x^2}\right]$. (We chose this simply because it looks like a crazy, random mess.) Describe the resulting motion.
    (e) How is the evolution of a string that starts in a normal mode different from the evolution of a string that starts in a different shape?

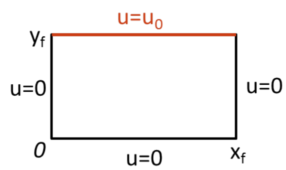
15. A string of length 1 obeys the wave equation with $v = 2$. The string is initially at rest at $y = 0$. The right side of the string is fixed, but the left side is given a quick jerk: $y(0,t) = e^{-100(t-1)^2}$. Solve the wave equation numerically with this boundary condition and animate the results out to $t = 5$. Describe the motion of the string.

16. A string of length π obeys the wave equation with $v = 2$. The right side of the string is fixed.

    Suppose the string starts at rest at $y = 0$ and you excite it by vibrating the left end of it: $y(0,t) = \sin(11t)$. Notice that this is *not* one of the normal mode frequencies.
    (a) Solve the wave equation numerically with this boundary condition out to $t = 10$. Describe the resulting motion.

    Next suppose the left end vibrates according to: $y(0,t) = \sin(10t)$.
    (b) Is this oscillation occurring at one of the normal mode frequencies?
    (c) Solve the wave equation numerically with this boundary condition out to $t = 10$. How is the resulting motion different from what you found in Part (a)?

17. Given enough time, any isolated region of space will tend to approach a steady state where the temperature at each point is unchanging. In this case the time derivative in the heat equation becomes zero and the temperature obeys Laplace's equation. In two dimensions this can be written as $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = 0$. In this problem you will use separation of variables to solve for the steady state temperature $u(x, y)$ on a rectangular slab subject to the boundary conditions $u(0, y) = u(x_f, y) = u(x, 0) = 0$, $u(x, y_f) = u_0$.



(a) Separate variables to get ordinary differential equations for the functions $X(x)$ and $Y(y)$.

(b) Solve the equation for $X(x)$ subject to the boundary conditions $X(0) = X(x_f) = 0$.

(c) Solve the equation for $Y(y)$ subject to the homogeneous boundary condition $Y(0) = 0$. Your solution should have one undetermined constant in it corresponding to the one boundary condition you have not yet imposed.

(d) Write the solution $u(x, y)$ as a sum of normal modes.

(e) Use the final boundary condition $u(x, y_f) = u_0$ to solve for the remaining coefficients and find the general solution.

(f) Check your solution by verifying that it solves Laplace's equation and meets each of the homogeneous boundary conditions given above.

(g) Use a computer to plot the 40^{th} partial sum of your solution. (You will have to choose some values for the constants in the problem.) Looking at your plot, describe how the temperature depends on y for a fixed value of x (other than $x = 0$ or $x = x_f$). You should see that it goes from $u = 0$ at $y = 0$ to $u = u_0$ at $y = y_f$. Does it increase linearly? If not describe what it does.

18. (a) Solve Laplace's equation in the cubic region $0 \leq x, y, z \leq L$ with $V(x, y, L) = \sin(\pi x/L)\sin(2\pi y/L) + \sin(2\pi x/L)\sin(\pi y/L)$ and $V = 0$ on the other five sides.

(b) Sketch $V(x, L/2, L)$ and $V(x, L/4, L)$, and $V(x, 0, L)$ as functions of x. How does changing the y value affect the plot?

(c) At $z = L$ sketch how V depends on x for many values of y. You can do this by making an animation or a series of still images, but either way you should have enough to see if it follows the behavior you predicted in Part (b). How would your sketches have changed if you had used $z = L/2$ instead? (You should not need a computer to answer that last question.)

The two problems below are a set; the first should be done without a computer and the second is a computer-based follow up.

19. (a) Use separation of variables to solve the 2D wave equation $\partial^2 z/\partial x^2 + \partial^2 z/\partial y^2 = (1/v^2)(\partial^2 z/\partial t^2)$ on the rectangle $x, y \in [0, L]$ with $z = 0$ on all four sides. Your solution will be a sum with arbitrary constants corresponding to the as-yet-unspecified initial conditions.

    (b) Now find the particular solution for the initial conditions $z(x, y, 0) = 0$, $\dot{z}(x, y, 0) = \begin{cases} c & L/3 < x, y < 2L/3 \\ 0 & otherwise \end{cases}$.

    Once again your answer will be an infinite sum, but this time with specified constants.

20. *[This problem depends on Problem 19.]* The equation you solved in Problem 19 might represent an oscillating square plate that was given a sudden blow in a region in the middle. This might represent a square drumhead hit by a square drumstick. The solution you got, however, was a fairly complicated looking double sum.

    (a) Have a computer plot the initial function $\dot{z}(x, y, 0)$ for the partial sum that goes up through $m = n = 5$, then again up through $m = n = 11$, and finally up through $m = n = 21$. As you add terms you should see the partial sums converging towards the shape of the initial conditions that you were solving for.

    (b) Take several of the nonzero terms in the series (the individual terms, not the partial sums) and for each one use a computer to make an animation of the shape of the drumhead (z, not $\dot{z}$) evolving over time. You should see periodic behavior. You should include $(m, n) = (1, 1)$, $(m, n) = (1, 3)$, $(m, n) = (3, 3)$, and at least one other term. Describe how the behaviors of these normal modes are different from each other.

    (c) Now make an animation of the partial sum that goes through $m = n = 21$. Describe the behavior. How is it similar to or different from the behavior of the individual terms you saw in the previous part?

    (d) How would your answers to Parts (b) and (c) have looked different if we had used a different set of initial conditions?

The two problems below are a set; the first should be done without a computer and the second is a computer-based follow up.

21. In this problem you will solve the partial differential equation $\partial y/\partial t - x^{3/2}(\partial^2 y/\partial x^2) - x^{1/2}(\partial y/\partial x) = 0$ subject to the boundary condition $y(1,t) = 0$ and the requirement that $y(0,t)$ be finite.
    (a) Begin by guessing a separable solution $y = X(x)T(t)$. Plug this guess into the differential equation. Then divide both sides by $X(x)T(t)$ and separate variables.
    (b) Find the general solution to the resulting ODE for $X(x)$ three times: with a positive separation constant k^2, a negative separation constant $-k^2$, and a zero separation constant. For each case you can solve the ODE by hand using the variable substitution $u = x^{1/4}$ or you can use a computer. Show which one of your solutions can match the boundary conditions without requiring $X(x) = 0$.
    (c) Apply the condition that $y(0,t)$ is finite to show that one of the arbitrary constants in your solution from Part (b) must be 0.
    (d) Apply the boundary condition $y(1,t) = 0$ to find all the possible values for k. There will be an infinite number of them, but you should be able to write them in terms of a new constant n that can be any positive integer. Writing k in terms of n will involve $\alpha_{p,n}$, the zeros of the Bessel functions.
    (e) Solve the ODE for $T(t)$, expressing your answer in terms of n.
    (f) Multiply $X(x)$ times $T(t)$ to find the normal modes of this system. You should be able to combine your two arbitrary constants into one. Write the general solution $y(x,t)$ as a sum over these normal modes. Your arbitrary constant should include a subscript n to indicate that they can take different values for each value of n.
    (g) Use the initial condition $y(x,0) = \sin(\pi x)$ to find the arbitrary constants in your solution, using the equations for a Fourier-Bessel series. Your answer will be in the form of an integral that you will not be able to evaluate. *Hint*: You may have to define a new variable to get the resulting equation to look like the usual form of a Fourier-Bessel series.

22. *[This problem depends on Problem 21.]*
    (a) Have a computer calculate the 10^{th} partial sum of your solution to Problem 21 Part (g). Describe how the function $y(x)$ is evolving over time.
    (b) You should have found that the function at $x = 0$ starts at zero, then rises slightly, and then asymptotically approaches zero again. Explain why it does that. *Hint*: think about why all the normal modes cancel out at that point initially, and what is happening to each of them over time.

In Problems 23–26 you will be given a PDE and a set of boundary and initial conditions.

(a) Solve the PDE with the given boundary conditions using separation of variables. You may solve the ODEs you get by hand or with a computer. The solution to the PDE should be an infinite series with undetermined coefficients.

(b) Plug in the given initial condition. The result should be a Fourier-Bessel series. Write an equation for the coefficients in your series. This equation will involve an integral that you may not be able to evaluate.

(c) Have a computer evaluate the integrals in part (b) either analytically or numerically to calculate the 20^{th} partial sum of your series solution and either plot the result at several times or make a 3D plot of $y(x,t)$. Describe how the function is evolving over time.

It may help to first work through Problem 21 as a model.

23. $\dfrac{\partial y}{\partial t} = \dfrac{\partial^2 y}{\partial x^2} + \dfrac{1}{x}\dfrac{\partial y}{\partial x} - \dfrac{y}{x^2}$, $y(3,t) = 0$, $y(x,0) = \begin{cases} 1 & 1 \le x \le 2 \\ 0 & elsewhere \end{cases}$. Assume y is finite for $0 \le x \le 3$.

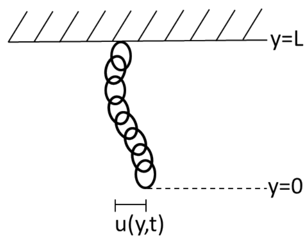
24. $\dfrac{\partial^2 y}{\partial t^2} = \dfrac{\partial^2 y}{\partial x^2} + \dfrac{1}{x}\dfrac{\partial y}{\partial x} - \dfrac{y}{x^2}$, $y(3,t) = 0$, $y(x,0) = 0$, $\dfrac{\partial y}{\partial t}(x,0) = \begin{cases} 1 & 1 \le x \le 2 \\ 0 & elsewhere \end{cases}$. Assume y is finite for $0 \le x \le 3$.

25. $\partial z/\partial t - \partial^2 z/\partial x^2 - (1/x)(\partial z/\partial x) - (1/x^2)(\partial^2 z/\partial y^2) = 0$, $z(1,y,t) = z(x,0,t) = z(x,1,t) = 0$, $z(x,y,0) = J_{2\pi}(\alpha_{2\pi,3}x)\sin(2\pi y)$. Assume z is finite throughout $0 \le x \le 1$.

26. $\sin^2 x \cos^2 x(\partial z/\partial t) - \sin^2 x(\partial^2 z/\partial x^2) - \tan x(\partial z/\partial x) + 4(\cos^2 x)z = 0$, $z(\pi/2,t) = 0$, $z(x,0) = \cos x$. Assume z is finite throughout $0 \le x \le \pi/2$. After you separate variables you will use the substitution $u = \sin x$ to turn an unfamiliar equation into Bessel's equation. (If your separation constant is the wrong sign you will get modified Bessel functions, which cannot meet the boundary conditions.)

The two problems below are a set; the first should be done without a computer and the second is a computer-based follow up.

27. Daniel Bernoulli first discovered Bessel functions in 1732 while working on solutions to the "hanging chain problem." A chain is suspended at $y = L$ and hangs freely with the bottom just reaching the point $y = 0$. The sideways motions of the chain $u(y,t)$ are described by the equation $\partial^2 u/\partial t^2 = g\left(y(\partial^2 u/\partial y^2) + \partial u/\partial y\right)$.



   (a) Separate variables and solve the equation for $T(t)$. Choose the sign of the separation constant that gives you oscillatory solutions.

   (b) The equation for $Y(y)$ can be turned into Bessel's equation with a substitution. You can start with $u = cy^q$ and find what values of c and q work, but we'll save you some algebra and tell you the correct substitution is $u = cy^{1/2}$. Plug that in and find the value of c needed to turn your $Y(y)$ equation into Bessel's equation.

   (c) Solve the equation for $Y(u)$ and plug the substitution you found back in to get a solution for $Y(y)$ subject to the boundary condition $u(L,t) = 0$ and the condition that Y remain finite in the range $0 \le y \le L$. You should find that your solutions are Bessel functions and that you can restrict the possible values of the separation constant.

$$Y(y) = AJ_0\left(\frac{2k}{\sqrt{g}}\sqrt{y}\right) + BY_0\left(\frac{2k}{\sqrt{g}}\sqrt{y}\right)$$

   Since $Y(0)$ must be finite, we discard the Y_0 solution. Since $Y(L) = 0$ we must have $2k\sqrt{L}/\sqrt{g} = \alpha_{0,n}$.

$$Y(y) = AJ_0\left(\frac{\alpha_{0,n}}{\sqrt{L}}\sqrt{y}\right)$$

   (d) Write the solution $u(y,t)$ as an infinite series and use the initial conditions $u(y,0) = f(y), \dot{u}(y,0) = h(y)$ to find the coefficients in this series.

28. *[This problem depends on Problem 27.]* In this problem you'll use the solution you derived in Problem 27 to model the motion of a hanging chain. For this problem you can take $g = 9.8$ m/s^2, $L = 1$m.

   (a) Calculate the first five coefficients of the series you derived for $u(y,t)$ in Problem 27 using the initial conditions $u(y,0) = d - (d/L)y, \dot{u}(y,0) = 0$ where $d = .5$m You can do this analytically by hand, use a computer to find it analytically, or use a computer to do it numerically. However you do it, though, you should get numbers for the five coefficients.

   (b) Using the fifth partial sum to approximate $u(y,t)$ make an animation showing $u(y)$ at different times or a 3D plot showing $u(y,t)$ at times ranging from $t = 0$ to $t = 5$. Does the behavior look reasonable for a hanging chain?

In Problems 29–31 you will be given a PDE, a domain, and a set of initial conditions. You should assume in each case that the function is finite in the given domain.

(a) Solve the PDE using separation of variables. You may solve the ODEs you get by hand or with a computer. The solution to the PDE should be an infinite series with undetermined coefficients.

(b) Plug in the given initial condition. The result should be a Fourier-Legendre series. Use this series to write an equation for the coefficients in your solution. You may not be able to evaluate this integral analytically: your solution will be in the form $\sum_{l=1}^{\infty} A_l$<*something*> where A_l is defined as an integral.

(c) Have a computer evaluate the integral in part (b) either analytically or numerically to calculate the 20^{th} partial sum of your series solution and plot the result at several times. Describe how the function is evolving over time.

29. $\partial y/\partial t - \left(1-x^2\right)\left(\partial^2 y/\partial x^2\right) + 2x\left(\partial y/\partial x\right) = 0, -1 \le x \le 1, y(x,0) = 1$, and $y(x)$ is finite everywhere in that domain
30. $\partial y/\partial t = \left(9-x^2\right)\left(\partial^2 y/\partial x^2\right) - 2x(\partial y/\partial x), -3 \le x \le 3, y(x,0) = \sin\left(\pi x/3\right)$
31. $\partial y/\partial t = \partial^2 y/\partial \theta^2 + \cot\theta(\partial y/\partial\theta), 0 \le \theta \le \pi, y(\theta,0) = \sin^2\theta$

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32. $\partial^2 u/\partial x^2 + x(\partial^2 u/\partial y^2) + \partial u/\partial x = 0$, $u(L,y) = u(x,H) = u(x,0) = 0$, $u(0,y) = \kappa$. Use eigenfunction expansion to reduce this to an ODE and use a computer to solve it with the appropriate boundary conditions. The solution $u(x,y)$ will be an infinite sum whose terms are hideous messes. Verify that it's correct by showing that each term individually obeys the original PDE, and by plotting a large enough partial sum of the series as a function of x and y to show that it matches the boundary conditions. (Making the plot will require choosing specific values for L, H, and κ.)
33. The air inside a flute obeys the wave equation with boundary conditions $\partial s/\partial x(0,t) = \partial s/\partial x(L,t) = 0$. The wave equation in this case is typically inhomogeneous because of someone blowing across an opening, creating a driving force that varies with x (position in the flute). Over a small period of time, it is reasonable to treat this driving function as a constant with respect to time. In this problem you will solve the equation $\partial^2 s/\partial x^2 - (1/c_s^2)(\partial^2 s/\partial t^2) = q(x)$ with the initial conditions $s(x,0) = \partial s/\partial t(x,0) = 0$.
    (a) Explain why, for this problem, a cosine expansion will be easier to work with than a sine expansion.
    (b) Expanding $s(x,t)$ and $q(x)$ into Fourier cosine series, write a differential equation for $a_{sn}(t)$.
    (c) Find the general solution to this differential equation. You should find that the solution for general n doesn't work for $n=0$ and you'll have to treat that case separately. Remember that $q(x)$ has no time-dependence, so each a_{qn} is a constant.
    (d) Now plug in the initial conditions to find the arbitrary constants and write the series solution $s(x,t)$.
    (e) As a specific example, consider $q(x) = ke^{-(x-L/2)^2/x_0^2}$. This driving term represents a constant force that is strongest at the middle of the tube and rapidly drops off as you move towards the ends. (This is not realistic in several ways, the most obvious of which is that the hole in a concert flute is not at the middle, but it nonetheless gives a qualitative idea of some of the behavior of flutes and other open tube instruments.) Use a computer to find the Fourier cosine expansion of this function and plot the 20^{th} partial sum of $s(x,t)$ as a function of x at several times t (choosing values for the constants). Describe how the function behaves over time. Based on the results you find, explain why this equation could not be a good model of the air in a flute for more than a short time.
34. An infinite rod being continually heated by a localized source at the origin obeys the differential equation $\partial u/\partial t - \alpha(\partial^2 u/\partial x^2) = ce^{-(x/d)^2}$ with initial condition $u(x,0) = 0$.
    (a) Solve for the temperature $u(x,t)$. Your answer will be in the form of a Fourier transform $\hat{u}(p,t)$.
    (b) Take the inverse Fourier transform of your answer to get the function $u(x,t)$. Plot the temperature distribution at several different times and describe how it is evolving over time.

35. A string of length π is given an initial blow so that it starts out with $y(x,0) = 0$ and

$$\frac{\partial y}{\partial t}(x,0) = \begin{cases} 0 & x < \pi/3 \\ s & \pi/3 \le x \le 2\pi/3 \\ 0 & 2\pi/3 < x \end{cases}$$

(a) Rewrite the initial velocity as a Fourier sine series.

(b) Write the solution $y(x,t)$. Your answer will be expressed as a series.

(c) Let $v = s = 1$ and have a computer numerically solve the wave equation with these initial conditions and plot the result at several different times. Then make a plot of this numerical solution and of the 10^{th} partial sum of the series solution at $t = 2$ on the same plot. Do they match?

For Problems 36–38

(a) Solve the given problem using separation of variables. The result will be an infinite series.

(b) Plot the first three nonzero *terms* (not partial sums) of the series at $t = 0$ and at least three other times. For each one describe the shape of the function and how it evolves in time.

(c) Plot successive partial sums at $t = 0$ until the plot looks like the initial condition for the problem. Examples are shown below of what constitutes a good match.

(d) Having determined how many terms you have to include to get a good match at $t = 0$, plot that partial sum at three or more other times and describe the evolution of the function. How is it similar to or different from the evolution of the individual terms in the series?

36. $\dfrac{\partial^2 y}{\partial t^2} - \dfrac{\partial^2 y}{\partial x^2} + y = 0,\ y(0,t) = y(4,t) = 0,\ y(x,0) = \begin{cases} 1 & 1 < x < 2 \\ -1 & 2 \le x < 3 \\ 0 & elsewhere \end{cases},\ \dfrac{\partial y}{\partial t}(x,0) = 0$

37. $\dfrac{\partial y}{\partial t} - t^2 \dfrac{\partial^2 y}{\partial x^2} = 0,\ y(0,t) = y(3,t) = 0,\ y(x,0) = \begin{cases} x & 0 \le x < 1 \\ 1 & 1 \le x \le 2 \\ 3 - x & 2x \le 3 \end{cases}$

38. $\partial^2 y/\partial t^2 - \partial y/\partial t - \partial^2 y/\partial x^2 + y/4 = 0$, $y(0,t) = y(1,t) = 0$, $y(x,0) = x(1-x)$, $\partial y/\partial t(x,0) = x(x-1)$.

39. You are conducting an experiment where you have a thin disk of radius R (perhaps a large Petri dish) with the outer edge held at zero temperature. The chemical reactions in the dish provide a steady, position-dependent source of heat. The steady state temperature in the disk is described by Poisson's equation in polar coordinates.

$$\rho^2 \frac{\partial^2 V}{\partial \rho^2} + \rho \frac{\partial V}{\partial \rho} + \frac{\partial^2 V}{\partial \phi^2} = \frac{\rho}{R} \sin\phi$$

(a) Begin by applying the variable substitution $\rho = Re^{-r}$ to rewrite Poisson's equation.

(b) What is the domain of the new variable r?

(c) Based on the domain you just described, the method of transforms is appropriate here. You are going to use a Fourier sine transform. Explain why this makes more sense for this problem than a Laplace transform or a Fourier cosine transform.

(d) Transform the equation. You can evaluate the integral using a formula from an integral table or just give it to a computer. Then solve the resulting ODE. Your general solution will have two arbitrary functions of p.

(e) The boundary conditions for ϕ are implicit, namely that $\hat{V}_s(\phi)$ and $\hat{V}_s'(\phi)$ must both have period 2π. You should be able to look at your solution and immediately see what values the arbitrary functions must take to lead to periodic behavior.

(f) Take the inverse transform. (You can use a computer to take the integral.) Then substitute back to find the solution to the original problem in terms of ρ.