CHAPTER 11

Partial Differential Equations

Before you read this chapter, you should be able to . . .

- solve ordinary differential equations with "initial" or "boundary" conditions (see Chapters 1 and 10).
- evaluate and interpret partial derivatives (see Chapter 4).
- find and use Fourier series—sine series, cosine series, both-sines-and-cosines series, and complex exponential forms, for one or more independent variables (see Chapter 9). Fourier transforms are required for Section 11.10 only.
- find and use Laplace transforms (for Section 11.11 only, see Chapter 10).

After you read this chapter, you should be able to . . .

- model real-world situations with partial differential equations and interpret solutions of these equations to predict physical behavior.
- use the technique "separation of variables" to find the normal modes for a partial differential equation. We focus particularly on trig functions, Bessel functions, Legendre polynomials, and spherical harmonics. You will learn about each of these in turn, but you will also learn to work more generally with any unfamiliar functions you may encounter.
- use those normal modes to create a general solution for the equation in the form of a series, and match that general solution to boundary and initial conditions.
- solve problems that have multiple inhomogeneous conditions by breaking them into subproblems with only one inhomogeneous condition each.
- solve partial differential equations by using the technique of "eigenfunction expansions."
- solve partial differential equations by using Fourier or Laplace transforms.

The differential equations we have used so far have been "ordinary differential equations" (ODEs) meaning they model systems with one independent variable. In this chapter we will use "partial differential equations" (PDEs) to model systems involving more than one independent variable.

We will begin by discussing the *idea* of a partial differential equation. How do you set up a partial differential equation to model a physical situation? Once you have solved such an equation, how can you interpret the results to understand and predict the behavior of a system? Just as with ordinary differential equations, you may find this part as or more challenging than the mechanics of finding a solution, but you will also find that understanding the equations is more important than solving them. The good news is, the work you have already done in understanding ordinary differential equations and partial derivatives will provide a strong foundation in understanding these new types of equations.

In the first three sections we will discuss a few key concepts including arbitrary functions, boundary and initial conditions, and normal modes. These discussions should be seen as extensions of work you have already done in earlier chapters. For instance, you know that the general solution to an ordinary differential equation (ODE) involves arbitrary constants.

A partial differential equation (PDE) may have an *infinite number* of arbitrary constants, or (equivalently) an *arbitrary function,* in the general solution. These constants or functions are determined from the initial and/or boundary conditions.

In the middle of the chapter—the largest part—we will deal with the engineer's and physicist's favorite tool for solving partial differential equations: "separation of variables." This technique replaces one *partial* differential equation with two or more *ordinary* differential equations that can be solved by hand or with a computer. The solutions to those equations will lead us to work with a wide variety of functions, some familiar and some new.

We will then discuss techniques that can be used when separation of variables fails. Some partial differential equations that cannot be solved by separation of variables can be solved with "eigenfunction expansion," which involves expanding your function into a series before you solve. If one of your variables has an infinite domain you can still use this trick, but it's called the "method of transforms" because you take a Fourier or Laplace transform instead of using a series expansion. Appendix I guides you through the process of looking at a new PDE and deciding which technique to use.

As you explore these techniques you will encounter many of the most important equations in engineering and physics: equations governing light and sound, heat, electric potential, and more. After this chapter you will be prepared to understand these equations, to solve them, and to interpret the solutions.

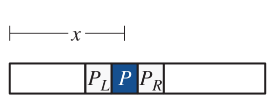
11.1 Motivating Exercise: The Heat Equation

A coin at temperature u_c is placed in a room at a constant temperature u_r.[1] "Newton's Law of Heating and Cooling" states that the rate of change of the coin's temperature is proportional to the *difference* between the temperatures of the room and the coin. (In other words, the coin will cool down faster in a freezer than in a refrigerator.) We can express this law as a differential equation: $du_c/dt = k(u_r - u_c)$ where k is a positive constant.

1. What does this differential equation predict will happen if a cold coin is placed in a hot room? Explain how you could get this answer from this differential equation, even if you didn't already know the answer on physical grounds.
2. Verify that $u_c(t) = u_r + Ce^{-kt}$ is the solution to this differential equation.

Now, suppose we replace the coin with a long insulated metal bar. We assumed above that the coin had a uniform temperature u_c, changing with time but not with position. A long bar, on the other hand, can have different temperatures at the left end, the right end, and every point in between. That means that temperature is now a function of time and position along the bar: $u(x, t)$.

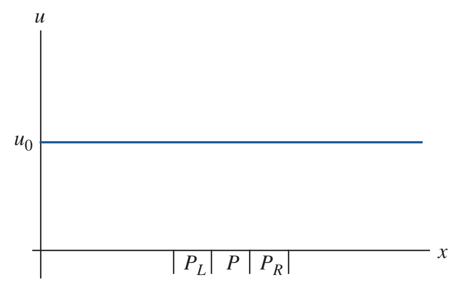
To write an equation for $u(x, t)$, consider how a small piece of the bar (call it P) at position x will behave in some simple cases.



We assume that piece P is so small that its temperature is roughly uniform. However, the pieces to the left and right of it (P_L and P_R) have their own temperatures. Piece P interacts with these adjacent pieces in the same way the coin interacted with the room: the rate of heat transfer between P and the pieces on each side depends on the temperature difference between them. We also assume that heat transfer between the different parts of the bar is fast enough that we can ignore any heat transfer between the bar and the surrounding air.

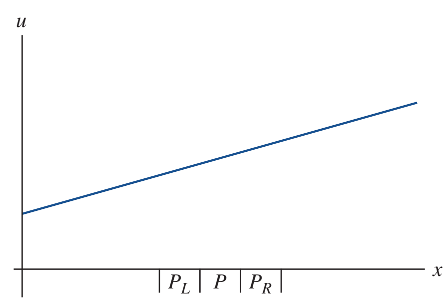
[1]Throughout this chapter we will use the letter u for temperature. Both u and T are commonly used in thermodynamics, but we need T for something different.

3. First, suppose we start with the temperature of the bar uniform. We use $u(x,0)$ to indicate the *initial temperature* of the bar—that is, the temperature at time $t = 0$—so we can express the condition "the initial temperature is a constant" by writing $u(x,0) = u_0$.



(a) Will P give heat to P_L, absorb heat from P_L, or neither?
(b) Will P give heat to P_R, absorb heat from P_R, or neither?
(c) Will the temperature of P go up, go down, or stay constant?

4. Now consider a linearly increasing initial temperature: $u(x,0) = mx + b$, with $m > 0$.



(a) Will P give heat to P_L, absorb heat from P_L, or neither?
(b) Will P give heat to P_R, absorb heat from P_R, or neither?
(c) Will the rate of heat transfer between P and P_L be faster, slower, or the same as the rate of heat transfer between P and P_R?
(d) Will the temperature of P go up, go down, or stay constant?

5. Now consider a parabolic initial temperature: $u(x,0) = ax^2 + bx + c$. Assume $a > 0$ so the parabola is concave up, and assume that P is on the increasing side of the parabola, as shown in Figure 11.1.

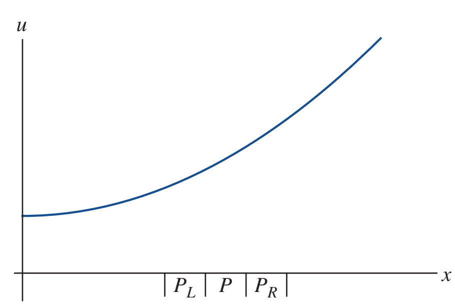


FIGURE 11.1

(a) Will P give heat to P_L, absorb heat from P_L, or neither?
(b) Will P give heat to P_R, absorb heat from P_R, or neither?

(c) Will the rate of heat transfer between P and P_L be faster, slower, or the same as the rate of heat transfer between P and P_R?
(d) Will the temperature of P go up, go down, or stay constant?

6. For each of the three cases you just examined, what were the signs (negative, positive, or zero) of each of the following quantities at point P: u, $\partial u/\partial x$, $\partial^2 u/\partial x^2$, and $\partial u/\partial t$? Just to be clear, you're giving 12 answers in all to this question. You'll get the signs of u, $\partial u/\partial x$, and $\partial^2 u/\partial x^2$ from our pictures, and $\partial u/\partial t$ from what was happening at point P in each case.

7. Which of the following differential equations would be consistent with the answers you gave to Part 6? In each one, k is a (real) constant, so k^2 is a positive number and $-k^2$ is a negative number.
(a) $\partial u/\partial t = k^2 u$
(b) $\partial u/\partial t = -k^2 u$
(c) $\partial u/\partial t = k^2(\partial u/\partial x)$
(d) $\partial u/\partial t = -k^2(\partial u/\partial x)$
(e) $\partial u/\partial t = k^2(\partial^2 u/\partial x^2)$
(f) $\partial u/\partial t = -k^2(\partial^2 u/\partial x^2)$

The "heat equation" you just found is an example of a "partial differential equation," which involves partial derivatives of a function of more than one variable. In this case, it involves derivatives of $u(x, t)$ with respect to both x and t. In general, partial differential equations are harder to solve than ordinary differential equations, but there are systematic approaches that enable you to solve many linear partial differential equations such as the heat equation analytically. For non-linear partial differential equations, the best approach is often numerical.

11.2 Overview of Partial Differential Equations

In Chapter 1 we discussed what differential equations are, how they represent physical situations, and what it means to solve them. We showed how the "general" solution has arbitrary constants which are filled in based on initial conditions to find a "particular" solution: a function.

When multiple independent variables are involved the derivatives become *partial* derivatives and the differential equations become *partial differential equations*. In this section we give an overview of these equations, showing how they are like ordinary differential equations and how they are different. The rest of the chapter will focus on techniques for solving these equations.

11.2.1 Discovery Exercise: Overview of Partial Differential Equations

We begin with an ordinary differential equation: that is, a differential equation with only one independent variable.

1. Consider the differential equation $dy/dx = y$. In words, "the function $y(x)$ is its own derivative."
(a) Verify that $y = 2e^x$ is a valid solution to this differential equation.
(b) Write another solution to this equation.
(c) Write the *general* solution to this equation. It should have one arbitrary constant in it.
(d) Find the only *specific* solution that meets the condition $y(0) = 7$.

Things become more complicated when differential equations involve functions of more than one variable, and therefore partial derivatives. For the following questions, suppose that z is a function of two independent variables x and y.

2. Consider the differential equation $\frac{\partial}{\partial x}z(x,y) = z(x,y)$. In words, "if you start with the function $z(x,y)$ and take its partial derivative with respect to x, you get the same function you started with." Note that $\partial z/\partial y$ is not specified by this differential equation, and may therefore be anything at all.

(a) Which of the following functions are valid solutions to this differential equation? Check all that apply.

i. $z = 5$
ii. $z = e^x$
iii. $z = e^y$
iv. $z = e^x e^y$
v. $z = ye^x$
vi. $z = xe^y$
vii. $z = e^x \sin y$

(b) Write a *general* solution to the differential equation $\partial z/\partial x = z$. Your solution will have an *arbitrary function* in it.

(c) Find the only *specific* solution that meets the condition $z(0,y) = \sin(y)$.

See Check Yourself #69 in Appendix L

3. Consider the differential equation $\frac{\partial}{\partial x}z(x,y) = \frac{\partial}{\partial y}z(x,y)$.

(a) Express this differential equation in words.

(b) Which of the following functions are valid solutions to this differential equation? Check all that apply.

i. $z = 5$
ii. $z = e^x$
iii. $z = e^y$
iv. $z = e^{x+y}$
v. $z = \sin(x+y)$
vi. $z = \sin(x-y)$
vii. $z = \ln(x+y)$

(c) Parts i, iv, v, and vii above are all specific examples of the general form $z = f(x+y)$. By plugging $z = f(x+y)$ into the original differential equation $\partial z/\partial x = \partial z/\partial y$, show that any function of this form provides a valid solution.

(d) Find the only *specific* solution that meets the condition $z(0,y) = \cos y$. *Hint*: It's not in the list above.

11.2.2 Explanation: Overview of Partial Differential Equations

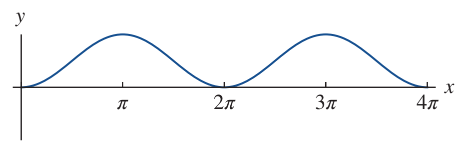
Chapter 1 stressed the importance of differential equations in modeling physical situations. Chapter 4 stressed the importance of *multivariate* functions: functions that depend on two or more variables.

Put the two together and you have differential equations with multiple independent variables. Because these equations are built from partial derivatives, they are called "partial differential equations." Investigate almost any field in physics and you will find a partial differential equation at the core: Maxwell's equations in electrodynamics, the diffusion equation in mass transfer operations, the Navier–Stokes equation in fluid dynamics, Schrödinger's equation in quantum mechanics, and the wave equation in optics, to name a few.

The acronym PDE is often used for "partial differential equation," as opposed to a single-variable "ordinary differential equation" or ODE.

Multivariate functions

Consider a guitar string, pinned to the x-axis at $x = 0$ and $x = 4\pi$, but free to move up and down between the two ends.



We can describe the motion by writing the height y as a function of the horizontal position x and the time t. We can look at such a $y(x, t)$ function in three different ways:

- At any given moment t there is a particular $y(x)$ function that describes the entire string. The string's motion is an evolution over time from one $y(x)$ to the next.
- Any given point x on the string oscillates according to a particular $y(t)$ function. The next point over (at $x + \Delta x$) oscillates according to a slightly different $y(t)$, and all the different $y(t)$ functions together describe the motion of the entire string.
- Finally, we can treat t as a spatial variable and plot y on the xt-plane.

The function $y(x, t)$ has two derivatives at any given point: $\partial y/\partial x$ gives the slope of the string at a given point, and $\partial y/\partial t$ gives the velocity of the string at that point. (A dot is often used for a time derivative, so $\dot{y}$ means $\partial y/\partial t$.) There are therefore four second derivatives. $\partial^2 y/\partial x^2$ is concavity, and $\partial^2 y/\partial t^2$ is acceleration. The "mixed partials" $\partial^2 y/\partial x \partial t$ and $\partial^2 y/\partial t \partial x$ generally come out the same.

All this is a quick reminder of how to think about multivariate functions and derivatives. It does not, however, address the question of what function a guitar string would actually follow. The function $y(x, t) = (1 - \cos x)\cos t$ looks fine, doesn't it? But in fact, no free guitar string would actually do that. In order to show that, and to find what it *would* do, we have to start with the equation that governs its motion.

Understanding partial differential equations

A guitar string will generally obey a partial differential equation called "the wave equation":

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} \qquad \text{The wave equation} \qquad (11.2.1)$$

where v is a constant. You will explore where this equation comes from in Problems 11.32 and 11.45, but here we want to focus on what it tells us. Let's begin by considering the function we proposed earlier.

EXAMPLE **Checking a Possible Solution to the Wave Equation**

Question: Does $y(x, t) = (1 - \cos x)\cos t$ satisfy the wave equation?

Answer:
We can answer this by taking the partial derivatives.

$$y(x, t) = (1 - \cos x)\cos t \quad \rightarrow \quad \frac{\partial^2 y}{\partial x^2} = \cos x \cos t \quad \text{and} \quad \frac{\partial^2 y}{\partial t^2} = -(1 - \cos x)\cos t$$

We see that $\partial^2 y/\partial x^2$ is not the same as $(1/v^2)(\partial^2 y/\partial t^2)$ (no matter what the constant v happens to be), so this function does not satisfy the wave equation. Left to its own devices, a guitar string will not follow that function.

We see with PDEs—just as we saw with ODEs in previous chapters—that it may be difficult to *find* a solution, but it is easy to *verify* that a solution does (or in this case does not) work.

We also saw with ODEs that we can often predict the behavior of a system directly from the differential equation, without ever finding a solution. This is also an important skill to develop with PDEs. Let's see what we can learn by looking at the wave equation.

The second derivative with respect to position, $\partial^2 y/\partial x^2$, gives the concavity: it is related to the shape of the string at one frozen moment in time. The second derivative of the height with respect to time, $\partial^2 y/\partial t^2$, is vertical acceleration: it describes how one point on the string is moving up and down, independent of the rest of the string. So Equation 11.2.1 tells us that wherever the string is concave up, it will accelerate upward; wherever the string is concave down, it will accelerate downward.

As an example, suppose our guitar string starts at rest in the shape $y(x, 0) = 1 - \cos x$. (Don't ask how it got there.) This function has points of inflection at $\pi/2$, $3\pi/2$, $5\pi/2$, and $7\pi/2$. So the two peaks, which are concave down, will accelerate downward; the middle, which is concave up, will accelerate upward. On the left and right are other concave up regions that will move upward, but remember that the guitar string is pinned down at $x = 0$ and $x = 4\pi$, so the very ends cannot move regardless of the shape. Based on all these considerations, we predict something like Figure 11.2.

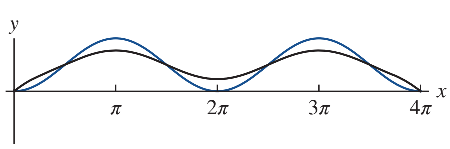


FIGURE 11.2 The blue plot shows the string at time $t = 0$ and the black plot shows the string a short time later.

The drawing shows the curve moving up where it was concave up, and down where it was concave down, while remaining fixed at the ends. We can state with confidence that the string will move from the blue curve to something kind of like the black curve, more or less. For exact solutions we have to actually find the function that matches the wave equation and all the conditions of this scenario. Within the next few sections you'll know how to do all that. But if you followed how we made that drawing, then you'll also know how to see if your answers make sense.

Boundary and Initial Conditions

In order to predict the behavior of a string, you need more than just the differential equation that governs its motion.

- You need the "initial conditions": that is, you need both the position and the velocity of every point on the string when it starts (at $t = 0$). In the example above we gave you the initial position as $y = 1 - \cos x$, and told you that the string started at rest.
- You also need the "boundary conditions." In the example above the guitar string was fixed at $y = 0$ for all time at the left and right sides. A different boundary condition (such as a moving end) would lead to different behavior over time, even if the initial conditions were unchanged.

When you solve a linear ODE you need one condition—one fact—for each arbitrary constant. For instance a second-order linear ODE has two arbitrary constants in the general solution, so you need two extra facts to find a specific solution. These might be the values of the function at two different points, or the value and derivative of the function at one point.

A PDE, on the other hand, requires an infinite number of facts. In our example the initial state occurs at an infinite number of x-positions and the boundary conditions occur at an infinite number of times. To match such conditions the solution must have an infinite number of arbitrary variables: in other words, an entire *arbitrary function*.

EXAMPLE **A Solution with an Arbitrary Function**

For the partial differential equation $\partial u/\partial x + x(\partial u/\partial y) = 0$, the general solution is $u = f\left(y - x^2/2\right)$.

Question: Give three *specific* examples of solutions to this equation.

Answer:
$\sqrt{y - x^2/2}$, $\sin\left(y - x^2/2\right)$, and $5/\left[2 + \ln(y - x^2/2)\right]$ are all solutions.

Question: Verify that *one* of the functions you just wrote solves this differential equation.

Answer:
$u(x, y) = \sin\left(y - \frac{x^2}{2}\right) \rightarrow \frac{\partial u}{\partial x} = \cos\left(y - \frac{x^2}{2}\right)(-x), \quad \frac{\partial u}{\partial y} = \cos\left(y - \frac{x^2}{2}\right)$
Since $\partial u/\partial x$ is $\partial u/\partial y$ multiplied by $-x$, the combination $\partial u/\partial x + x(\partial u/\partial y)$ is equal to zero.

Question: Prove that all functions of the form $u = f\left(y - x^2/2\right)$ solve this differential equation.

Answer:
The specific solution we tested above was valid because of the chain rule, which required the x derivative to be multiplied by $-x$ while the y derivative was just multiplied by 1. This generalizes to any function f.

$$u(x, y) = f\left(y - \frac{x^2}{2}\right) \rightarrow \frac{\partial u}{\partial x} = f'\left(y - \frac{x^2}{2}\right)(-x), \quad \frac{\partial u}{\partial y} = f'\left(y - \frac{x^2}{2}\right)$$
$$\frac{\partial u}{\partial x} + x\frac{\partial u}{\partial y} = 0$$

Several students have objected to our use of the notation f' in this example: it obviously means a derivative, but with respect to what? (This is the kind of question that only excellent students ask.) In this case the derivative is being taken with respect to $\left(y - x^2/2\right)$ but more generally it means "the derivative of the f function." For instance in the previous example f was a sine so f' was a cosine.

The question of exactly what information you need in order to specify the arbitrary function and find a specific solution to a PDE turns out to be surprisingly complicated. We will return to that question when we discuss Sturm-Liouville theory in Chapter 12. Here we will offer just one guideline, which is to look at the order of the equation. For instance, the heat equation $\partial u/\partial t = \alpha(\partial^2 u/\partial x^2)$ is second order in space, and therefore requires two boundary conditions (such as the function $u(t)$ at each end). It is first order in time, and therefore requires only one initial condition (usually the function $u(x)$ at $t = 0$). The wave equation $\partial^2 y/\partial x^2 = (1/v^2)(\partial^2 y/\partial t^2)$, on the other hand, requires two boundary conditions *and* two initial conditions (usually position and velocity).

Remember that a linear ODE is referred to as "homogeneous" if every term in the equation includes the dependent variable or one of its derivatives. If a linear equation is homogeneous then a linear combination of solutions is itself a solution: a very helpful property, when you have it! The same rule applies to PDEs: Laplace's equation $\partial^2 V/\partial x^2 + \partial^2 V/\partial y^2 + \partial^2 V/\partial z^2 = 0$ is linear and homogeneous, so any linear combination of solutions is itself a solution. Poisson's equation $\partial^2 V/\partial x^2 + \partial^2 V/\partial y^2 + \partial^2 V/\partial z^2 = f(x, y, z)$ is inhomogeneous as long as $f \neq 0$.

However, *unlike* with ODEs, we will now be making the same distinction with our boundary and initial conditions.

Definition: Homogeneous

A linear differential equation, boundary condition, or initial condition is referred to as "homogeneous" if it has the following property:

If the functions f and g are both valid solutions to the equation or condition, then the function $Af + Bg$ is also a valid solution for any constants A and B.

This definition should be clearer with an example:

EXAMPLE

Homogeneous and Inhomogeneous Boundary Conditions

If the end of our waving string is fixed at $y(0, t) = 0$, we have a *homogeneous boundary condition*. In other words, if $f(0, t) = 0$ and $g(0, t) = 0$, then $(f + g)(0, t) = 0$ too.

However, $y(0, t) = 1$ represents an *inhomogeneous* condition. If $f(0, t) = 1$ and $g(0, t) = 1$, then $(f + g)(0, t) = 2$, so the function $f + g$ does not meet the boundary condition.

As a final note on conditions, you may be surprised that we distinguish between "initial" (time) and "boundary" (space) conditions. You can graph a $y(x, t)$ function on the xt-plane, so doesn't time act mathematically like just another spatial dimension? Sometimes it does, but initial conditions often look different from boundary conditions, and must be treated differently. In our waving string example the differential equation is second order in both space and time, so we need two spatial conditions and two temporal—but look at the ones we got! The boundary conditions specify y on both the left and right ends of the string. The initial conditions, on the other hand, say nothing about the "end time": instead, they specify both y and $\dot{y}$ at the beginning. This common (though not universal) pattern—boundary conditions on both ends, initial conditions on only one end—leads to significant differences in the ways initial and boundary conditions affect our solutions.

A Few Important PDEs

As you read through this chapter you will probably notice that we keep coming back to the same partial differential equations. Our purpose is *not* to make you memorize a laundry list of PDEs, or to treat each one as a different case: on the contrary, we want to give you a set of tools that you can use on just about *any* linear PDE you come across.

Nonetheless, the examples are important. Each of the equations listed below comes up in many different contexts, and their solutions describe many of the most important quantities in engineering and physics.

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} \qquad \textit{the wave equation} \qquad (11.2.2)$$

We discussed the one-dimensional wave equation above. In two dimensions we could write $\partial^2\Psi/\partial x^2 + \partial^2\Psi/\partial y^2 = (1/v^2)(\partial^2\Psi/\partial t^2)$ and in three dimensions we would add a third spatial derivative term. We can write this equation very generally using the "Laplacian" operator as $\nabla^2\Psi = (1/v^2)(\partial^2\Psi/\partial t^2)$, which is represented by different differential equations as we change dimensions and coordinate systems. You may recall the Laplacian from vector calculus but we will supply the necessary equations as we go through this chapter.

The wave equation describes the propagation of light waves through space, of sound waves through a medium, and many other physical phenomena. You will explore where it comes from in Problems 11.32 and 11.45.

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2} \quad (\alpha > 0) \qquad \textit{the heat equation} \tag{11.2.3}$$

Like Equation 11.2.2, Equation 11.2.3 (which you derived in the motivating exercise) is a one-dimensional version of a more general equation where the second derivative is replaced by a Laplacian. The equation is used to model both conduction of heat and diffusion of chemical species. (In the latter case it's called the "diffusion equation.")

$$\nabla^2 V = f(x, y, z) \qquad \textit{Poisson's equation} \tag{11.2.4}$$

Poisson's equation is used to model spatial variation of electric potential, gravitational potential, and temperature. In general, f is a function of position—for instance, it may represent electrical charge distribution. Poisson's equation therefore represents many *different* PDEs with different solutions, depending on the function $f(x, y, z)$. In the special case $f = 0$ it reduces to Laplace's equation:

$$\nabla^2 V = 0 \qquad \textit{Laplace's equation} \tag{11.2.5}$$

We have chosen to express Equations 11.2.4 and 11.2.5 in their general multidimensional form: the one-dimensional versions are not very interesting, and in fact are not partial differential equations at all. These equations relate different spatial derivatives, but no time derivative. Problems involving these equations therefore have boundary conditions but no initial conditions.

$$-\frac{\hbar^2}{2m}\nabla^2\Psi + V\left(\vec{x}\right)\Psi = i\hbar\frac{d\Psi}{dt} \qquad \textit{Schrödinger's equation} \tag{11.2.6}$$

Schrödinger's equation serves a role in quantum mechanics analogous to $\vec{F} = m\vec{a}$ in Newtonian mechanics: it is the starting point for solving almost any problem. Like Poisson's equation, Schrödinger's equation actually represents a wide variety of differential equations with different solutions, depending in this case on the potential function $V\left(\vec{x}\right)$.

11.2.3 Problems: Overview of Partial Differential Equations

For Problems 11.1–11.6 indicate which of the listed functions are solutions to the given PDE. *Choose all of the valid solutions; there may be more than one.*

11.1 $\partial z/\partial y + \partial^2 z/\partial x^2 = 0$

(a) $z(x, y) = x^2 - y$

(b) $z(x, y) = x^2 - 2y$

(c) $z(x, y) = e^x e^{-y}$

(d) $z(x, y) = e^{-x} e^y$

11.2 $\partial^2 u/\partial y^2 + k\left(\partial u/\partial y\right) + \alpha\left(\partial^2 u/\partial x^2\right) = 0$

(a) $u(x, y) = x^2 - 2\alpha y/k$

(b) $u(x, y) = x^2 - 2\alpha y/k + C$

(c) $u(x, y) = \sin\left(kx\sqrt{2/\alpha}\right) e^{ky}$

(d) $u(x, y) = e^{kx\sqrt{2/\alpha}} e^{ky}$

11.3 $\partial u/\partial t = \alpha\left(\partial^4 u/\partial x^4\right)$

(a) $u(x, t) = \alpha t^2 - x^5/120$

(b) $u(x, t) = e^{\alpha t} + e^x$

(c) $u(x, t) = e^{\alpha t} e^x$

(d) $u(x, t) = e^{\alpha t}\sin x$

11.4 $t^2\left(\partial^2 f/\partial t^2\right) + t\left(\partial f/\partial t\right) - 2x^2\left(\partial^2 f/\partial x^2\right) = 0$

(a) $f(x, t) = x^2 t^2$

(b) $f(x, t) = x^2 t^2 + C$

(c) $f(x, t) = e^t e^x$

(d) $f(x, t) = e^t + e^x$

11.5 $\partial^2 f/\partial t^2 - v^2\left(\partial^2 f/\partial x^2\right) = 0$
- **(a)** $f(x,t) = (x+vt)^5$
- **(b)** $f(x,t) = (x+vt)^5 + C$
- **(c)** $f(x,t) = (x+vt)^5 + kt^2/2$
- **(d)** $f(x,t) = e^{vt}e^x - kx^2/(2v^2)$
- **(e)** $f(x,t) = \ln(vt)\ln(x) - kx^2/(2v^2)$

11.6 $\partial^2 f/\partial t^2 - v^2\left(\partial^2 f/\partial x^2\right) = k$
- **(a)** $f(x,t) = (x+vt)^5$
- **(b)** $f(x,t) = (x+vt)^5 + kt^2/2$
- **(c)** $f(x,t) = (x+vt)^5 + kt^2/2 + C$
- **(d)** $f(x,t) = e^{vt}e^x - kx^2/(2v^2)$
- **(e)** $f(x,t) = \ln(vt)\ln(x) - kx^2/(2v^2)$

For Problems 11.7–11.8 indicate which of the listed functions are solutions to the given PDE *and* the given boundary conditions. You will need to know that $\sinh x = \left(e^x - e^{-x}\right)/2$. *List all of the valid solutions; there may be more than one.*

11.7 $\partial^2 z/\partial y^2 + \partial^2 z/\partial x^2 = 0$, $z(x,0) = z(x,\pi) = z(0,y) = 0$, $z(\pi,y) = \sin y$
- **(a)** $z(x,y) = \sin x \sin y$
- **(b)** $z(x,y) = \sinh x \sin y$
- **(c)** $z(x,y) = \sinh x \sin y/\sinh \pi$
- **(d)** $z(x,y) = \sinh x \sin y/\sinh \pi + C$

11.8 $\partial^2 y/\partial t^2 - v^2\left(\partial^2 y/\partial x^2\right) = 0$, $y(0,t) = y(L,t) = 0$
- **(a)** $y(x,t) = \sin x \sin(vt)$
- **(b)** $y(x,t) = \sinh(\pi x)\sin(\pi vt)$
- **(c)** $y(x,t) = \sin(\pi x/L)\sin(\pi vt/L)$
- **(d)** $y(x,t) = \sin(\pi x/L)\sin(\pi vt/L) + C$

11.9 For the differential equation $x(\partial z/\partial x) + y\left(\partial z/\partial y\right) = 0$:
- **(a)** Show that $z = \ln x - \ln y$ is a valid solution.
- **(b)** Show that $z = \sin\left(x/y\right)$ is a valid solution.
- **(c)** Show that, for any function f, the function $z = f\left(x/y\right)$ is a valid solution.
- **(d)** Is $z = f\left(x/y\right) + C$ a solution? Why or why not?
- **(e)** Is $z = f\left(y/x\right)$ a solution? Why or why not?

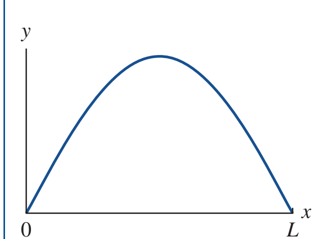
11.10 For the differential equation $2y\left(\partial^2 z/\partial x^2\right) + \partial^2 z/(\partial x \partial y) = 0$:
- **(a)** Show that $z = f(x - y^2)$ is a valid solution for any function f.
- **(b)** Is $z = f(x^2 - y)$ a solution? Why or why not?
- **(c)** Is $z = Af(y^2 - x)$ a solution? Why or why not?

11.11 For the differential equation $\frac{\partial}{\partial x}(xz) = \partial z/\partial y + z$:
- **(a)** Show that $z = \ln x + y$ is a valid solution.
- **(b)** Show that $z = f(xe^y)$ is a solution for any function f.
- **(c)** Is $z = f(x^2 e^{2y}) + C$ a solution? Why or why not?
- **(d)** Is $z = Cf(e^x y)$ a solution? Why or why not?

The sketches below show possible initial values of a function, each with a corresponding set of boundary conditions. For Problems 11.12–11.18, copy each of these initial condition sketches onto your paper and then, on the same sketch, show what the function will look like a short time later. Clearly label which sketch is the initial condition and which is the function at a later time. When an initial velocity is needed, assume that it is zero everywhere.

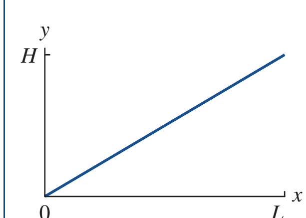
See Figure 11.2 for an example of what an answer should look like.

(a)



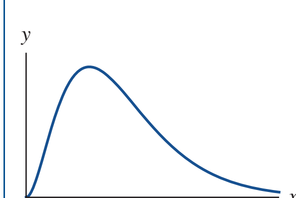
The function is fixed at zero at the two ends.

(b)



The function is fixed at zero on the left and at H on the right.

(c)



The function is fixed at zero on the left. The domain extends to infinity on the right.

11.12 $\partial^2 y/\partial x^2 = (1/v^2)\left(\partial^2 y/\partial t^2\right)$

11.13 $\partial y/\partial t = -\alpha^2\left(\partial^2 y/\partial x^2\right)$

11.14 $\partial^2 y/\partial t^2 = -\alpha^2 x^2\left(\partial^2 y/\partial x^2\right)$

11.15 $\partial y/\partial t = \alpha^2 y\left(\partial^2 y/\partial x^2\right)$

11.16 Equation 11.2.3 (the heat equation)

11.17 $\partial y/\partial t = -k^2\left(\partial y/\partial x\right)$

11.18 $\partial^2 y/\partial t^2 = k^2\left(\partial y/\partial x\right)$

11.19 Figure (a) above shows the initial *velocity* of a string attached at $y = 0$ on both ends. (For this problem pretend the vertical axis is labeled $\partial y/\partial t$ instead of y.) The initial position is $y = 0$ everywhere. For this problem you do not need to copy the sketch. Instead draw a sketch that shows the position of the string a short time after the initial moment, assuming the string obeys the wave equation, Equation 11.2.2.

11.20 An infinite string obeys the wave equation everywhere. You do not need boundary conditions for this case.

(a) Consider the initial conditions $u(x, 0) = \sin x, \partial u/\partial t(x, 0) = 0$. Sketch this initial condition and on the same sketch show how it would look a short time later if $u(x, t)$ obeys the wave equation.

(b) Describe how you would expect the function to behave over longer times.

(c) Repeat Parts (a) and (b) for the initial condition $u(x, 0) = 1 + \sin x, \partial u/\partial t(x, 0) = 0$. What are the similarities and the differences between the long term behavior in these two cases?

11.21 *[This problem depends on Problem 11.20.]* The "Klein-Gordon" equation $\left(1/v^2\right)\left(\partial^2 u/\partial t^2\right) - \partial^2 u/\partial x^2 + \omega^2 u = 0$ arises frequently in field theory. It's similar to the wave equation, but with an added term. Repeat Problem 11.20 for the Klein-Gordon equation. In what ways is the behavior described by these two equations similar and in what ways is it different?

11.22 State whether the given condition is homogeneous or inhomogeneous. Assume that the function $f(x, t)$ is defined on the domain $0 \le x \le x_f,\ 0 \le t < \infty$.

(a) $f(0, t) = f(x_f, t) = 0$

(b) $f(0, t) = 3$

(c) $f(x, 0) = 0$

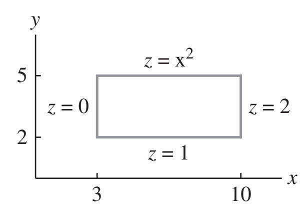
(d) $f(x, 0) = \sin\left(\pi x/x_f\right)$

(e) $\lim_{t\to\infty} f(x, t) = 0$

(f) $\lim_{t\to\infty} f(x, t) = 1$

(g) $\lim_{t\to\infty} f(x, t) = \infty$

11.23 The function $z(x, y)$ is defined on $x \in [3, 10],\ y \in [2, 5]$. The boundary conditions are: $z(3, y) = 0,\ z(x, 2) = 1,\ z(10, y) = 2$, and $z(x, 5) = x^2$. Which of these four boundary conditions are homogeneous, and which are not?



11.24 The function f is defined for all real values of x, and is *periodic:* that is, it is subject to the condition $f(x, t) = f(x + 2\pi, t)$ for all x. Is this a homogeneous condition, or not? Explain.

11.25 The function g is subject to the condition that $g(0)$ must be finite. (This condition turns out to be both common and important.) Is this a homogeneous condition, or not? Explain.

In Problems 11.26–11.28 you will be given a partial differential equation.

(a) Suppose f is defined on the domain $-\infty < x < \infty$. Give one example of a sufficient set of initial conditions for this equation. At least one of your initial conditions must be non-zero somewhere.

(b) Describe in words how $f(x, t)$ will behave for a short time after $t = 0$ for that set of initial conditions.

11.26 $\partial f/\partial t = k^2\left(\partial f/\partial x\right)$

11.27 $\partial f/\partial t = -k^2\left(\partial f/\partial x\right)$

11.28 $\partial^2 f/\partial t^2 = k^2\left(\partial f/\partial x\right)$

Problems 11.29–11.35 depend on the Motivating Exercise (Section 11.1).

11.29 In the Motivating Exercise you used physical arguments to write the equation for the evolution of the temperature distribution in a thin bar. In that problem we ignored the surrounding air, assuming that heat transfer within the bar takes place much faster than transfer with the environment. Now write a different PDE for the temperature $u(x, t)$ in a thin bar with an external heat source that provides heat proportional to distance from the left end of the bar.

11.30 The "specific heat" of a material measures how much the temperature changes in response to heat flowing in or out. If the same amount of heat is supplied to two bricks of the same size, one of which has twice as high a specific heat as the other, the one with the higher specific heat will increase its temperature by half as much as the other one.

(Be careful about that difference; *higher* specific heat means *smaller* change in temperature.)

The motivating exercise tacitly assumed that the specific heat of the bar was constant. Now consider instead a bar whose specific heat is proportional to distance from the left edge of the bar: $h = kx$. Derive the PDE for the temperature $u(x, t)$ across the bar.

11.31 The Motivating Exercise was based on a practically one-dimensional object. Now consider heat flowing through a three-dimensional object. The outer surface of that object is held at a fixed temperature distribution. (One simple example would be a cube, where the top is held at temperature $u = 100°$ and the other five sides are all held at $u = 0°$. But this problem refers to the *general* three-dimensional heat equation with fixed boundary conditions, not to any specific example.)

(a) Write the heat equation: the partial differential equation that governs the temperature $u(x, y, z, t)$ inside the object.

(b) Under these circumstances, the heat equation will approach a "steady state" that, if reached, will never change. Write a partial differential equation for the steady-state solution $u_0(x, y, z)$.

11.32 **The wave equation**: In the Motivating Exercise you used physical arguments to write down the heat equation. In this problem you'll use similar arguments to explain the form of the wave equation. Consider a string with a tension T. As you did in the motivating exercise, you should focus on a small piece of string (P) at some position x and how it interacts with the pieces to the left and right of it (P_L and P_R). *You should justify your answers to all the parts of this problem based on the physical description, not based on the wave equation, since that equation is what we're trying to derive.*

(a) If the string is initially flat, $y(x, 0) =$<*constant*>, will P_L exert an upward, downward, or zero force on P? What about the force of P_R on P? Will the net force on P be upward, downward, or zero?

(b) Repeat Part (a) if the initial shape of the string is linear, $y(x, 0) = mx + b$. (Assume $m > 0$.)

(c) Repeat Part (a) if the initial shape of the string is parabolic, $y(x, 0) = ax^2 + bx + c$ as shown in Figure 11.1 (the increasing part of a concave up curve).

(d) Does the net force on P depend on y, $\partial y/\partial x$, or $\partial^2 y/\partial x^2$?

(e) Does the force on P determine y, $\partial y/\partial t$, or $\partial^2 y/\partial t^2$?

(f) Explain in words why Equation 11.2.2 is the correct description for the motion of a string.

11.33 *[This problem depends on Problem 11.32.]* Write the PDE for a string with a drag force, e.g. a string vibrating underwater. The drag force on each piece of the string is proportional to the velocity of the string at that point, but opposite in direction. *Hint*: Go through the derivation in Problem 11.32 and see at what point the drag force would be added to what you did there.

11.34 *[This problem depends on Problem 11.32.]* Write the PDE for a string whose density is some function $\rho(x)$. (*Hint*: Think about what this implies for the mass of each small segment, and what that means for the acceleration of that segment.)

11.35 The chemical gas Wonderflonium has accumulated in a pipe. When the density of Wonderflonium is even throughout the pipe it stays constant, but if there's more Wonderflonium in one part of the pipe than another, it will tend to flow from the region of high Wonderflonium density to the region of low Wonderflonium density.

(a) Write a PDE that could describe the concentration of Wonderflonium in the pipe.

(b) Give the sign and units of any constants in your equation.

For Problems 11.36–11.44 use a computer to numerically solve the given PDEs and graph the results. You can make plots of $f(x)$ at different times, make a 3D plot of $f(x, t)$, or do an animation of $f(x)$ evolving over time. For each one describe the late term behavior (steady state, oscillating, growing without bound), based on your computer results. Then explain, based on the given equations, why you would expect that behavior even if you didn't have a computer.

11.36 $\partial^2 f/\partial t^2 = \partial^2 f/\partial x^2$, $0 < x < 1$, $0 < t < 10$, $f(0, t) = 0$, $f(1, t) = e - 1$, $f(x, 0) = e^x - 1$, $\dot{f}(x, 0) = 0$

11.37 $\partial^2 f/\partial t^2 = \partial^2 f/\partial x^2$, $0 < x < 1$, $0 < t < 10$, $f(0, t) = 0$, $f(1, t) = 0$, $f(x, 0) = 0$, $\dot{f}(x, 0) = \sin(\pi x)$

11.38 $\partial f/\partial t = \partial^2 f/\partial x^2$, $0 < x < 1$, $0 < t < 10$, $f(0,t) = 0$, $f(1,t) = e - 1$, $f(x,0) = e^x - 1$

11.39 If you are unable to numerically solve this equation explain why.
$\partial^2 f/\partial t^2 = -\partial^2 f/\partial x^2$, $0 < x < 1$, $0 < t < 0.5$, $f(0,t) = 0$, $f(1,t) = e - 1$, $f(x,0) = e^x - 1$, $\dot{f}(x,0) = 0$

11.40 $\partial f/\partial t = \partial f/\partial x$, $0 < x < 1$, $0 < t < 1$, $f(1,t) = 0$, $f(x,0) = e^{-20(x-.5)^2}$

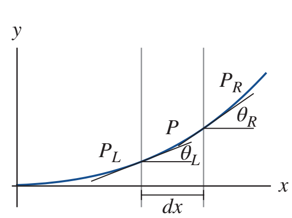
11.41 $\partial f/\partial t = \partial f/\partial x$, $0 < x < 1$, $0 < t < 3$, $f(1,t) = \sin(10t)$, $f(x,0) = 0$

11.42 $\partial f/\partial t = -\partial f/\partial x$, $0 < x < 1$, $0 < t < 1$, $f(0,t) = 0$, $f(x,0) = e^{-20(x-.5)^2}$

11.43 Try solving this out to several different final times and see if you get consistent behavior. If you are unable to numerically solve this equation explain why.
$\partial f/\partial t = -\partial f/\partial x$, $0 < x < 1$, $f(1,t) = 0$, $f(x,0) = e^{-20(x-.5)^2}$

11.44 $\partial f/\partial t = x\left(\partial f/\partial x\right)$, $0 < x < 1$, $0 < t < 1$, $f(1,t) = 0$, $f(x,0) = e^{-20(x-.5)^2}$

11.45 Exploration: The Wave Equation, Quantitatively
In Problem 11.32 you gave qualitative arguments for the form of the wave equation. In this problem you will derive it more rigorously. Consider a string of uniform tension T and density (mass per unit length) λ. At some moment t the string has shape $y(x)$. Focus on a small piece of string at position x with length dx. As before we will call this small piece of string P.



(a) The string to the right of P exerts a force T_R on P. Find the y-component of this force. Your answer will depend on the angle θ_R in the figure above.

(b) The angle θ_R is related to the slope of $y(x)$ at the right edge of the string. Rewrite your answer to Part (a) in terms of this slope: $\left(\partial y/\partial x\right)_R$.

(c) The wave equation we discuss in this chapter is only valid for small displacements of a string. Assuming the slope is small, find the linear terms of the Maclaurin series for your answer to Part (b). Use this approximate expression for the rest of the problem. If you need a derivative that you don't know you can look it up in a book or online. Or, if you prefer, you can simply have a computer generate the terms you need of the Maclaurin series.

(d) Write the y-component of the force exerted on P by the string to the left of it. You should once again assume that the slope is small and your answer should again be a linear function of the slope: $\left(\partial y/\partial x\right)_L$.

(e) Write the y-component of the net force on P.

(f) What is the mass of P in terms of quantities given in this problem?

(g) Use your answers to Parts (e) and (f) to write an equation for the vertical acceleration of P.

(h) Take the limit of your answer as $dx \to 0$ and show that this reduces to Equation 11.2.2. Express v in that equation as a function of T and λ.

11.46 Exploration: Large Waves *[This problem depends on Problem 11.45.]* In this problem you will redo Problem 11.45 *without* the assumption of small vibrations.

(a) In Part (b) of Problem 11.45 you derived the expression for the y-component of the force on P from the segment of string to the right of it. Simplify this expression as much as possible without assuming that the slope is small. Eliminate all trig and inverse trig functions from the expression.

(b) Repeat steps d-g in Problem 11.45 without assuming the slopes are small. The resulting expression for acceleration should have dx in the denominator and a complicated function of $\left(\partial y/\partial x\right)_R$ and $\left(\partial y/\partial x\right)_L$ in the numerator. This expression should not include any trig functions.

(c) Even though the slope may be anything, dx is small, so $\left(\partial y/\partial x\right)_R \approx \left(\partial y/\partial x\right)_L$.

Replace every occurrence of $\left(\partial y/\partial x\right)_L$ in your equation with $\left(\partial y/\partial x\right)_R - \epsilon$. Since ϵ must approach zero in the limit $dx \to 0$, you can take a Maclaurin series in ϵ of your acceleration equation and only keep the linear term. The terms that don't have ϵ should cancel, leaving $\epsilon/(dx)$ times an expression that doesn't contain any infinitesimal terms.

(d) Recalling that $\epsilon = \left(\partial y/\partial x\right)_R - \left(\partial y/\partial x\right)_L$, what is $\lim_{dx\to 0} \epsilon/(dx)$?

(e) Write the differential equation for large amplitude vibrations of a string. Since we are taking the limit $dx \to 0$, you can drop the subscripts on the slopes now and just write them as $\partial y/\partial x$. The result should be a non-linear differential equation involving $\partial^2 y/\partial t^2$, $\partial^2 y/\partial x^2$, and $\partial y/\partial x$.

(f) Show that your equation reduces to the wave equation when $\partial y/\partial x = 0$. How small does the slope have to be for the right hand side of this non-linear equation to be within 1% of the right hand side of the wave equation?

(g) The equation you just derived is more general than the wave equation because you dropped the assumption of small slopes. There are still some important approximations being used in this derivation, however. List at least two assumptions/approximations you made in deriving this equation. (*Note*: Saying that dx is small is *not* an assumption. It's part of the definition of dx.)

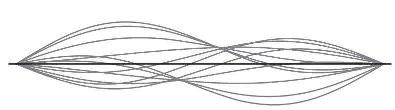
11.3 Normal Modes

We have seen that a vibrating string can be represented by a function $y(x, t)$ that obeys a partial differential equation called the "wave equation." The resulting motion depends on the string's initial position and velocity, and can sometimes be so complicated that it appears almost random. This section presents two different ways to see order behind the chaos. In the first approach the behavior is seen to be the sum of two different functions: one that holds its shape but moves to the left, and one that holds its shape but moves to the right.

The second approach—which will dominate, not only this section, but most of the rest of the chapter—builds up the behavior from special solutions called "normal modes." Finding simple normal modes that match your boundary conditions, and then summing those normal modes to describe more complicated behavior, is the key to understanding a wide variety of systems.

11.3.1 Discovery Exercise: Normal Modes

A guitar string extends from $x = 0$ to $x = \pi$. It is fixed at both ends, but free to vibrate in between.



The position of the string $y(x, t)$ is subject to the equation:

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{9}\frac{\partial^2 y}{\partial t^2} \qquad (11.3.1)$$

The phrase "fixed at both ends" in the problem statement gives us the boundary conditions:

$$y(0, t) = y(\pi, t) = 0 \qquad (11.3.2)$$

1. One solution to this problem is $y = 5\sin(2x)\cos(6t)$. (Later in the chapter you will find such solutions for yourself; right now we are focusing on understanding the equation and its solutions.)
   (a) Confirm that this solution solves the differential equation by plugging $y(x, t)$ into both sides of Equation 11.3.1 and showing that you get the same answer.
   (b) Confirm that this solution meets the boundary conditions, Equation 11.3.2.
   (c) Draw graphs of the shape of the string between $x = 0$ and $x = \pi$ at times $t = 0$, $t = \pi/12$, $t = \pi/6$, $t = \pi/4$, and $t = \pi/3$. Then describe the resulting motion in words.
   (d) Which of the three numbers in this solution—the 5, the 2, and the 6—is an arbitrary constant? That is, if you change that number to any other constant, the function will still solve the differential equation and meet the boundary conditions.
2. Another solution to this equation is $y = -(1/2)\sin(10x)\cos(kt)$, if you choose the correct value of k.
   (a) Plug this function into both sides of Equation 11.3.1 and solve for k.
   (b) Does the resulting function also meet the boundary conditions?
   (c) What is the period of this function in space (i.e. the distance between adjacent peaks, or the wavelength)?
   (d) What is the period of this function in time (i.e. how long do you have to wait before the string returns to its initial position)?

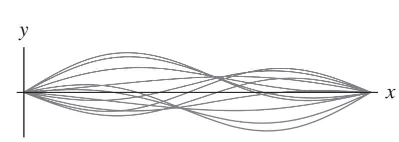
   See Check Yourself #70 in Appendix L
3. For the value of k that you calculated in 2, is the function $y = 5\sin(2x)\cos(6t) - (1/2)\sin(10x)\cos(kt)$ a valid solution? (Make sure to check whether it meets both the differential equation and the boundary conditions!)
4. Next consider the solution $y = A\sin(px)\cos(kt)$.
   (a) For what values of k will this function solve Equation 11.3.1? Your answer will depend on p.
   (b) For what values of p will this solution match the boundary conditions?
5. Write the solution to this differential equation in the most general form you can.

11.3.2 Explanation: Normal Modes

A vibrating string such as a guitar string obeys the *one-dimensional wave equation:*

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} \tag{11.3.3}$$

The dependent variable y represents the displacement of the string from its relaxed height. (Note that y can be positive, negative, or zero.) The constant v is related to the tension and linear density of the string.



When you are solving an Ordinary Differential Equation (ODE) you often try to find the "general solution," a closed-form function that represents all possible solutions, with a few arbitrary constants to be filled in based on initial conditions. The same can sometimes be done for a Partial Differential Equation (PDE), and below we present the general solution to the wave equation, valid for all possible initial and boundary conditions. However, it is not always practical to solve problems using this general solution. Instead we will find a special set of particular solutions known as "normal modes" and build up solutions for different initial and boundary conditions using these normal modes.

In this section we'll simply present the normal modes for the wave equation and use combinations of them to build other solutions. In later sections we'll show you how to find the normal modes for other PDEs.

The General Solution, aka d'Alembert's Solution

The functions $\sin\left[(x+vt)^2\right]$, $3/(x+vt)$, and $6\ln(x+vt)$ are all valid solutions to the wave equation. You may not have any idea how we just came up with them, but you can easily verify that they work. More generally, *any* function of the form $f(x+vt)$ will solve the wave equation: you can confirm this in general just as you can for the specific cases.

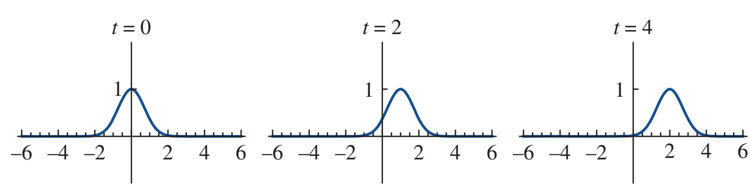
$$y=f(x+vt) \quad \rightarrow \quad \frac{\partial^2 y}{\partial x^2}=f''(x+vt),\ \frac{\partial^2 y}{\partial t^2}=v^2 f''(x+vt) \quad \rightarrow \quad \frac{\partial^2 y}{\partial x^2}=\frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$

For similar reasons, any function $g(x-vt)$ will also be a solution. Any sum of such functions will be a solution as well, so we can write the general solution of this equation as:

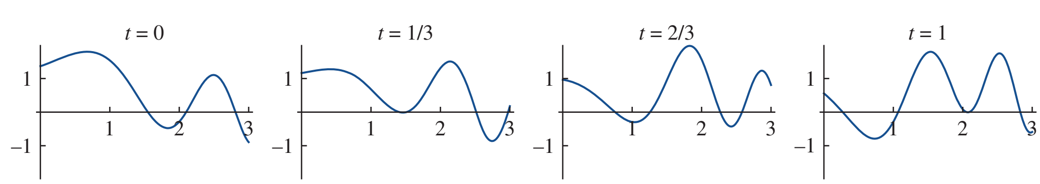
$$y(x,t)=f(x+vt)+g(x-vt)$$

(We don't need arbitrary constants in front because f and g can include any constants.) This form of the general solution to the wave equation is known as "d'Alembert's Solution." The constant v in this solution is not arbitrary; it was part of the original differential equation. However, f and g are *arbitrary functions:* replace them with any functions at all and you have a solution. For instance, the function $y=1/(x+vt)^2-3\ln(x-vt)$ solves the equation (as you will demonstrate in Problem 11.47).

What does all that tell us about vibrating strings? $g(x-vt)$ represents a function that does not change its shape over time: it only moves to the right with speed v. For instance, the function $y(x,t)=e^{-(x-vt)^2}$ describes the behavior you might see if you grab one end of a rope and give it a quick jerk.



Similarly, $f(x+vt)$ represents an arbitrary curve moving steadily to the left. But the general solution $f(x+vt)+g(x-vt)$ does *not* describe a static curve that moves: it can change its shape over time in ways that might surprise you. For instance, we show here the function $\cos\left[(x+t)^2\right]+e^{-(x-t-1)^2}$.



The correct combination of functions f and g describe any possible solution to the wave equation. But while f retains its shape while moving to the left, and g retains its shape while moving to the right, a combination of the two may evolve in complicated ways.

Simple Solutions for Simple Cases

Now that we have the general solution, it's just a matter of matching it to the boundary and initial conditions, and we can solve any wave equation problem, right? In principle, that's correct. In practice, it can be hard. For instance, below we consider a string that is fixed at both ends (simple boundary conditions), and begins at rest in the shape of a sine wave (simple initial conditions). Before we present the solution, you may want to try to find a function of the form $f(x+vt)+g(x-vt)$ that fits these conditions. If you don't get very far, you may be interested to hear about a very different approach—more practical and, surprisingly, no less general in the end.

Consider a string subject to the following conditions.

1. The string is fixed at both ends. If the length of the string is π this imposes the conditions $y(0,t)=0$ and $y(\pi,t)=0$.
2. The initial shape of the string is half a sine wave $y(x,0)=\sin x$ as shown in Figure 11.3, and the initial velocity is zero.

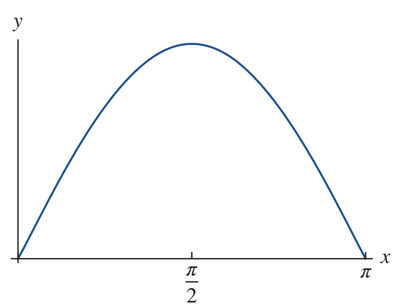


FIGURE 11.3 $y(x,0)$.

You can determine the motion of such a string experimentally by plucking a tight rubber band. (This is easy to do, but you have to watch carefully.) Even without such an experiment you can imagine the behavior based on your physical intuition. The sine wave will decrease in amplitude until it momentarily flattens out, and then begin to open up again on the negative side until it reaches an upside-down half-wave $y=-\sin x$. Then it will start moving up again, and so on.

What function would describe that kind of motion? You may be able to guess the answer yourself, at least to within a constant or two. We'll fill in those constants and tell you that the solution is:

$$y=\sin(x)\cos(vt) \tag{11.3.4}$$

Every point on the string oscillates with a period of $2\pi/v$ and an amplitude given by its original height (Figure 11.4).

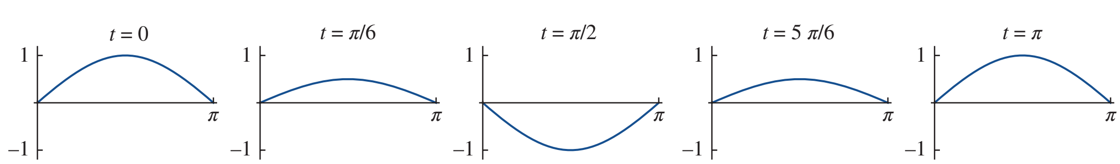


FIGURE 11.4 The function $y=\sin(x)\cos(vt)$ with $v=2$.

Is it really that simple? The "rubber band" argument may not be sufficiently rigorous for you; in fact this solution may remind you of a similar function that *didn't* work in the previous section. But the initial condition is different now, and our new solution works perfectly. We leave it to you to confirm that the function $y=\sin(x)\cos(vt)$ satisfies the wave equation (11.3.3), our boundary conditions $y(0,t)=y(\pi,t)=0$, and our initial conditions $y(x,0)=\sin x$ and $\dot{y}(x,0)=0$. Those confirmations are the acid test of a solution, however we arrived at it.

And what about our general solution? Can this function be rewritten in the form $y(x,t)=f(x+vt)+g(x-vt)$? It must be possible, because *all* solutions to the wave equation must have this form. In this particular case it can be done using trig identities, but it isn't necessary. It was easier to start with a physically motivated guess than to find the right functions f and g.

Making an educated guess works great if the initial condition happens to be $y = \sin x$, but can we do it for other cases? You probably won't be surprised to hear that guessing a solution is just as easy if $y(x, 0) = \sin(2x)$, or $\sin(10x)$, or any other sine wave that fits comfortably into the total length: the string just oscillates, retaining its original shape but changing amplitude. You'll show in the problems that if the initial conditions are zero velocity and $y(x, 0) = \sin(nx)$ (where n is an integer), then the solution is $y(x, t) = \sin(nx)\cos(nvt)$.

Solutions of this form are called the "normal modes" of the string.

Definition: Normal Mode

A "normal mode" is a solution that evolves in time by changing its amplitude while leaving its basic shape unchanged. For instance, if all points x on a string oscillate with the same frequency and phase as each other, the result is a standing wave that grows and shrinks.

To see more clearly what this definition means, consider the example $y(x, t) = \sin(x)\cos(vt)$ discussed above. At $x = \pi/6$ this becomes $y(\pi/6, t) = .5\cos(vt)$, which is an oscillation with frequency $v/(2\pi)$ and amplitude .5. (Remember that *frequency* just means one over the period.) At the point $x = \pi/3$ the string oscillates with frequency $v/(2\pi)$ and amplitude $\sqrt{3}/2$. Since every point on the string oscillates at the same frequency, this solution is a normal mode of the system. This may remind you of the "normal modes" of coupled oscillators in Chapter 6. In that case the system consisted of a finite number of discrete oscillators instead of an infinite number of oscillating points, but the definition of normal mode is the same in both cases.

If the string starts out in the curve $y(x, 0) = \sin(nx)$, we know exactly how it will evolve over time. Surprisingly, that insight turns out to be the key to the motion of our string under *any* initial conditions.

More Complicated Solutions for More Complicated Cases

What if the string doesn't happen to start in a sine wave? The bad news is that, in general, the string will *not* keep its overall shape while stretching and compressing vertically. The good news is that we can apply our finding from one very special case—the normal modes—to find the solution for almost any initial condition. The key, as it often is, is the ability to write a general solution as a *sum* of specific solutions.

EXAMPLE A Sum of Two Normal Modes

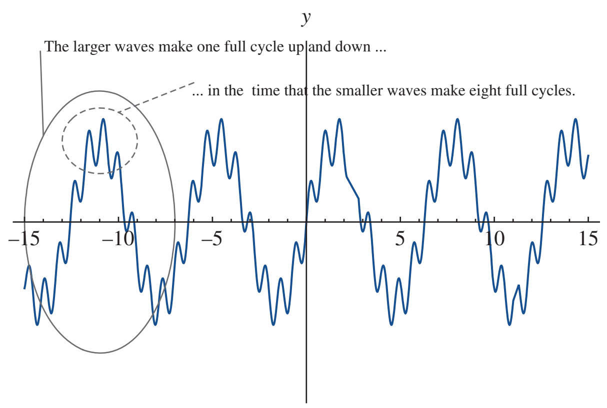
Problem:
Consider a string subject to the same boundary condition we used above, $y(0, t) = y(\pi, t) = 0$, but starting in the initial form $y(x, 0) = 7\sin x + 2\sin(8x)$. How will such a string evolve over time?

Solution:
The wave equation is linear and homogeneous, which means that any linear combination of solutions is also a solution. Our boundary conditions are also homogeneous, which means that any sum of solutions will match the

boundary conditions as well. Since we know that $y(x,t) = \sin(x)\cos(vt)$ and $y(x,t) = \sin(8x)\cos(8vt)$ are both solutions, it must also be true that:

$$y(x,t) = 7\sin(x)\cos(vt) + 2\sin(8x)\cos(8vt)$$

is a solution. Since it matches the boundary and initial conditions, it is *the* solution for this case.[2]



Once again, this should look familiar if you've studied coupled oscillators. There too you can solve for any initial condition by writing it as a sum of normal modes.

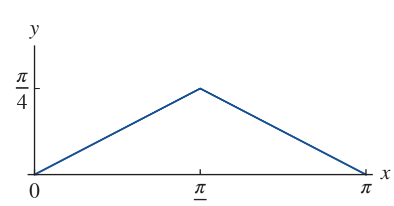
You may object that our example was too easy: the initial condition wasn't exactly a normal mode, but it was the next best thing. How do we find the solution for an initial condition that doesn't just "happen" to be the sum of a few sine waves?

The answer is that practically *any* function on a finite domain "happens" to be the sum of sine waves—or at least, we can write it as the sum of sine waves if we want to. That's what Fourier series are all about! And that insight leads us to a general approach. First you decompose your initial condition into a sum of sine waves (or, more generally, into a sum of normal modes). Then you write your solution as a sum of individual solutions for each of these normal modes.

EXAMPLE A Plucked Guitar String

Problem:

Solve the wave equation with boundary conditions $y(0,t) = 0$ and $y(\pi,t) = 0$, and initial conditions $\dot{y}(x,0) = 0$, $y(x,0) = \begin{cases} x/2 & 0 < x < \pi/2 \\ (\pi - x)/2, & \pi/2 < x < \pi \end{cases}$



[2]As we said earlier, we are not going to formally discuss exactly what boundary and initial conditions are sufficient to conclude that a solution is unique. In this case, however, we can make the argument on purely physical grounds: we know everything there is to know about this particular string, and it can only do one thing!

Solution:
We begin by writing the initial condition as a sum of normal modes. In other words, we write the Fourier sine series for the function. The answer (after some calculations) is:

$$y(x,0) = \sum_{n=1}^{\infty} (-1)^{(n-1)/2} \frac{2}{n^2\pi} \sin(nx),\ n \text{ odd } = \frac{2}{\pi}\sin(x) - \frac{1}{9\pi}\sin(3x) + \frac{2}{25\pi}\sin(5x) + \dots$$

That infinite sum may look intimidating, but if you look term by term, you can see that for each value of n this is just a constant times a sine function. We can do the same thing we did above with the sum of just two sine functions; we multiply each sine function of x by the corresponding cosine function of t. So the solution is:

$$\begin{aligned} y(x,t) &= \sum_{n=1}^{\infty} (-1)^{(n-1)/2} \frac{2}{n^2\pi} \sin(nx)\cos(nvt),\ n \text{ odd} \\ &= \frac{2}{\pi}\sin(x)\cos(vt) - \frac{1}{9\pi}\sin(3x)\cos(3vt) + \dots \end{aligned} \tag{11.3.5}$$

If you had trouble with the step where we took the Fourier sine series of $y(x,0)$ you should review Fourier series now; we're going to use them a lot in this chapter. All the relevant formulas are in Appendix G.

Even if you didn't have trouble with the calculation, you may still find an infinite series to be an unsatisfying answer. However, that $1/n^2$ will make the series converge pretty quickly, so for most purposes the first few terms will give you a pretty good approximate answer. As we go through the chapter, you'll see that most analytical solutions for PDEs come in the form of series; expressions in closed form are the exceptions.

Now you know how to solve for the motion of a vibrating string, at least if:

- The ends of the string are at $x = 0$ and $x = \pi$.
- The ends of the string are held fixed at $y = 0$.
- The initial velocity of the string is zero everywhere.

In the problems you will use the same technique with these three conditions remaining constant—only the initial position, and therefore the Fourier series, will vary. Then you will do other problems that change all three of these conditions, and you will find that although the forms of the solutions vary, the basic idea carries through. The example below, for instance, is the same in length and initial velocity, but different in boundary conditions.

EXAMPLE Air in a Flute

The air inside a flute obeys the wave equation $\partial^2 s/\partial x^2 = (1/c_s^2)(\partial^2 s/\partial t^2)$, where $s(x,t)$ is displacement of the air and the constant c_s is the speed of sound. For reasons we're not going to get into here, s is not constrained to go to zero at the edges, but $\partial s/\partial x$ is. Hence, we are solving the same differential equation with different boundary conditions.

For consistency, let's consider a flute that extends from $x = 0$ to $x = \pi$, and let's take as a simple initial condition $s(x,0) = \cos x$, $\partial s/\partial t(x,0) = 0$. (Why can't we start with the same initial condition we used for our string above? Because it doesn't meet our new boundary conditions!)

The solution to our new system is $s(x, t) = \cos(x)\cos(c_s t)$. You can confirm this solution by verifying the following requirements:

1. It satisfies the wave equation $\partial^2 s/\partial x^2 = (1/c_s^2)(\partial^2 s/\partial t^2)$.
2. It satisfies the boundary condition $\partial s/\partial x = 0$ at $x = 0$ and $x = \pi$.
3. It satisfies the initial condition $s(x, 0) = \cos x$. (Did you expect our solution to have $\sin(c_s t)$ instead of the cosine? This would meet the first two requirements, but not the initial condition. Try it!)

This solution is a normal mode since every point in the flute vibrates with the same frequency. More generally, $s(x, t) = \cos(nx)\cos(nc_s t)$ is the normal mode of this system for the initial condition $s(x, 0) = \cos(nx)$, $\partial s/\partial t(x, 0) = 0$.

Therefore, if the system happened to start in the state $s(x, 0) = 10\cos(3x) + 2\cos(8x)$, $\partial s/\partial t(x, 0) = 0$, its motion would be described by the function $s(x, t) = 10\cos(3x)\cos(3c_s t) + 2\cos(8x)\cos(8c_s t)$. More generally, we can decompose any initial condition into a Fourier cosine series, and simply write the solution from there. (For a function defined on a finite interval you create an "odd extension" of that function to write a Fourier sine series or an "even extension" to write a Fourier cosine series: see Chapter 9.)

Normal modes provide a very general approach that can be used to solve many problems in partial differential equations, provided you can take two key steps.

1. Figure out what the normal modes are. In this case we figured them out using physical intuition—or, as it may seem to you, improbably lucky guesswork.
2. Rewrite any initial condition as a sum of normal modes. In this case we used Fourier series.

In the next section we will see a more general approach to the first step, *finding* the normal modes. In the sections that follow, we will see that the second step is often possible even when the normal modes do not involve sines and cosines.

11.3.3 Problems: Normal Modes

11.47 Demonstrate that the function $y = 1/(x + vt)^2 - 3\ln(x - vt)$ is a valid solution to the wave equation (11.3.3).

11.48 Demonstrate that any function of the form $f(x - vt) + g(x + vt)$ is a valid solution to the wave equation (11.3.3).

11.49 Consider the function $y = (x - vt)^2$.

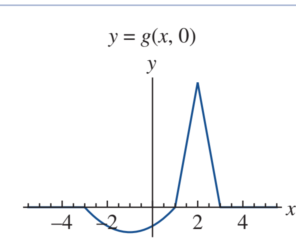
(a) For the case $v = 1$, draw graphs of this function at times $t = 0$, $t = 1$, and $t = 2$.

(b) In general, how does this function vary over time?

(c) Repeat parts (a) and (b) for $v = 2$.

(d) In general, how does the constant v affect the behavior of this function?

11.50 Consider the function $g(x + 3t)$. At time $t = 0$, the function looks like this, stretching along the x-axis toward infinity in both directions.



(a) Draw graphs of this function at times $t = 1$, $t = 2$, and $t = 3$.

(b) In general, how does this function evolve over time?

11.51 A string that obeys the wave equation (11.3.3) is tacked down at the ends, so $y(0, t) = y(\pi, t) = 0$. The string starts at rest, so $\dot{y}(x, 0) = 0$. If the initial position of the string happens to be $y(x, 0) = \sin(2x)$, then the string will follow the function $y = \sin(2x)\cos(2vt)$.

(a) Guess at a solution to the same differential equation with the same boundary conditions, but with initial position $y(x, 0) = \sin(5x)$.

(b) Confirm that your guess to Part (a) satisfies the differential Equation 11.3.3, the boundary conditions $y(0, t) = y(\pi, t) = 0$, and the initial conditions $y(x, 0) = \sin(5x)$ and $\dot{y}(x, 0) = 0$. (If it doesn't, keep guessing until you find one that does.)

(c) Now guess at the solution $y(x, t)$ for a string that is identical to the strings above, except that its initial position is $y(x, 0) = 3\sin(2x)$. Once again, confirm that your solution solves the wave equation and all required conditions.

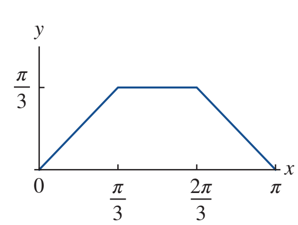
(d) Solve the wave equation subject to all the same conditions above, except that this time, $y(x, 0) = 6\sin(x) - 5\sin(7x)$. Once again, confirm that your solution solves the wave equation and all required conditions.

(e) Solve the wave equation for the same conditions one last time, but this time with the initial position $y(x, 0) = \sum_{n=1}^{\infty}(1/n^2)\sin(nx)$. Leave your answer in the form of a series. (This part should not take much more work than the other ones.) You do not need to verify this answer.

11.52 A string of length π is fixed at both ends, so $y(0, t) = y(\pi, t) = 0$. You pull the string up at two points and then let go, so the initial conditions are:

$$y(x, 0) = \begin{cases} x & x < \pi/3 \\ \pi/3 & \pi/3 \le x \le 2\pi/3 \\ \pi - x & 2\pi/3 < x \end{cases}$$

$$\text{and} \quad \frac{\partial y}{\partial t}(x, 0) = 0$$



(a) Rewrite the initial condition as a Fourier sine series.

(b) Write the solution $y(x, t)$. Your answer will be expressed as a series and will include a constant v.

(c) Take $v = 2$ and have a computer calculate the 20^{th} partial sum of the solution you found Part (b). Plot the solution at a series of times and describe its evolution.

11.53 A great deal of what we want you to learn in this section can be expressed in one concise mathematical statement: "Any function of the form $y(x, t) = A\sin(x)\cos(vt) + B\sin(2x)\cos(2vt) + C\sin(3x)\cos(3vt) + \ldots$ is a solution of the wave equation (11.3.3) and the boundary conditions $y(0, t) = y(\pi, t) = 0$."

(a) Prove that this result is true.

(b) Explain why this result can be used to solve the wave equation for almost any initial conditions, whether they are sinusoidal or not.

In Problems 11.54–11.59 a string of length π, initially at rest, has boundary conditions $y(0, t) = y(\pi, t) = 0$. For the initial shape given in the problem:

(a) Find the solution $y(x, t)$ to the wave equation (11.3.3), taking $v = 2$. Your answer will be in the form of a series.

(b) If your answer is in the form of an infinite series make plots of the 1^{st} partial sum, the 3^{rd} partial sum, and the 20^{th} partial sum of the series solution at several times.

11.54 $y(x, 0) = \sin(10x)$

11.55 $y(x, 0) = \sin(2x) + (1/10)\sin(10x)$

11.56 $y(x, 0) = (1/10)\sin(2x) + \sin(10x)$

11.57 $y(x, 0) = \begin{cases} 1 & \pi/3 < x < 2\pi/3 \\ 0 & \text{elsewhere} \end{cases}$

11.58 $y(x, 0) = \pi^2/4 - (x - \pi/2)^2$

11.59 Make up an initial position $y(x, 0)$. You may use any function that obeys the boundary conditions *except* the trivial case $y(x, 0) = 0$, or any function we have already used in the Explanation (Section 11.3.2) or the problems above.

11.60 Section 11.2 looked at a string pinned to the x-axis at $x = 0$ and $x = 4\pi$, with initial position $y(x, 0) = 1 - \cos x$. We found that this is not a normal mode; it will not just oscillate. Now you find out what such a string actually will do. (*Hint*: when finding the Fourier series you will be faced with a difficult integral. The easiest approach is rewriting the trig functions as complex exponentials.)

11.61 In the Explanation (Section 11.3.2), we make a big deal of the fact that functions of the form $y(x, t) = \sin(nx)\cos(nvt)$ are "normal modes" of the wave equation (11.3.3). This does *not* simply mean that these functions are valid solutions of the differential equation; it means something much stronger than that. Explain in your own words what a "normal mode" means, and why it is important.

11.62 A string of length 1 is fixed at both ends and obeys the wave equation (11.3.3) with $v = 2$. For each of the initial conditions given below assume the initial velocity of the string is zero.

(a) Have a computer numerically solve the wave equation for this string with initial condition $y(x, 0) = \sin(2\pi x)$ and animate the resulting motion of the string. Solve to a late enough time to see the string oscillate at least twice, using trial and error if necessary. Describe the resulting motion.

(b) Have a computer numerically solve for and animate the motion of the string for initial condition $y(x, 0) = \sin(20\pi x)$, using the same final time you used in Part (a). How is this motion different from what you found in Part (a)?

(c) Consider the initial condition $y(x, 0) = \sin(2\pi x) + (.1)\sin(20\pi x)$. What would you expect the motion of the string to look like in this case? Solve the wave equation with this initial condition numerically and animate the results. Did the results match your prediction?

(d) Finally, make an animation of the solution to the wave equation for the case $y(x, 0) = .2\sin(2\pi x)\left[7 + 6x - 100x^2 + 100x^3 + \cos(36x) - e^{-36x^2}\right]$. (We chose this simply because it looks like a crazy, random mess.) Describe the resulting motion.

(e) How is the evolution of a string that starts in a normal mode different from the evolution of a string that starts in a different shape?

11.63 In the Explanation (Section 11.3.2), we discussed a string of length π fixed at both ends with initial shape $y(x, 0) = \sin x$ and no initial velocity. Based on physical arguments we guessed that the initial shape of the string would oscillate sinusoidally in time. We then jumped to the exact correct function with very little justification. (Did you notice?) In this problem, your job is to fill in the missing steps. Start with a "guess" that represents the initial function oscillating: $y(x, t) = \sin(x)\left(A\sin(\alpha t) + B\cos(\beta t)\right)$. Plug this guess into the wave equation (11.3.3) along with the initial conditions $y(x, 0) = \sin x$ and $\dot{y}(x, 0) = 0$, and solve for A, B, α, and β.

In Problems 11.64–11.68 you will solve for the displacement of air inside a flute of length π. The displacement $s(x, t)$ obeys the wave equation $\partial^2 s/\partial x^2 = (1/c_s^2)(\partial^2 s/\partial t^2)$, just like a vibrating string, but the boundary conditions for the flute are $\partial s/\partial x(0, t) = \partial s/\partial x(\pi, t) = 0$. This leads to a different set of normal modes, which we found in the example on Page 561. The initial condition for $s(x, 0)$ is given below, and you should assume in each case that $\partial s/\partial t(x, 0) = 0$. For each of these initial conditions find the solution $s(x, t)$ to the wave equation, taking $c_s = 3$.

11.64 $s(x, 0) = \cos(5x)$

11.65 $s(x, 0) = \cos(x) + (1/10)\cos(10x)$

11.66 $s(x, 0) = (1/10)\cos(x) + \cos(10x)$

11.67 $s(x, 0) = \left(x^2 - \pi^2\right)^2$. Your answer will be an infinite series. Make plots of the 20^{th} partial sum of the series solution at three or more different times. Describe how the function $s(x)$ is evolving over time.

11.68 Make up an initial position $s(x, 0)$. You may use any function that obeys the boundary conditions $\partial s/\partial x(0, t) = \partial s/\partial x(\pi, t) = 0$ *except* the trivial case $s(x, 0) = 0$, or any function we have already used in the Explanation (Section 11.3.2) or the problems above.

11.69 In the Explanation (Section 11.3.2), we considered a string of length π with fixed ends and zero initial velocity. We wrote an expression for the normal modes of this system and showed how to solve for the motion if the string started in one of the normal modes. More importantly, we showed how to write any other given initial condition as a sum of normal modes using Fourier series and thus solve the wave equation. In this problem you will perform a similar analysis for a string of length L.

(a) If a string starts in the initial position $y(x, 0) = \sin(nx)$ where n is any integer, it is guaranteed to meet the boundary conditions $y(0, 0) = y(\pi, 0) = 0$. What must be true of the constant k if the initial position meets the boundary conditions $y(0, 0) = y(L, 0) = 0$? (Your answer will once again end in the phrase "where n is any integer.")

(b) For a string of length π with fixed ends and zero initial velocity, the normal modes can be written as $y(x,t) = \sin(nx)\cos(nvt)$, $n = 0, 1, 2, \ldots$. Based on the initial position you wrote in Part (a), write a similar expression for the normal modes $y(x,t)$ for a string of length L. Confirm that your solution solves the wave equation (11.3.3).

(c) Find the solution $y(x,t)$ if the string starts at rest in position $y(x,0) = \sin(3\pi x/L)$. Make sure your solution satisfies the wave equation and matches the initial and boundary conditions.

(d) Find the solution $y(x,t)$ if the string starts at rest in position $y(x,0) = 2\sin(3\pi x/L) + 5\sin(8\pi x/L)$.

(e) Now find the solution if the string starts at rest in position $y(x,0) = \begin{cases} 1 & L/3 < x < 2L/3 \\ 0 & \textit{elsewhere} \end{cases}$. (*Hint*: You will need to start by expanding this initial function in a Fourier sine series.) Your answer will be in the form of an infinite series.

11.70 In the Explanation (Section 11.3.2), we considered a string with zero initial velocity and non-zero initial displacement. We wrote an expression for the normal modes of this system and showed how to solve for the motion if the string started in one of the normal modes. More importantly, we showed how to write a more complicated initial position as a sum of normal modes using Fourier series and thus solve the wave equation.

In this problem you will perform a similar analysis for a string with the same boundary conditions, $y(0,t) = y(\pi,t) = 0$, but with different initial conditions: your string has zero initial *displacement* and non-zero initial *velocity*.

(a) Regardless of its initial velocity, this string has zero initial acceleration. How do we know that?

(b) Suppose that the string has initial velocity $\partial y/\partial t(x,0) = \sin x$. Sketch the shape of the string a short time later after the initial time.

(c) Describe in words how you would expect the string to evolve over time. (To answer this, you will need to take into account not only the initial velocity, but also the wave equation (11.3.3) that dictates its acceleration over time.)

(d) Express your guess as a mathematical function $y(x,t)$ and verify that it solves the wave equation and matches the initial and boundary conditions. (If at first your guess doesn't succeed: try, try again.)

(e) Next consider the initial velocity $\partial y/\partial t(x,0) = \sin(3x)$. Find the solution $y(x,t)$ for this case and make sure your solution satisfies the wave equation and the initial and boundary conditions.

(f) In the Explanation (Section 11.3.2), we found that the normal modes for a string of length π with fixed ends and zero initial velocity can all be written as $y(x,t) = \sin(nx)\cos(nvt)$, $n = 0, 1, 2, \ldots$. Write a similar expression for the normal modes of a string of length π with fixed ends and zero initial *displacement*.

(g) Find the solution $y(x,t)$ if the string starts at $y(x,0) = 0$ with initial velocity $\partial y/\partial t(x,0) = 2\sin(3x) + 4\sin(5x)$.

(h) Find the solution if the string starts at $y(x,0) = 0$ with initial velocity $\partial y/\partial t(x,0) = \begin{cases} 1 & \pi/3 < x < 2\pi/3 \\ 0 & \textit{elsewhere} \end{cases}$. This might occur if the middle of the string were suddenly struck with a hammer. (*Hint*: You will need to start by expanding this initial function in a Fourier sine series.) Your answer will be in the form of an infinite series.

11.71 In the Explanation (Section 11.3.2), we found the normal modes for a vibrating string that obeys the wave equation (11.3.3) with fixed boundaries and zero initial velocity. If the string is infinitely long we no longer have those boundary conditions. What are all the possible normal modes for a system obeying the wave equation (11.3.3) with zero initial velocity on the real line: $-\infty < x < \infty$? To answer this you should look at the Explanation and see what restriction the boundary conditions imposed on our normal modes, and then remove that restriction.

11.72 A function $y(x,t)$ may be said to be a "normal mode" if the initial shape $y(x,0)$ evolves in time by changing amplitude—that is, by stretching or compressing vertically—but does not change in any other way. We have seen that the normal modes for the wave equation are sines and cosines, but different equations may have very different normal modes. Express with no words (just a simple equation) the statement "$y(x,t)$ is a normal mode" as defined above.

11.73 A string of length 1 obeys the wave equation with $v = 2$. The string is initially at rest at $y = 0$. The right side of the string is fixed, but the left side is given a quick jerk: $y(0, t) = e^{-100(t-1)^2}$. Solve the wave equation numerically with this boundary condition and animate the results out to $t = 5$. Describe the motion of the string.

11.74 A string of length π obeys the wave equation with $v = 2$. The right side of the string is fixed.

Suppose the string starts at rest at $y = 0$ and you excite it by vibrating the left end of it: $y(0, t) = \sin(11t)$. Notice that this is *not* one of the normal mode frequencies.

(a) Solve the wave equation numerically with this boundary condition out to $t = 10$. Describe the resulting motion.

Next suppose the left end vibrates according to: $y(0, t) = \sin(10t)$.

(b) Is this oscillation occurring at one of the normal mode frequencies given by Equation 11.3.4?

(c) Solve the wave equation numerically with this boundary condition out to $t = 10$. How is the resulting motion different from what you found in Part (a)?

11.75 **Two different general solutions.** The general solution for a vibrating string of length π with fixed ends and zero initial velocity is

$$y(x, t) = \sum_{n=0}^{\infty} b_n \sin(nx)\cos(nvt)$$

We know, however, that any solution to the wave equation can be written in the form of d'Alembert's solution

$$y(x, t) = f(x + vt) + g(x - vt)$$

so it must be possible to rewrite the normal mode solution in this form.

(a) Use trig identities to rewrite $\sin(nx)\cos(nvt)$ in terms of $(x + vt)$ and $(x - vt)$.

(b) Find the functions f and g such that $f(x + vt) + g(x - vt) = \sum_{n=0}^{\infty} b_n \sin(nx)\cos(nvt)$.

11.76 **Exploration: Wind Instruments**

When air blows across an opening in a pipe it excites the air inside the pipe at many different frequencies, but the only ones that get amplified by the pipe are the normal modes of the pipe. So the normal modes of a wind instrument determine the notes you hear. Generally the longer wavelength modes are louder, so the dominant tone you hear from the pipe is that of the normal mode with the longest wavelength (lowest frequency). This is called the "fundamental note" or "fundamental frequency" of a wind instrument.

Sound waves in a pipe obey the wave equation $\partial^2 s/\partial x^2 = (1/c_s^2)(\partial^2 s/\partial t^2)$, where $c_s = 345 m/s$ is the speed of sound. When the end of the pipe is open s is not constrained to go to zero at the edges of the pipe, but $\partial s/\partial x$ is. Real wind instruments change the effective length of the pipe by opening and closing valves or holes in different places, but for this problem we will consider the simplest case of a cylindrical instrument such as a flute or clarinet with all the holes closed except at the ends.

(a) A flute is open at both ends. What are all of the possible normal modes for a flute of length L? (This is similar to the example on Page 561, but this time the flute is of length L instead of length π. How does this change the normal modes?) Be sure to include both the space and time parts of the normal mode. For simplicity you can assume that $(\partial s/\partial t)(x, 0) = 0$.

(b) A typical flute might be 66 cm. What is the fundamental frequency of such a flute? Remember that frequency is defined as 1/period, so you will need to start by figuring out the period of the normal mode.

(c) What is the frequency of the next normal mode in the series? Look up what notes these frequencies correspond to. (For example, a frequency of 2500 Hz is roughly a note of E, two octaves above middle C.)

(d) What are all of the possible normal modes for a pipe of length L that is closed at one end $(s(0, t) = 0)$ and open at the other $(\partial s/\partial x(L, t) = 0)$? A clarinet is a typical example.

(e) If a clarinet and a flute were the same length, which one would play a higher fundamental note and why?

(f) One common type of clarinet is 60 cm long. What is the fundamental frequency of such a clarinet? What is the frequency of the next normal mode in the series? Look up what notes these frequencies correspond to.

11.4 Separation of Variables—The Basic Method

This section is the heart of the chapter. By reducing a *partial* differential equation to two or more *ordinary* differential equations, the technique we introduce here allows you to find the normal modes of the system and thereby a solution.

The three sections that follow this are all elaborations of the basic method presented here. If you carefully follow the algebra in this section and see how the steps fit together to form a big picture, you will be well prepared for much of the rest of this chapter. At the end, in an unusually long "Stepping Back," we discuss the variations we will present so you can see how the chapter fits together.

11.4.1 Discovery Exercise: Separation of Variables—The Basic Method

Consider a bar going from $x = 0$ to $x = L$. The "heat equation" (which you derived in Section 11.1) governs the evolution of the temperature distribution:

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2} \tag{11.4.1}$$

where $u(x, t)$ is the temperature and α is a positive constant called "thermal diffusivity" that reflects how efficiently the bar conducts heat. Both ends are immersed in ice water, so $u(0, t) = u(L, t) = 0$, but the initial temperature may be non-zero at other interior points. You are going to solve for the temperature $u(x, t)$. Your strategy will be to guess a solution of the form $u(x, t) = X(x)T(t)$ where X is a function of the variable x (but not of t), and T is a function of the variable t (but not of x).

1. Plug the function $u(x, t) = X(x)T(t)$ into Equation 11.4.1. Note that if $u(x, t) = X(x)T(t)$ then $\partial u/\partial x = X'(x)T(t)$.
2. "Separate the variables" in your answer to Part 1. In other words, algebraically rearrange the equation so that the left side contains all the functions that depend on t, and the right side contains all the functions that depend on x. In this problem—in fact in many problems—you can separate the variables by dividing both sides of the equation by $X(x)T(t)$. You should also divide both sides of the equation by α, which will bring α to the side with t; this step is not necessary but it will make the equations a bit simpler later on.
3. The next step relies on this key result: if the left side of the equation depends on t (but not on x), and the right side depends on x (but not on t), then both sides of the equation *must equal a constant*. Explain why this result must be true.
4. Based on the result from Part 3, you can now turn your *partial* differential equation from Part 1 into two *ordinary* differential equations: "the left side of the equation equals a constant" and "the right side of the equation equals *the same* constant." Write both equations. Call the constant P.

   See Check Yourself #71 in Appendix L

The next step, finding real solutions to these two ordinary differential equations, depends on the sign of the constant P. We shall therefore handle the three cases separately. We'll start by solving the equation for $X(x)$. Remember that the constant α is necessarily positive. Avoid complex answers.

5. Assuming $P > 0$, we can replace P with k^2 where k is any real number. Solve the equation for $X(x)$ for this case. Your solution will have two arbitrary constants in it: call them A and B.
6. Assuming $P = 0$, solve the equation for $X(x)$, once again using A and B for the arbitrary constants.
7. Assuming $P < 0$, we can replace P with $-k^2$ where k is any real number. Solve the equation for $X(x)$ for this case, once again using A and B for the arbitrary constants.
8. For two of the three solutions you just found the only way to match the boundary conditions is with the "trivial" solution $X(x) = 0$. Only one of the three allows for non-trivial solutions that match the boundary conditions. Based on that fact, what must the sign of P be?

See Check Yourself #72 in Appendix L

9. The boundary condition $u(0, t) = 0$ implies that $X(0) = 0$. Plug this into your solution for $X(x)$ to find the value of one of the two arbitrary constants.
10. Find the values of k that match the second boundary condition $u(L, t) = 0$. *Hint*: there are infinitely many such values. We will return to these values when we match initial conditions. For the rest of this exercise we will continue to just write k.

Having found $X(x)$, we now turn our attention to the ODE you wrote for $T(t)$ way back in Part 4.

11. Replace P with $-k^2$ in the $T(t)$ differential equation, as you did with the $X(x)$ one. (Why? Because both differential equations were set equal to the *same constant* P.)
12. Having done this replacement, solve the equation for $T(t)$. Your solution will introduce a new arbitrary constant: call it C.
13. Write the solution $u(x, t) = X(x)T(t)$ based on your answers. This solution should depend on k.
14. Explain why, when you combine your $X(x)$ and $T(t)$ functions into one $u(x, t)$ function, you can combine the arbitrary constants from the two functions into one arbitrary constant.

The solution you just found is a normal mode of this system. In the Explanation that follows we will write the general solution to such an equation as a linear superposition of all the normal modes, and use initial conditions to solve for the arbitrary constants.

11.4.2 Explanation: Separation of Variables—The Basic Method

In the last section we found that, if a vibrating string with fixed ends happens to start at rest in a sinusoidal shape, it will evolve very simply over time, changing amplitude only. We called such a solution a "normal mode." If the string doesn't happen to start in such a fortuitous position, we can model its motion by writing the position as a sum of these normal modes (a Fourier series).

That's a great result, but it all started with a lucky guess. How could we have solved that problem if we hadn't thought of using sine waves? More importantly, how do you solve other problems?

It turns out that you can solve a wide variety of partial differential equations by writing the general solution as a sum of normal modes. "Separation of variables" is the most important technique for finding normal modes and solving partial differential equations. (You may recall a technique called "separation of variables" for solving ordinary differential equations. Both techniques involve some variables that have to be separated from each other, but beyond that, they have nothing to do with each other. Sorry about that.)

Before we dive into the details, let's start with an analogy to a familiar problem. To solve the *ordinary* differential equation $y'' - 6y' + 5y = 0$, we might take the following steps:

- *Guess a solution with an unknown constant.* In Chapter 1 we saw that the correct guess for such an equation would be $y = e^{px}$.
- *Plug your guess into the original equation, to solve for the unknown constant.* Plugging $y = e^{px}$ into the original differential equation, we can solve to find two solutions $y = e^x$ and $y = e^{5x}$. (Try it.)
- *Sum the solutions.* If a differential equation is linear and homogeneous, then any linear combination of solutions is also a solution. So we write $y = Ae^x + Be^{5x}$. Since we are solving a second-order linear equation and we have a solution with two arbitrary constants, this is the general solution.
- *Plug in initial conditions.* "Initial conditions" in this context means two additional pieces of information beside the original differential equation. For instance, if we know that $f(2) = 3$ then we can write $3 = Ae^2 + Be^{10}$. If we know one other piece of information—such as another point, or the derivative at that point—we can solve for the arbitrary constants, finding the specific solution we want.

Plugging in "guesses" in this way reduces ordinary differential equations to algebra equations, which are generally easier to solve.

Separation of variables allows you to solve partial differential equations in much the same way. We will show below that by "guessing" a solution that changes in amplitude only, you turn your partial differential equation into several ordinary differential equations. Solving these equations gives you your normal modes with an infinite number of arbitrary constants to match based on "initial" (time) and "boundary" (space) conditions. With the variables separated, you can approach these two kinds of conditions separately.

The Problem

We're going to demonstrate this new technique with a familiar problem. When we arrive at the solution, we will recognize it from the previous sections. But this time we will derive the solution in a way that can be applied to different problems.

Here, then, is our familiar problem: a string of length L obeys the wave equation:

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} \tag{11.4.2}$$

The string is fixed at both ends, which gives us our boundary conditions:

$$y(0,t) = y(L,t) = 0 \tag{11.4.3}$$

To fully solve for the motion of the string, we need to know its initial position, and its initial velocity. For the moment we will keep both of those generic:

$$y(x,0) = f(x) \quad \text{and} \quad \frac{dy}{dt}(x,0) = g(x) \tag{11.4.4}$$

Solve for the motion of the string.

Step 1: The Guess

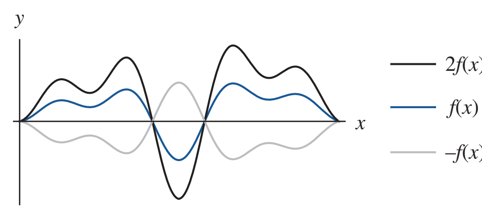
You could imagine y as any function of x and t, such as x^t or $3\ln(xt)/(x+t)$ or even more hideous-looking combinations, but almost none of them would work as solutions to the wave equation. Our hopeful guess is a solution of the form:

$$y(x,t) = X(x)T(t)$$

where X is a function of the variable x (but not of t), and T is a function of the variable t (but not of x).

It's worth taking a moment to consider what sort of functions we are looking at. $X(x)$ represents the shape of the string at a given moment t. Since we have placed no restrictions on the form of that function, it could turn out to be simple or complicated.

But how does that function evolve over time? $T(t)$ is simply a number, positive or negative, at any given moment. When you multiply $X(x)$ by a number, you stretch it vertically. You may also turn it upside down. But *beyond those simple transformations, the function $X(x)$ will not alter in any way.*



Wherever $X(x) = 0$, it will stay zero forever: that is to say, the places where the string crosses the x-axis will never change. (The only exception is that, whenever $T(t)$ happens to be zero, the entire string is uniformly flat.) The x-values of the critical points, where the string reaches a local maximum or minimum, will likewise never change.

A function of that form—changing in amplitude, but not in shape—is called a "normal mode." In guessing a solution of the form $y(x,t) = X(x)T(t)$, we are asserting "There exists a set of normal modes for this differential equation." We will then plug in this function to find the normal modes. If the initial state does not happen to correspond perfectly to a normal mode (which it usually doesn't) we will build it up as a *sum* of normal modes; since we understand how each of them evolves in time, we can compute how the entire assemblage evolves.[3]

Step 2: Plug In the Guess

What do we get if we plug $y(x,t) = X(x)T(t)$ into Equation 11.4.2? When we take the derivative with respect to x, the function $T(t)$ (which by definition has no x-dependence) acts as a constant. Similarly, $X(x)$ is a constant when we take derivatives with respect to t. So the equation becomes:

$$X''(x)T(t) = \frac{1}{v^2}X(x)T''(t)$$

Now comes the step that gives the technique of "separation of variables" its name. We rearrange the equation so that all the t-dependency is on the right, and all the x-dependency is on the left. We accomplish this (in this case and in fact in many cases) by dividing both sides by $X(x)T(t)$.

$$\frac{X''(x)}{X(x)} = \frac{1}{v^2}\frac{T''(t)}{T(t)} \tag{11.4.5}$$

If x changes, does the right side of this equation change? The answer must be "no" since $T(t)$ has no x-dependency. That means—since the two sides of the equation must stay equal for *all values* of x and t—that changing x cannot change the left side of the equation either.

Similarly, changing t has no effect on the left side of the equation, and must therefore have no effect on the right side. If both sides of the equation are equal, and neither one

[3]When can you build up your initial conditions as a sum of normal modes? The answer, to make a long story short, is "almost always." We'll make that story long again when we discuss Sturm-Liouville theory in Chapter 12.

depends on x or t, then they must be…*(drum roll please)*…a constant! Calling this constant P, we write:

$$\frac{X''(x)}{X(x)} = P \textit{ and } \frac{1}{v^2}\frac{T''(t)}{T(t)} = P \tag{11.4.6}$$

Step 3: Solve the Spatial Equation and Match the Boundary Conditions

Instead of one partial differential equation, we now have two *ordinary* differential equations—and pretty easy ones at that. We start with the spatial equation:

$$X''(x) = PX(x)$$

The solution to this equation depends on the sign of P. In the following analysis we define a new real number k. Because k is real we use k^2 to mean "any positive P" and $-k^2$ to mean "any negative P."

$P > 0$	$P = k^2$	$X''(x) = k^2 X(x)$	$X(x) = Ae^{kx} + Be^{-kx}$
$P = 0$		$X''(x) = 0$	$X(x) = mx + b$
$P < 0$	$P = -k^2$	$X''(x) = -k^2 X(x)$	$X(x) = A\sin(kx) + B\cos(kx)$

The exponential[4] and linear solutions can only satisfy our boundary condition ($y = 0$ at both ends) with the "trivial" solution $X(x) = 0$. Unless our initial conditions place the string in an unmoving horizontal line, this solution is inadequate.

On the other hand, sines and cosines are flexible enough to meet all our demands. We can tailor them to meet our boundary conditions, and then we can sum the result to meet whatever initial condition the string throws at us. We therefore declare that P is negative and replace it with $-k^2$.

Our first boundary condition is $y(0, t) = 0$. Plugging $x = 0$ into $A\sin(kx) + B\cos(kx)$ and setting it equal to zero gives $B = 0$, so we have a sine without a cosine.

The next condition is $y(L, t) = 0$.

$$A\sin(kL) = 0$$

One way to match this condition would be to set $A = 0$, which would bring us back to the trivial $y(x, t) = 0$. The alternative is $\sin(kL) = 0$, which can only be satisfied if $kL = n\pi$ where n is an integer. The case $n = 0$ gives us the trivial solution again, and negative values of n give us the same solutions as positive values (just with different values of the arbitrary constant A), so the complete set of non-trivial functions $X(x)$ is:

$$X(x) = A\sin(kx) \quad \left(k = \frac{n\pi}{L} \text{ for all positive integers } n\right) \tag{11.4.7}$$

We need to stress that we are *not* saying that our string must, at any given time, take the shape $x = A\sin(n\pi x/L)$. We are saying, instead, that Equation 11.4.7 defines the normal modes of the string. If the string is described by that equation then it will evolve simply in time—we'll figure out exactly how in a moment. If the string is not in such a shape then we will write it as a sum of such functions and then evolve each one independently.

[4]The exponential solution can also be written with hyperbolic trig functions as $A\sinh(kx) + B\cosh(kx)$. This is often more convenient for matching boundary conditions, but it does not change anything fundamental such as the inability to meet these particular conditions.

Step 4: Solve the Time Equation

Equation 11.4.6 introduced one new constant P, not two. We found that P must be negative and wrote $P = -k^2$ in the $X(x)$ equation; the P in the $T(t)$ equation is the same variable.

$$\frac{1}{v^2}\frac{T''(t)}{T(t)} = -k^2$$

We can quickly rewrite this as $T''(t) = -k^2 v^2 T(t)$, which has a similar form—and therefore a similar solution—to our spatial equation.

$$T(t) = C\sin(vkt) + D\cos(vkt)$$

We found that to match the boundary conditions for $X(x)$ we needed $k = n\pi/L$, and (again) this is the same k. The solution we are looking for is a product of the $X(x)$ and $T(t)$ functions.

$$X(x)T(t) = \sin\left(\frac{n\pi}{L}x\right)\left[C\sin\left(\frac{nv\pi}{L}t\right) + D\cos\left(\frac{nv\pi}{L}t\right)\right] \quad \text{where} \quad n = 1, 2, 3\ldots \tag{11.4.8}$$

We have absorbed the arbitrary constant A into the arbitrary constants C and D. Why can we do that? Remember that A simply means "you can put any constant here" and C also means "you can put any constant here," so AC simply stands for "any constant." We can call this new constant C with no loss of generality.

For any positive integer n and any values of C and D the function 11.4.8 is a valid solution to the differential equation 11.4.2 and matches the boundary conditions 11.4.3.

Step 5: Sum the Solutions

Equation 11.4.8 is a family of different solutions, each representing a normal mode of the string. For instance one normal mode looks like this.

$$y(x,t) = \sin\left(\frac{3\pi}{L}x\right)\cos\left(\frac{3v\pi}{L}t\right) \tag{11.4.9}$$

Equation 11.4.9 has two different frequencies[5] and it's important not to get them confused. $3\pi/L$ is a quantity in space, not time—it means that the wavelength of the string is $2L/3$. The wavelengths of our normal modes were determined by the boundary conditions $y(0,t) = y(L,t) = 0$. $3v\pi/L$ is a quantity in time, not space—it means that the string will return to its starting position every $2L/(3v)$ seconds. The fact that this frequency is v times the spatial one is a result of the PDE we are solving.

Equation 11.4.9 tells us that if the string starts at rest in a perfect sine wave with spatial frequency $3\pi/L$ then it is in a normal mode and will retain its basic shape while its amplitude oscillates with temporal frequency $3v\pi/L$.

That was a normal mode with $n = 3$. Every positive integer n corresponds to a different wavelength. Every such wavelength fits our boundary conditions, and every such mode will oscillate with a different period in time. The amplitude of each normal mode can be anything, represented by the arbitrary constants C and D.

Because we are solving a linear homogeneous PDE with homogeneous boundary conditions, any linear combination of these solutions is itself a solution. So $\sin(\pi x/L)\left[3\sin(v\pi t/L) + 4\cos(v\pi t/L)\right]$ is a solution, and $\sin(2\pi x/L)\left[5\sin(2v\pi t/L) - 8\cos(2v\pi t/L)\right]$

[5] Our word "frequency," whether in space or time, is shorthand for the quantity that should more properly be termed "angular frequency."

is another solution, and if you sum those two functions you get yet another solution. To find the *general* solution we sum all possible solutions.

$$y(x,t)=\sum_{n=1}^{\infty}\sin\left(\frac{n\pi}{L}x\right)\left[C_n\sin\left(\frac{nv\pi}{L}t\right)+D_n\cos\left(\frac{nv\pi}{L}t\right)\right] \tag{11.4.10}$$

We've added a subscript to the constants C and D because they can take on different values for each value of n.

Step 6: Match the Initial Conditions

Finally—*after* we write one solution that is a sum of all the solutions we have found so far—we impose the initial conditions to find the values of the arbitrary constants. Note that in the heat equation that you solved in the Discovery Exercise (Section 11.4.1) the time dependence was first order, so there was only one initial condition and one arbitrary constant per solution. The wave equation is second order in time, and so requires two initial conditions (such as initial position and velocity) and two arbitrary constants per solution (C and D). But since we have an infinite number of solutions, "two constants per solution" is actually an *infinite number* of arbitrary constants: C_1, D_1, C_2, D_2, and so on. We have to find them all to match our initial conditions $y(x,0)=f(x)$ and $dy/dt(x,0)=g(x)$.

From Equation 11.4.10 we can write:

$$\frac{\partial y}{\partial t}=\sum_{n=1}^{\infty}\sin\left(\frac{n\pi}{L}x\right)\left[C_n\left(\frac{nv\pi}{L}\right)\cos\left(\frac{nv\pi}{L}t\right)-D_n\left(\frac{nv\pi}{L}\right)\sin\left(\frac{nv\pi}{L}t\right)\right] \tag{11.4.11}$$

Plugging $t=0$ into Equations 11.4.10 and 11.4.11,

$$y(x,0)=\sum_{n=1}^{\infty}D_n\sin\left(\frac{n\pi}{L}x\right)=f(x) \tag{11.4.12}$$

$$\frac{\partial y}{\partial t}(x,0)=\sum_{n=1}^{\infty}C_n\left(\frac{nv\pi}{L}\right)\sin\left(\frac{n\pi}{L}x\right)=g(x) \tag{11.4.13}$$

It is now time to relate these equations to the results we saw in the previous section. We said that if the string is in a *normal mode,* its motion will be very simple. If it is not in a normal mode, we can understand its motion by building the initial conditions as a sum of normal modes.

Let's start with a very simple example: the initial position $f(x)=3\sin(5\pi x/L)$, and the initial velocity $g(x)=0$. Can you see what this does to our coefficients in Equations 11.4.12 and 11.4.13? It means that $D_5=3$, that $D_n=0$ for all n other than 5, and that $C_n=0$ for *all* n. In other words, $y(x,t)=3\cos(5v\pi t/L)\sin(5\pi x/L)$.

On the other hand, what if $f(x)$ is not quite so convenient? We are left with using Equation 11.4.12 to solve for all the D_n coefficients to match the initial position. But that is exactly what we do when we create a Fourier sine series: we find the coefficients to build an arbitrary function as a series of sine waves. You may want to quickly review that process in Chapter 9. Here we are going to jump straight to the answer: $D_n=(2/L)\int_0^L f(x)\sin(n\pi x/L)\,dx$.

Similar arguments apply to the initial velocity and C_n, with the important caveat that the Fourier coefficients in Equation 11.4.13 are $C_n(nv\pi/L)$. So we can write $C_n(nv\pi/L)=(2/L)\int_0^L g(x)\sin(n\pi x/L)\,dx$.

We're done! The complete solution to Equation 11.4.2 subject to the boundary conditions 11.4.3 and the initial conditions 11.4.4 is:

$$y(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{L}x\right)\left[C_n \sin\left(\frac{nv\pi}{L}t\right) + D_n \cos\left(\frac{nv\pi}{L}t\right)\right] \tag{11.4.14}$$

$$C_n = \frac{2}{nv\pi}\int_0^L g(x)\sin\left(\frac{n\pi}{L}x\right)dx, \quad D_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx$$

In the problems you'll evaluate this solution analytically and numerically for various initial conditions $f(x)$ and $g(x)$. For some simple initial conditions you may be able to evaluate this sum explicitly, but for others you can use partial sums to get numerical answers to whatever accuracy you need.

EXAMPLE Separation of Variables

Solve the partial differential equation

$$4\frac{\partial z}{\partial t} - 9\frac{\partial^2 z}{\partial x^2} - 5z = 0$$

on the domain $0 \le x \le 6$, $t \ge 0$ subject to the boundary conditions $z(0,t) = z(6,t) = 0$ and the initial condition $z(x,0) = \sin^2(\pi x/6)$.

1. Assume a solution of the form $z(x,t) = X(x)T(t)$.
2. Plug this solution into the original differential equation and obtain

$$4X(x)T'(t) - 9X''(x)T(t) - 5X(x)T(t) = 0$$

Rearrange to separate the variables:

$$\frac{4T'(t)}{T(t)} - 5 = \frac{9X''(x)}{X(x)}$$

Both sides must now equal a constant:

$$\frac{4T'(t)}{T(t)} - 5 = P \quad \text{and} \quad \frac{9X''(x)}{X(x)} = P$$

3. If $P > 0$ the solution is exponential and if $P = 0$ the solution is linear. Neither of these can match the boundary conditions except in the trivial case $X(x) = 0$. We therefore conclude that $P < 0$, replace it with $-k^2$, and obtain the solution $X(x) = A\sin(kx/3) + B\cos(kx/3)$. The boundary condition $z(0,t) = 0$ implies $B = 0$. The boundary condition $z(6,t) = 0$ implies $k = \pi n/2$. So $X(x) = A\sin(\pi nx/6)$.
4. The time equation can be rewritten as $T'(t) = (P+5)T/4$ and the solution is $T(t) = Ce^{(P+5)t/4}$. We can combine this with our spatial solution and replace P with $-(\pi n/2)^2$ to get

$$X(x)T(t) = D\sin\left(\frac{\pi n}{6}x\right)e^{(-\pi^2 n^2+20)t/16}$$

5. Because both our equation and our boundary conditions are homogeneous, any linear combination of solutions is a solution:

$$z(x,t)=\sum_{n=1}^{\infty} D_n \sin\left(\frac{\pi n}{6}x\right) e^{\left(-\pi^2 n^2+20\right)t/16}$$

6. Finally, our initial condition gives

$$\sin^2\left(\frac{\pi x}{6}\right)=\sum_{n=1}^{\infty} D_n \sin\left(\frac{\pi n}{6}x\right)$$

This is a Fourier sine series. The formula for the coefficients is in Appendix G. The resulting integral looks messy but it can be solved by using Euler's formula and some trig identities or by just plugging it into a computer program. The result is

$$D_n=\frac{2}{6}\int_0^6 \sin^2\left(\frac{\pi x}{6}\right)\sin\left(\frac{n\pi x}{6}\right)dx=\begin{cases} 8/\left(4n\pi-n^3\pi\right) & n \text{ odd} \\ 0 & n \text{ even}\end{cases}$$

So the solution is

$$z(x,t)=\sum_{n=1}^{\infty}\frac{8}{4n\pi n^3\pi}\sin\left(\frac{\pi n}{6}x\right)e^{\left(-\pi^2 n^2+20\right)t/16},\ n \text{ odd}$$

Stepping Back Part I: Solving Problems that are Just Like this One

It may seem like the process above involved a number of tricks that might be hard to apply to other problems. But the method of separation of variables is actually very general, and the steps you take are pretty consistent.

1. Begin by *assuming* a solution which is a simple product of single-variable functions: in other words, a normal mode.
2. Substitute this assumed solution into the original PDE. Then try to "separate the variables," algebraically rearranging so that each side of the equation depends on only one variable. (This often involves dividing both sides of the equation by $X(x)T(t)$.) Set both sides of the equation equal to a constant, thus turning one partial differential equation into two ordinary differential equations. Depending on the boundary conditions, you may be able to quickly identify the sign of the separation constant.
3. Solve the spatial equation and substitute in your boundary conditions to restrict your solutions. This step may eliminate some solutions (such as the cosines in the example above) and/or restrict the possible values of the separation constant.
4. Solve the time equation and multiply it by your spatial solution. Since both solutions will have arbitrary constants in front, you can often absorb the arbitrary constants from the spatial solution into those from the time solution. The resulting functions are the normal modes of your system.
5. Since your equation is linear and homogeneous, the linear combination of all these normal modes provides the most general solution that satisfies the differential equation and the boundary conditions. This solution will be an infinite series.
6. Finally, match your solution to your initial conditions to find the coefficients in that series. For instance, if your normal modes were sines and cosines, then you are building the initial conditions as Fourier series.

In the examples above the boundary conditions were homogeneous while the initial conditions were not. If two solutions y_1 and y_2 satisfy the boundary conditions $y(0, t) = y(L, t) = 0$ then the sum $y_1 + y_2$ also satisfies these boundary conditions. That's why we can apply these boundary conditions to each normal mode individually, confident that the sum of all normal modes will meet the same conditions.

By contrast if two solutions y_1 and y_2 individually satisfy the initial condition $y(x, 0) = f(x)$, their sum $(y_1 + y_2)(x, 0)$ will add up to $2f(x)$. So we have to apply the condition $y(x, 0) = f(x)$ to the entire series *after* summing, rather than to the individual parts.

It's fairly common for the boundary conditions to be homogeneous and the initial conditions inhomogeneous. In such cases, you will follow the pattern we gave here: first apply the boundary conditions to $X(x)$, then sum all solutions into a series, and then apply the initial conditions to $\sum X(x)T(t)$.

Stepping Back Part II: Solving Problems that are a Little Bit Different

Many differential equations *cannot* be solved by the exact method described above. We list here the primary variations you may encounter, many of which are discussed later in this chapter.

- **You can't separate the variables.** If you cannot algebraically separate your variables—for instance, if the differential equation is inhomogeneous—this particular technique will obviously not work. You may still be able to solve the differential equation using other techniques, some of which are discussed later in this chapter and some of which are discussed in textbooks on partial differential equations.
- **Your boundary conditions are inhomogeneous.** Just as with ODEs, you can add a "particular" solution to a "complementary" solution. We discuss this situation in Section 11.8.
- **Your initial conditions are homogeneous.** You'll show in Problem 11.97 why separation of variables doesn't generally work for homogeneous initial conditions. This problem forces you to find an alternative method, some of which are discussed in this chapter.
- **You have more than two independent variables.** Do separation of variables multiple times until you have one ordinary differential equation for each independent variable. We'll solve such an example in Section 11.5.
- **You have more than one spatial variable and no time variable.** If you have inhomogeneous boundary conditions along one boundary, and homogeneous boundary conditions along the others, you can treat the inhomogeneous boundary the way we treated the initial conditions above. (Apply the homogeneous boundary conditions, then sum, and then apply the inhomogeneous boundary condition.) You already know everything you need for such a problem, so you'll try your hand at it in Problem 11.96. If you have more than one inhomogeneous boundary condition, you need to apply the method described in Section 11.8.
- **Your normal modes are not sines and cosines.** Separation of variables gives you an ordinary differential equation that you can solve to find the normal modes of the partial differential equation you started with. Besides trig functions these normal modes may take the form of Bessel functions, Legendre polynomials, spherical harmonics, and many other categories of functions. If you can find the normal modes, and if you can build your initial conditions as a sum of normal modes, the overall process remains exactly the same. See Sections 11.6 and 11.7.
- **Your differential equation is non-linear.** You're pretty much out of luck. Separation of variables will not work for a non-linear equation because it relies on writing the solution as a linear combination of normal modes. In fact, there are very few non-linear partial differential equations that can be analytically solved by any method. When scientists and engineers need to solve a non-linear partial differential equation, which they do

quite often, they generally have to find a way to approximate it with a linear equation or solve it numerically.

11.4.3 Problems: Separation of Variables—The Basic Method

In the Explanation (Section 11.4.2) we found the general solution to the wave equation 11.4.2 with boundary conditions $y(0,t) = y(L,t) = 0$. In Problems 11.77–11.81 plug the given initial conditions into the general solution 11.4.14 and solve for the coefficients C_n and D_n to get the complete solution $y(x,t)$. In some cases your answers will be in the form of infinite series.

11.77 $y(x,0) = F\sin(2\pi x/L), \partial y/\partial t(x,0) = 0$

11.78 $y(x,0) = 0, \partial y/\partial t(x,0) = c\sin(\pi x/L)$

11.79 $y(x,0) = H\sin(2\pi x/L), \partial y/\partial t(x,0) = c\sin(\pi x/L)$

11.80 $y(x,0) = \begin{cases} x & 0 < x < L/2 \\ L-x & L/2 < x < L \end{cases}, \dfrac{\partial y}{\partial t}(x,0) = 0$

11.81 $y(x,0) = \begin{cases} x & 0 < x < L/2 \\ L-x & L/2 < x < L \end{cases},$

$$\frac{\partial y}{\partial t}(x,0) = \begin{cases} 0 & 0 < x < L/3 \\ c & L/3 < x < 2L/3 \\ 0 & 2L/3 < x < L \end{cases}$$

11.82 Walk-Through: Separation of Variables. In this problem you will use separation of variables to solve the equation $\partial^2 y/\partial t^2 - \partial^2 y/\partial x^2 + y = 0$ subject to the boundary conditions $y(0,t) = y(1,t) = 0$.

(a) Begin by guessing a separable solution $y = X(x)T(t)$. Plug this guess into the differential equation. Then divide both sides by $X(x)T(t)$ and separate variables so that all the t dependence is on the left and all the x dependence is on the right. Put the constant term on the t-dependent side. (You could put it on the other side, but the math would get a bit messier later on.)

(b) Explain in your own words why both sides of the separated equation you just wrote must equal a constant.

(c) Set the x-equation equal to a constant named P and find the general solution to the resulting ODE three times: for $P > 0$, $P = 0$, and $P < 0$. Explain why you cannot match the boundary conditions with the solutions for $P > 0$ or $P = 0$ unless you set $X(x) = 0$. Since you now know P must be negative, you can call it $-k^2$. Write your solution for $X(x)$ in terms of k.

(d) Apply the boundary condition $y(0,t) = 0$ to show that one of the arbitrary constants in your solution from Part (c) must be 0.

(e) Apply the boundary condition $y(1,t) = 0$ to find all the possible values for k. There will be an infinite number of them, but you should be able to write them in terms of a new constant n, which can be any positive integer.

(f) Solve the ODE for $T(t)$, expressing your answer in terms of n.

(g) Multiply $X(x)$ times $T(t)$ to find the normal modes of this system. You should be able to combine your three arbitrary constants into two. Write the general solution $y(x,t)$ as a sum over these normal modes. Your arbitrary constants should include a subscript n to indicate that they can take different values for each value of n.

11.83 *[This problem depends on Problem 11.82.]* In this problem you will plug the initial conditions $y(x,0) = \begin{cases} 1 & 1/3 < x < 2/3 \\ 0 & \textit{elsewhere} \end{cases}, \dfrac{\partial y}{\partial t}(x,0) = 0$ into the solution you found to Problem 11.82.

(a) The condition $\partial y/\partial t(x,0) = 0$ should allow you to set one of your arbitrary constants to zero. The remaining condition should give you the equation $y(x,0) = \sum_{n=1}^{\infty} D_n \sin(n\pi x)$, which is a Fourier sine series for the function $y(x,0)$. Find the coefficients D_n. The appropriate formula is in Appendix G. (*Of course you may not have called this constant D in your solution, but you should get an equation of this form.*)

(b) Plug the coefficients that you found into your general solution to write the complete solution $y(x,t)$ for this problem. The result should be an infinite series.

(c) Have a computer calculate the 100^{th} partial sum of your solution and plot it at a variety of times. Describe how the function $y(x)$ is evolving over time.

Solve Problems 11.84–11.90 using separation of variables. For each problem use the boundary conditions $y(0,t) = y(L,t) = 0$. When the initial conditions are given as arbitrary functions, write the solution as a series and write expressions for the coefficients in the series, as we did for the wave equation. When specific initial conditions are given, solve for the coefficients. The solution may still be in the form of a series. It may help to first work through Problem 11.82 as a model.

11.84 $\partial y/\partial t = c^2(\partial^2 y/\partial x^2)$, $y(x,0) = f(x)$

11.85 $\partial y/\partial t = c^2(\partial^2 y/\partial x^2)$, $y(x,0) = \sin(\pi x/L)$

11.86 $\partial y/\partial t + y = c^2(\partial^2 y/\partial x^2)$, $y(x,0) = f(x)$

11.87 $\partial^2 y/\partial t^2 + y = c^2(\partial^2 y/\partial x^2)$, $y(x,0) = f(x)$, $\partial y/\partial t(x,0) = 0$

11.88 $\dfrac{\partial^2 y}{\partial t^2} + y = c^2\dfrac{\partial^2 y}{\partial x^2}$,
$$y(x,0) = \begin{cases} x & 0 \le x \le L/2 \\ L-x & L/2 < x \le L \end{cases},$$
$$\frac{\partial y}{\partial t}(x,0) = \begin{cases} -x & 0 \le x \le L/2 \\ x-L & L/2 < x \le L \end{cases}$$

11.89 $\partial^4 y/\partial t^4 = -c^2(\partial^2 y/\partial x^2)$, $y(x,0) = \dot{y}(x,0) = \ddot{y}(x,0) = 0$, $\partial^3 y/\partial t^3(x,0) = \sin(3\pi x/L)$ *Hint*: the algebra in this problem will be a little easier if you use hyperbolic trig functions. If you aren't familiar with them you can still do the problem without them.

11.90 $\partial y/\partial t = c^2 t(\partial^2 y/\partial x^2)$, $y(x,0) = y_0 \sin(\pi x/L)$

Problems 11.91–11.94 refer to a rod with temperature held fixed at the ends: $u(0,t) = u(L,t) = 0$. For each set of initial conditions write the complete solution $u(x,t)$. You will need to begin by using separation of variables to solve the heat equation (11.2.3), but if you do more than one of these you should just find the general solution once.

11.91 $u(x,0) = u_0 \sin(2\pi x/L)$

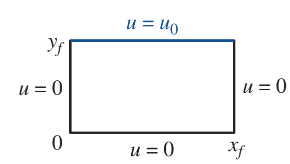
11.92 $u(x,0) = \begin{cases} cx & 0 < x < L/2 \\ c(L-x) & L/2 < x < L \end{cases}$

11.93 $u(x,0) = \begin{cases} 0 & 0 < x < L/3 \\ u_0 & L/3 < x < 2L/3 \\ 0 & 2L/3 < x < L \end{cases}$

11.94 $u(x,0) = 0$. Show how you can find the solution to this the same way you would for Problems 11.91–11.93, and also explain how you could have predicted this solution without doing any calculations.

11.95 As we solved the "string" problem in the Explanation (Section 11.4.2) we determined that a positive separation constant P leads to the solution $X(x) = Ae^{kx} + Be^{-kx}$. We then discarded this solution based on the argument that it cannot meet the boundary conditions $X(0) = X(L) = 0$. Show that there is no possible way for that solution to meet those boundary conditions unless $A = B = 0$. Then explain why we discard *that* solution too.

11.96 Given enough time, any isolated region of space will tend to approach a steady state where the temperature at each point is unchanging. In this case the time derivative in the heat equation becomes zero and the temperature obeys Laplace's equation 11.2.5. In two dimensions this can be written as $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = 0$. In this problem you will use separation of variables to solve for the steady-state temperature $u(x,y)$ on a rectangular slab subject to the boundary conditions $u(0,y) = u(x_f,y) = u(x,0) = 0$, $u(x,y_f) = u_0$.



(a) Separate variables to get ordinary differential equations for the functions $X(x)$ and $Y(y)$.

(b) Solve the equation for $X(x)$ subject to the boundary conditions $X(0) = X(x_f) = 0$.

(c) Solve the equation for $Y(y)$ subject to the homogeneous boundary condition $Y(0) = 0$. Your solution should have one undetermined constant in it corresponding to the one boundary condition you have not yet imposed.

(d) Write the solution $u(x,y)$ as a sum of normal modes.

(e) Use the final boundary condition $u(x,y_f) = u_0$ to solve for the remaining coefficients and find the general solution.

(f) Check your solution by verifying that it solves Laplace's equation and meets each of the homogeneous boundary conditions given above.

(g) Use a computer to plot the 40^{th} partial sum of your solution. (You will have to choose some values for the constants in the problem.) Looking at your plot, describe how the temperature depends on y for a fixed value of x (other than $x = 0$ or $x = x_f$). You should see that it goes from $u = 0$ at $y = 0$ to $u = u_0$ at $y = y_f$. Does it increase linearly? If not describe what it does.

11.97 We said in the Explanation (Section 11.4.2) that separation of variables generally doesn't work for an initial value problem when the initial conditions are homogeneous. To illustrate why, consider once again the wave equation 11.4.2 for a string stretched from $x = 0$ to $x = L$. Assume the initial position and velocity of the string are both zero.

(a) If the boundary conditions are homogeneous, $y(0, t) = y(L, t) = 0$, what is the solution? (You can figure this one out without doing any math: just think about it.)

Note that this case is easy to solve but not very interesting or useful. For the rest of the problem we will therefore assume that the boundary conditions are not entirely homogeneous, and for simplicity we'll take the boundary conditions $y(0, t) = 0$, $y(L, t) = H$.

(b) Explain why you need to apply the initial conditions to each $T(t)$ function separately, and then apply the boundary conditions to the entire sum.

(c) Separate variables and write the resulting ODEs for $X(x)$ and $T(t)$.

(d) Solve the equation for $T(t)$ with the initial conditions given above. (Choose the sign of the separation constant to lead to sinusoidal solutions, not exponential or linear.) Using your solution, explain why separation of variables cannot be used to find any non-trivial solutions to this problem.

11.98 Solve Problem 11.97 using the heat equation (11.2.3) instead of the wave equation (11.4.2).

11.99 In quantum mechanics a particle is described by a "wavefunction" $\Psi(x, t)$ which obeys the Schrödinger equation:

$$-\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V(x)\Psi = i\hbar\frac{\partial\Psi}{\partial t}$$

(We are considering only one spatial dimension for simplicity.) This is really a whole family of PDEs, one for each possible potential function $V(x)$. Nonetheless we can make good progress without knowing anything about the potential function.

(a) Plug in a trial solution of the form $\Psi(x, t) = \psi(x)T(t)$ and separate variables. Call the separation constant E. (It turns out that each normal mode—called an "energy eigenstate" in quantum mechanics—represents the state of the particle with energy E.)

(b) Solve the ODE for $T(t)$. Describe in words the time dependence of an energy eigenstate.

Your work above applies to any one-dimensional Schrödinger equation problem; further work depends on the particular potential function. For the rest of the problem you'll consider a "particle in a box" that experiences no force inside the box but cannot move out of the box. In one dimension that means $V(x) = 0$ on $0 \le x \le L$ with the boundary conditions $\psi(0) = \psi(L) = 0$,

(c) Solve your separated equation using this $V(x)$ and boundary conditions to find all the allowable values of E—in other words find the energy levels of this system.

(d) Write the general solution for $\psi(x)$ as a sum over all individual solutions. Your answer will not explicitly involve E.

(e) Write the general solution for $\Psi(x, t)$. Use the values of E you found so that your answer has n in it but not E.

(f) Find $\Psi(x, t)$ for the initial condition $\Psi(x, 0) = \psi_0$ (a constant) for $L/4 \le x \le 3L/4$ and 0 elsewhere.

11.100 Exploration: Laplace's Equation on a Disk The steady-state temperature in an isolated region of space obeys Laplace's equation (11.2.5). In this problem you are going to solve for the steady-state temperature on a disk of radius a, where the temperature on the boundary of the disk is given by $u(a, \phi) = u_0 \sin(k\phi)$. Laplace's equation will be easiest to solve in polar coordinates, so you are looking for a function $u(\rho, \phi)$. (Feel free to try this problem in Cartesian coordinates. You'll feel great about it until you try to apply the boundary conditions.) In polar coordinates Laplace's equation can be written as:

$$\frac{\partial^2 u}{\partial\rho^2} + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial\phi^2} + \frac{1}{\rho}\frac{\partial u}{\partial\rho} = 0$$

(a) This problem has an "implicit," or unstated, boundary condition that $u(\phi + 2\pi) = u(\phi)$. Explain how we know this boundary condition must be followed even though it was never stated in the problem. Because of that implicit boundary condition the constant k in the boundary conditions cannot be just any real number. What values of k are allowed, and why?

(b) Plug in the initial guess $R(\rho)\Phi(\phi)$ and separate variables in Laplace's equation.

(c) Setting both sides of the separated equation equal to a constant P, find the general real solution to the

ODE for $\Phi(\phi)$ for the three cases $P > 0$, $P = 0$, and $P < 0$.

(d) Use the implicit boundary condition from Part (a) to determine which of the three solutions you found is the correct one. Now that you know what sign the separation constant must have, rename it either p^2 or $-p^2$. Use the period of 2π to constrain the possible values of p.

(e) Write the ODE for $R(\rho)$. Solve it by plugging in a guess of the form $R(\rho) = \rho^c$ and solve for the two possible values of c. Since the ODE is linear you can then combine these two solutions with arbitrary constants in front. Your answer should have p in it.

(f) There is another implicit condition: $R(0)$ has to be finite. Use that condition to show that one of the two arbitrary constants in your solution for $R(\rho)$ must be zero. (You can assume that $p > 0$.)

(g) Multiply your solutions for $\Phi(\phi)$ and $R(\rho)$, combining arbitrary constants as much as possible, and write the general solution $u(\rho, \phi)$ as an infinite series.

(h) Plug in the boundary condition $u(a, \phi) = u_0 \sin(k\phi)$ and use it to find the values of the arbitrary constants. (You could use the formula for the coefficients of a Fourier sine series but you can do it more simply by inspection.)

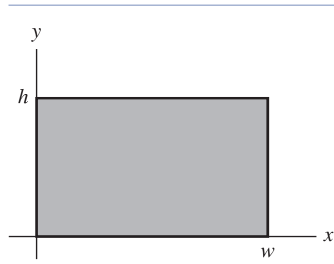
(i) Write the solution $u(\rho, \phi)$.

11.5 Separation of Variables—More than Two Variables

We have seen that with one independent variable you have an ordinary differential equation; with two independent variables you have a partial differential equation. What about three independent variables? (Or four or five or eleven?) Does each of those need its own chapter?

Fortunately, the process scales up. With each new variable you have to separate one more time, resulting in many ordinary differential equations. In the end your solution is expressed as a series over more than one variable.

11.5.1 Discovery Exercise: Separation of Variables—More than Two Variables



A thin rubber sheet is stretched on a rectangular frame. The sheet is glued to the frame; however, it is free to vibrate inside the frame. (This system functions as a rectangular drum; the more common circular drum will be taken up in Section 11.6.) For convenience we place our axes in the plane of the frame, with the lower-left-hand corner of the rectangle at the origin.

The motion of this sheet can be described by a function $z(x, y, t)$ which will obey the wave equation in two dimensions:

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{1}{v^2}\frac{\partial^2 z}{\partial t^2} \quad \textit{the wave equation in two dimensions} \tag{11.5.1}$$

The boundary condition is that $z = 0$ on all four edges of the rectangle. The initial conditions are $z(x, y, 0) = z_0(x, y)$, $\dot{z}(x, y, 0) = v_0(x, y)$.

1. Begin with a "guess" of the form $z(x, y, t) = X(x)Y(y)T(t)$. Substitute this expression into Equation 11.5.1.
2. Separate the variables so that the left side of the equation depends on x (not on y or t), and the right side depends on y and t (not on x). (This will require two steps.)
3. Explain why both sides of this equation must now equal a constant.

4. The function $X(x)$ must satisfy the conditions $X(0) = 0$ and $X(w) = 0$. Based on this restriction, is the separation constant positive or negative? Explain your reasoning. If positive, call it k^2; if negative, $-k^2$.
5. Solve the resulting differential equation for $X(x)$.
6. Plug in both boundary conditions on $X(x)$. The result should allow you to solve for one of the arbitrary constants and express the real number k in terms of an integer-valued n.

See Check Yourself #73 in Appendix L

7. You have another equation involving $Y(y)$ and $T(t)$ as well as k. Separate the variables in this equation so that the $Y(y)$ terms are on one side, and the $T(t)$ and k terms on the other. Both sides of this equation must equal a constant, but that constant is *not* the same as our previous constant.
8. The function $Y(y)$ must satisfy the conditions $Y(0) = 0$ and $Y(h) = 0$. Based on this restriction, is the *new* separation constant positive or negative? If positive, call it p^2; if negative, $-p^2$.
9. Solve the resulting differential equation for $Y(y)$.
10. Plug in both boundary conditions on $Y(y)$. The result should allow you to solve for one of the arbitrary constants and express the real number p in terms of an integer-valued m.

Important: p is not necessarily the same as k, and m is not necessarily the same as n. These new constants are unrelated to the old ones.

11. Solve the differential equation for $T(t)$. (The result will be expressed in terms of both k and p, or in terms of both n and m.)
12. Write the solution $X(x)Y(y)T(t)$. Combine arbitrary constants as much as possible. If your answer has any real-valued k or p constants, replace them with the integer-valued n and m constants by using the formulas you found from boundary conditions.
13. The general solution is a sum over *all possible* n- and m-values. For instance, there is one solution where $n = 3$ and $m = 5$, and another solution where $n = 12$ and $m = 5$, and so on. Write the double sum that represents the general solution to this equation.

See Check Yourself #74 in Appendix L

14. Write equations relating the initial conditions $z_0(x, y)$ and $v_0(x, y)$ to your arbitrary constants. These equations will involve sums. Solving them requires finding the coefficients of a double Fourier series, but for now it's enough to just write the equations.

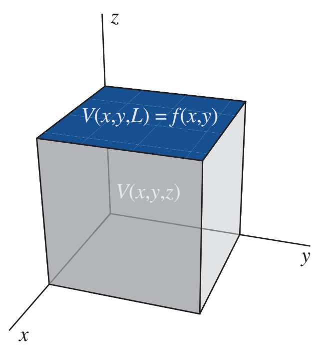
11.5.2 Explanation: Separation of Variables—More than Two Variables

When you have more than two independent variables you have to separate variables more than once to isolate them. Each time you separate variables you introduce a new arbitrary constant, resulting in multiple series.

The Problem

The steady-state electric potential in a region with no charged particles follows Laplace's equation $\nabla^2 V = 0$. In three-dimensional Cartesian coordinates, this equation can be written:

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \tag{11.5.2}$$



Consider a cubic box of side length L. Five sides of this box are grounded, which means they are held at potential $V = 0$. The sixth side has some arbitrary potential function that is also held fixed over time. There are no charged particles inside the box. What is the potential $V(x, y, z)$ at all points inside the box?

For convenience we choose our axes so that one corner of the box is at the origin and the non-grounded side is the top $(z = L)$. We can therefore describe the potential on that side by some function $V(x, y, L) = f(x, y)$.

Note that this problem has five homogeneous boundary conditions and one inhomogeneous boundary condition. As we discussed in the last section, we will apply the homogeneous conditions before we sum all the solutions, and the inhomogeneous one afterwards. In other words, we will treat the inhomogeneous boundary condition much as we treated the *initial* conditions for the vibrating string.

Separating the Variables, and Solving for the First One

Our guess is a fully separated function, involving three functions of one variable each.

$$V(x, y, z) = X(x)Y(y)Z(z)$$

We plug that guess into Laplace's equation (11.5.2) and divide both sides by $X(x)Y(y)Z(z)$ to separate variables.

$$X''(x)Y(y)Z(z) + X(x)Y''(y)Z(z) + X(x)Y(y)Z''(z) = 0 \quad \rightarrow \quad \frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} + \frac{Z''(z)}{Z(z)} = 0$$

We now bring one of these terms to the other side. We've chosen the y-term below, but we could have chosen x just as easily. It would be a bit tougher if we chose z (the variable with an inhomogeneous boundary condition); we'll explain why in a moment.

$$\frac{Y''(y)}{Y(y)} = -\frac{X''(x)}{X(x)} - \frac{Z''(z)}{Z(z)} \tag{11.5.3}$$

You may now object that the variables are not entirely separated. (How could they be, with three variables and only two sides of the equation?) But the essential relationship still holds: the left side of the equation depends only on y, and the right side of the equation depends only on x and z, so the only function both sides can equal is a constant.

When you set the left side of Equation 11.5.3 equal to a constant you get the same problem we solved for $X(x)$ in Section 11.4, so let's just briefly review where it goes.

- We are solving $Y''(y)/Y(y) =$ <*a constant*>, with the boundary conditions $Y(0) = Y(L) = 0$.
- A positive constant would lead to an exponential solution and a zero constant to a linear solution, neither of which could meet those boundary conditions.
- We therefore call the constant $-k^2$ and after a bit of algebra arrive at the solution.

$$Y(y) = A\sin(ky) \text{ where } k = \frac{n\pi}{L} \text{ where } n \text{ can be any positive integer}$$

Now you can see why we start by isolating a variable with homogeneous boundary conditions. The inhomogeneous condition for $Z(z)$ cannot be applied until after we build our series, so it would not have determined the sign of our separation constant.

Separating Again, and Solving for the Second Variable

The left side of Equation 11.5.3 equals $-k^2$, so the right side must equal the same constant. We rearrange terms in that equation to separate variables a second time.

$$-\frac{X''(x)}{X(x)} - \frac{Z''(z)}{Z(z)} = -k^2 \quad \rightarrow \quad \frac{X''(x)}{X(x)} = k^2 - \frac{Z''(z)}{Z(z)}$$

Both sides of this equation must equal a constant, and this constant must be negative. (Can you explain why?) However, this new constant does *not* have to equal our existing $-k^2$; the two are entirely independent. We will call the new constant $-p^2$.

$$\frac{X''(x)}{X(x)} = -p^2 \quad \text{and} \quad k^2 - \frac{Z''(z)}{Z(z)} = -p^2$$

The $X(x)$ differential equation and boundary conditions are the same as the $Y(y)$ problem that we solved above. The solution is therefore also the same, with one twist: p cannot also equal $n\pi/L$ because that would make the two constants the same. Instead we introduce a new variable m that must be a positive integer, but not necessarily the same integer as n.

$$X(x) = B\sin(px) \text{ where } p = \frac{m\pi}{L} \text{ where } m \text{ can be any positive integer}$$

The Variable with an Inhomogeneous Boundary Condition

We have one ordinary differential equation left.

$$k^2 - \frac{Z''(z)}{Z(z)} = -p^2$$

Rewriting this as $Z''(z) = \left(k^2 + p^2\right) Z(z)$, we see that the positive constant requires an exponential solution.

$$Z(z) = Ce^{\sqrt{k^2+p^2}z} + De^{-\sqrt{k^2+p^2}z}$$

Our fifth *homogeneous* boundary condition $Z(0) = 0$ leads to $C + D = 0$, so $D = -C$. We can therefore write:

$$Z(z) = C\left(e^{\sqrt{k^2+p^2}z} - e^{-\sqrt{k^2+p^2}z}\right) = C\left(e^{(\pi/L)\sqrt{n^2+m^2}z} - e^{-(\pi/L)\sqrt{n^2+m^2}z}\right)$$

(You can write $\sinh\left(\sqrt{k^2+p^2}\,z\right)$ instead of $e^{\sqrt{k^2+p^2}z} - e^{-\sqrt{k^2+p^2}z}$. It amounts to the same thing with a minor change in the arbitrary constant.)

When we solved for $X(x)$ and $Y(y)$, we applied their relevant boundary conditions as soon as we had solved the equations. Remember, however, that the boundary condition at $z = L$ is inhomogeneous; we therefore cannot apply it until we combine the three solutions and write a series.

The Solution so far, and a Series

When we multiply all three solutions, the three arbitrary constants combine into one.

$$X(x)Y(y)Z(z) = A\sin\left(\frac{m\pi}{L}x\right)\sin\left(\frac{n\pi}{L}y\right)\left(e^{(\pi/L)\sqrt{n^2+m^2}z} - e^{-(\pi/L)\sqrt{n^2+m^2}z}\right)$$

Remember that this is a solution for any positive integer n and any positive integer m. For instance, there is one solution with $n = 5$ and $m = 3$ and this solution can have any coefficient A. There is another solution with $n = 23$ and $m = 2$, which could have a different coefficient A. In order to sum all possible solutions, we therefore need a *double* sum.

$$V(x, y, z) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{mn} \sin\left(\frac{m\pi}{L}x\right) \sin\left(\frac{n\pi}{L}y\right) \left(e^{(\pi/L)\sqrt{n^2+m^2}z} - e^{-(\pi/L)\sqrt{n^2+m^2}z}\right) \tag{11.5.4}$$

We have introduced the subscript A_{mn} to indicate that for each choice of m and n there is a different free choice of A.

The Last Boundary Condition

Applying our last boundary condition $V(x, y, L) = f(x, y)$ to the entire series, we write:

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{mn} \sin\left(\frac{m\pi}{L}x\right) \sin\left(\frac{n\pi}{L}y\right) \left(e^{\pi\sqrt{n^2+m^2}} - e^{-\pi\sqrt{n^2+m^2}}\right) = f(x, y)$$

At this point in our previous section, building up the (one-variable) function on the right as a series of sines involved writing a Fourier Series. In this case we are building a two-variable function as a double Fourier Series. Once again the formula is in Appendix G.

$$A_{mn}\left(e^{\pi\sqrt{n^2+m^2}} - e^{-\pi\sqrt{n^2+m^2}}\right) = \frac{4}{L^2}\int_0^L \int_0^L f(x, y) \sin\left(\frac{m\pi}{L}x\right) \sin\left(\frac{n\pi}{L}y\right) dxdy \tag{11.5.5}$$

As before, if $f(x, y)$ is particularly simple you can sometimes find the coefficients A_{mn} explicitly. In other cases you can use numerical approaches. You'll work examples of each type for different boundary conditions in the problems.

EXAMPLE **The Two-Dimensional Heat Equation**

Problem:
Solve the two-dimensional heat equation:

$$\frac{\partial u}{\partial t} = \alpha\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

on the domain $0 \le x \le 1$, $0 \le y \le 1$, $t \ge 0$ subject to the boundary conditions

$$u(0, y, t) = u(1, y, t) = u(x, 0, t) = u(x, 1, t) = 0$$

and the initial condition

$$u(x, y, 0) = u_0(x, y).$$

Solution:
We begin by assuming a solution of the form:

$$u(x, y, t) = X(x)Y(y)T(t)$$

Plugging this into the differential equation yields:

$$X(x)Y(y)T'(t) = \alpha\left[X''(x)Y(y)T(t) + X(x)Y''(y)T(t)\right]$$

Dividing through by $X(x)Y(y)T(t)$, we have:

$$\frac{T'(t)}{T(t)} = \alpha\left[\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)}\right]$$

It might seem natural at this point to work with $T(t)$ first: it is easier by virtue of being first order, and it is already separated. But you always have to start with the variables that have homogeneous boundary conditions. As usual, we will handle the boundary conditions *before* building a series, and tackle the initial condition last. We must separate one of the spatial variables: we choose X quite arbitrarily.

$$\frac{T'(t)}{T(t)} - \alpha\frac{Y''(y)}{Y(y)} = \alpha\frac{X''(x)}{X(x)}$$

The separation constant must be negative. (Do you see why?) Calling it $-k^2$, we solve the equation and boundary conditions for X to find: $X(x) = A\sin\left(kx/\sqrt{\alpha}\right)$ where $k = n\pi\sqrt{\alpha}$ for $n = 1, 2, 3\ldots$

Meanwhile we separate the other equation.

$$\frac{T'(t)}{T(t)} - \alpha\frac{Y''(y)}{Y(y)} = -k^2 \quad \rightarrow \quad \frac{T'(t)}{T(t)} + k^2 = \alpha\frac{Y''(y)}{Y(y)}$$

Once again the separation constant must be negative: we will call it $-p^2$. The equation for Y looks just like the previous equation for X, yielding $Y(y) = B\sin\left(py/\sqrt{\alpha}\right)$ where $p = m\pi\sqrt{\alpha}$ for $m = 1, 2, 3\ldots$

Finally, we have $T'(t) = -\left(k^2 + p^2\right)T(t)$, so

$$T(t) = Ce^{-(k^2+p^2)t}$$

Writing a complete solution, collecting arbitrary constants, and summing, we have:

$$u(x, y, t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{mn}\sin(n\pi x)\sin(m\pi y)e^{-\pi^2\alpha(n^2+m^2)t}$$

Finally we are ready to plug in our inhomogeneous initial condition, which tells us that

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{mn}\sin(n\pi x)\sin(m\pi y) = u_0(x, y)$$

and therefore, once again using a formula from Appendix G:

$$A_{mn} = 4\int_0^1\int_0^1 u_0(x, y)\sin(n\pi x)\sin(m\pi y)dxdy \tag{11.5.6}$$

For example, suppose the initial temperature is 5 in the region $1/4 < x < 3/4, 1/4 < y < 3/4$ and 0 everywhere outside it. From Equation 11.5.6:

$$A_{mn} = 4\int_0^1\int_0^1 u_0(x, y)\sin(n\pi x)\sin(m\pi y)dxdy = 20\int_{1/4}^{3/4}\int_{1/4}^{3/4}\sin(n\pi x)\sin(m\pi y)dxdy$$

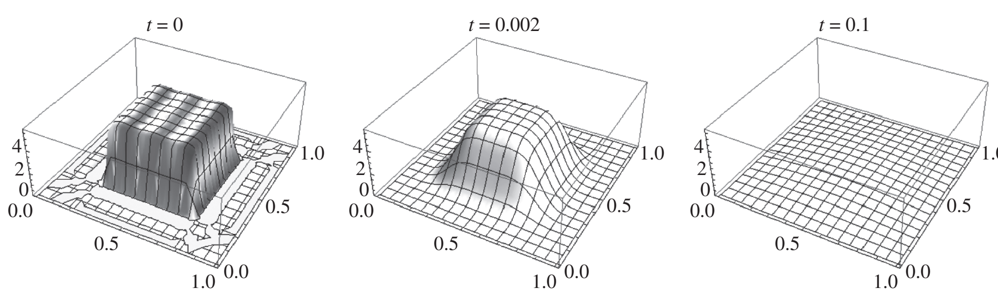
This integral is simple to evaluate, and after a bit of algebra gives

$$A_{mn} = \frac{80}{mn\pi^2} \sin\left(\frac{m\pi}{4}\right) \sin\left(\frac{m\pi}{2}\right) \sin\left(\frac{n\pi}{4}\right) \sin\left(\frac{n\pi}{2}\right)$$

Writing all the terms with $n, m < 4$ gives

$$u(x, y, t) = \frac{80}{\pi^2}\left(\frac{1}{2}\sin(\pi x)\sin(\pi y)e^{-2\pi^2\alpha t} - \frac{1}{6}\sin(\pi x)\sin(3\pi y)e^{-10\pi^2\alpha t}\right.$$
$$\left.-\frac{1}{6}\sin(3\pi x)\sin(\pi y)e^{-10\pi^2\alpha t} + \frac{1}{18}\sin(3\pi x)\sin(3\pi y)e^{-18\pi^2\alpha t} + \ldots\right)$$

If you plot the first few terms you can see that they represent a bump in the middle of the domain that flattens out over time due to the decaying exponentials. With computers, however, you can go much farther. The plots below show this series (with $\alpha = 1$) with all the terms up to $n, m = 100$ (10,000 terms in all, although many of them equal zero).



At $t = 0$ you can see that the double Fourier series we constructed accurately models the initial conditions. A short time later the temperature is starting to even out more, and at much later times it relaxes towards zero everywhere. (Do you see why we call $t = 0.1$ "much later"? See Problem 11.105.)

11.5.3 Problems: Separation of Variables—More than Two Variables

Problems 11.101–11.105 follow up on the example of Laplace's equation in a cubic region from the Explanation (Section 11.5.2).

11.101 The solution is written in terms of sines of x and y and exponentials of z. Does this mean that for any given boundary condition the solution $V(x)$ at a fixed y and z will look sinusoidal? If so, explain how you know. If not, explain what you can conclude about what $V(x)$ at fixed y and z will look like from this solution?

11.102 Solve Laplace's equation $\partial^2 V/\partial x^2 + \partial^2 V/\partial y^2 + \partial^2 V/\partial z^2 = 0$ in the cubic region $x, y, z \in [0, L]$ with $V(x, y, L) = V_0$ and $V = 0$ on the other five sides.

11.103 **(a)** Solve Laplace's equation in the cubic region $0 \le x, y, z \le L$ with $V(x, y, L) = \sin(\pi x/L)\sin\left(2\pi y/L\right) + \sin(2\pi x/L)\sin\left(\pi y/L\right)$ and $V = 0$ on the other five sides.

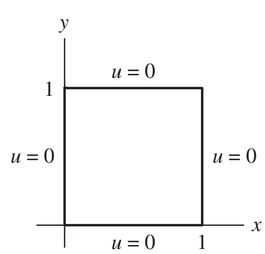
(b) Sketch $V(x, L/2, L)$ and $V(x, L/4, L)$, and $V(x, 0, L)$ as functions of x. How does changing the y-value affect the plot?

(c) At $z = L$ sketch how V depends on x for many values of y. You can do this by making an animation or a series of still images, but either way you should have enough to see if it follows the behavior you predicted in Part (b). How would your sketches have changed if you had used $z = L/2$ instead? (You should not need a computer to answer that last question.)

11.104 Solve Laplace's equation for $V(x, y, z)$ in the region $0 \le x, y, z \le L$ with $V(x, y, 0) = f(x, y)$ and $V = 0$ on the other five sides. (This is the same problem that was solved in the Explanation (Section 11.5.2) except that the side with non-zero potential is at $z = 0$ instead of $z = L$.)

11.105 The example on Page 584 ended with a series, which we plotted at several times. From those plots it's clear that for $\alpha = 1$ the initial temperature profile hadn't changed much by $t = 0.002$ but had mostly relaxed towards zero by $t = 0.1$. How could you have predicted this by looking at the series solution?

11.106 **Walk-Through: Separation of Variables—More Than Two Variables.** In this problem you will use separation of variables to solve the equation $\partial^2 u/\partial t^2 - \partial^2 u/\partial x^2 - \partial^2 u/\partial y^2 + u = 0$ subject to the boundary conditions $u(0, y, t) = u(1, y, t) = u(x, 0, t) = u(x, 1, t) = 0$.



(a) Begin by guessing a separable solution $u = X(x)Y(y)T(t)$. Plug this guess into the differential equation. Then divide both sides by $X(x)Y(y)T(t)$ and separate variables so that all the y and t dependence is on the left and the x dependence is on the right. Put the constant term on the left side. (You could put it on the right side, but the math would get a bit messier later on.)

(b) Set both sides of that equation equal to a constant. Given the boundary conditions above for u, what are the boundary conditions for $X(x)$? Use these boundary conditions to explain why the separation constant must be negative. You can therefore call it $-k^2$.

(c) Separate variables in the remaining ODE, putting the t-dependent terms and the constant terms on the left, and the y-dependent term on the right. Set both sides equal to a constant and explain why this separation constant also must be negative. Call it $-p^2$.

(d) Solve the equation for $X(x)$. Use the boundary condition at $x = 0$ to show that one of the arbitrary constants must be 0 and apply the boundary condition at $x = 1$ to find all the possible values for k. There will be an infinite number of them, but you should be able to write them in terms of a new constant m, which can be any positive integer.

(e) Solve the equation for $Y(y)$ and apply the boundary conditions to eliminate one arbitrary constant and to write the constant p in terms of a new integer n.

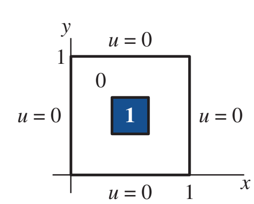
(f) Solve the equation for $T(t)$. Your answer should depend on m and n and should have two arbitrary constants.

(g) Multiply $X(x)$, $Y(y)$, and $T(t)$ to find the normal modes of this system. You should be able to combine your four arbitrary constants into two. Write the general solution $u(x, y, t)$ as a sum over these normal modes. Your arbitrary constants should include a subscript mn to indicate that they can take different values for each combination of m and n.

11.107 *[This problem depends on Problem 11.106.]* In this problem you will plug the initial conditions $u(x, y, 0) =$

$$\begin{cases} 1 & 1/3 < x < 2/3,\ 1/3 < y < 2/3 \\ 0 & \textit{elsewhere} \end{cases},$$

$\dfrac{\partial u}{\partial t}(x, y, 0) = 0$ into the solution you found to Problem 11.106.

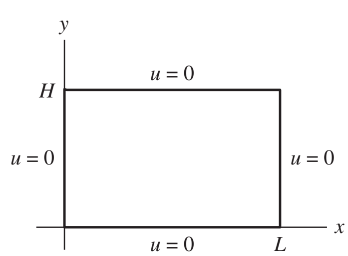


(a) Use the condition $\partial u/\partial t(x, y, 0) = 0$ to show that one of your arbitrary constants must equal zero.

(b) The remaining condition should give you an equation that looks like $u(x, y, 0) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} F_{mn} \sin(m\pi x)\sin(n\pi y)$ (with a different letter if you didn't use F_{mn} for the same arbitrary constant we did). This is a double Fourier sine series for the function $u(x, y, 0)$. Find the coefficients F_{mn}. The appropriate formula is in Appendix G.

(c) Have a computer calculate a partial sum of your solution including all terms up to $m, n = 20$ (400 terms in all) and plot it at a variety of times. Describe how the function u throughout the region is evolving over time.

Solve Problems 11.108–11.112 using separation of variables. For each problem use the boundary conditions $u(0, y, t) = u(L, y, t) = u(x, 0, t) = u(x, H, t) = 0$. When the initial conditions are given as arbitrary functions, write the solution as a series and write expressions for the coefficients in the series. When specific initial conditions are given, solve for the coefficients. It may help to first work through Problem 11.106 as a model.



11.108 $\partial u/\partial t = a^2(\partial^2 u/\partial x^2) + b^2(\partial^2 u/\partial y^2)$, $u(x, y, 0) = f(x, y)$

11.109 $\partial u/\partial t = a^2(\partial^2 u/\partial x^2) + b^2(\partial^2 u/\partial y^2)$, $u(x, y, 0) = \sin(2\pi x/L)\sin(3\pi y/H)$

11.110 $\partial u/\partial t + u = a^2(\partial^2 u/\partial x^2) + b^2(\partial^2 u/\partial y^2)$, $u(x, y, 0) = f(x, y)$

11.111 $\partial^2 u/\partial t^2 + u = \partial^2 u/\partial x^2 + \partial^2 u/\partial y^2$, $u(x, y, 0) = 0, \partial u/\partial t(x, y, 0) = g(x, y)$

11.112 $\partial^2 u/\partial t^2 + \partial u/\partial t - \partial^2 u/\partial x^2 - \partial^2 u/\partial y^2 + u = 0$, $u(x, y, 0) = 0, \partial u/\partial t(x, y, 0) = \sin(\pi x/L)\sin(2\pi y/H)$

For Problems 11.113–11.114 solve the 2D wave equation $\partial^2 z/\partial x^2 + \partial^2 z/\partial y^2 = (1/v^2)(\partial^2 z/\partial t^2)$ on the rectangle $x, y \in [0, L]$ with $z = 0$ on all four sides and initial conditions given below. If you have not done the Discovery Exercise (Section 11.5.1) you will need to begin by using separation of variables to solve the wave equation.

11.113 $z(x, y, 0) = \sin(\pi x/L)\sin\left(\pi y/L\right)$, $\dot{z}(x, y, 0) = 0$

11.114 $z(x, y, 0) = 0$,
$$\dot{z}(x, y, 0) = \begin{cases} c & L/3 < x < 2L/3, L/3 < y < 2L/3 \\ 0 & \textit{otherwise} \end{cases}$$

11.115 *[This problem depends on Problem 11.114.]* The equation you solved in Problem 11.114 might represent an oscillating square plate that was given a sudden blow in a region in the middle. This might represent a square drumhead hit by a square drumstick.[6] The solution you got, however, was a fairly complicated looking double sum.

(a) Have a computer plot the initial function $\dot{z}(x, y, 0)$ for the partial sum that goes up through $m = n = 5$, then again up through $m = n = 11$, and finally up through $m = n = 21$. As you add terms you should see the partial sums converging towards the shape of the initial conditions that you were solving for.

(b) Take several of the non-zero terms in the series (the individual terms, not the partial sums) and for each one use a computer to make an animation of the shape of the drumhead (z, not $\dot{z}$) evolving over time. You should see periodic behavior. You should include $(m, n) = (1, 1)$, $(m, n) = (1, 3)$, $(m, n) = (3, 3)$, and at least one other term. Describe how the behaviors of these normal modes are different from each other.

(c) Now make an animation of the partial sum that goes through $m = n = 21$. Describe the behavior. How is it similar to or different from the behavior of the individual terms you saw in the previous part?

(d) How would your answers to Parts (b) and (c) have looked different if we had used a different set of initial conditions?

Solve Problems 11.116–11.118 using separation of variables. Write the solution as a series and write expressions for the coefficients in the series, as we did for Laplace's equation in the Explanation (Section 11.5.2), Equations 11.5.4 and 11.5.5.

11.116 Solve the wave equation $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 + \partial^2 u/\partial z^2 = (1/v^2)(\partial^2 u/\partial t^2)$ in a 3D cube of side L with $u = 0$ on all six sides and initial conditions $u(x, y, z, 0) = f(x, y, z)$, $\dot{u}(x, y, z, 0) = 0$.

[6] If "square drumhead" sounds a bit artificial, don't worry. In Section 11.6 we'll solve the wave equation on a more conventional circular drumhead.

11.117 In the wave equation the parameter v is the sound speed, meaning the speed at which waves propagate in that medium. "Anisotropic" crystals have a different sound speed in different directions. Solve the anisotropic wave equation $\partial^2 u/\partial t^2 = v_x^2(\partial^2 u/\partial x^2) + v_y^2(\partial^2 u/\partial y^2)$ in a square box of side length L, with $u = 0$ on all four sides and initial conditions $u(x, y, 0) = 0$, $\dot{u}(x, y, 0) = g(x, y)$.

11.118 Solve the heat equation $\partial u/\partial t = \alpha\left(\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 + \partial^2 u/\partial z^2\right)$ in a 3D cube of side L with $u = 0$ on all six sides and initial condition $u(x, y, z, 0) = f(x, y, z)$.

11.6 Separation of Variables—Polar Coordinates and Bessel Functions

We have seen how separation of variables finds the "normal modes" of an equation. If the initial conditions happen to match the normal modes, the solution will evolve simply in time; if the initial conditions are more complicated, we build them as sums of normal modes.

In the examples we have seen so far, the normal modes have been sines and cosines. In general, the normal modes may be Bessel functions, Legendre polynomials, spherical harmonics, associated Laguerre polynomials, Hankel Functions, and many others. This section will provide a brief introduction to Bessel functions, and show how they arise in solutions to PDEs in polar and cylindrical coordinates. The next section will go through a similar process, showing how PDEs in spherical coordinates lead to Legendre polynomials. In both sections our main point will be that the normal modes may change, but the process of separating variables remains consistent.

11.6.1 Explanation: Bessel Functions—The Unjustified Essentials

The solutions to many important PDEs in polar and cylindrical coordinates involve "Bessel functions"—a class of functions that may be totally unfamiliar to you. Our presentation of Bessel functions comes in two parts. In this chapter we present all the properties you need to use Bessel functions in solving PDEs; in Chapter 12 we will show where those properties come from.

As an analogy, consider how little information about sines and cosines you have actually needed to make it through this chapter so far.

- The ordinary differential equation $d^2y/dx^2 + k^2y = 0$ has two real, linearly independent solutions, which are called $\sin(kx)$ and $\cos(kx)$. Because the equation is linear and homogeneous, its general solution is $A\sin(kx) + B\cos(kx)$.
- The function $\sin(kx)$ has zeros at $x = n\pi/k$ where n is any integer. The function $\cos(kx)$ has zeros at $(\pi/2 + n\pi)/k$. (These facts are important for matching boundary conditions.)
- Sines and cosines form a "complete basis," which is a fancy way of saying that you can build up almost any function $f(x)$ as a linear combination of sines and cosines (a Fourier series). All you need is the formula for the coefficients. (This information is important for matching initial conditions.)

As an introduction to trigonometry, those three bullet points are hopelessly inadequate. They don't say a word about how the sine and cosine functions are defined (SOHCAHTOA or the unit circle, for instance). They don't discuss why these functions are "orthogonal," and how you can use that fact to *derive* the coefficients of a Fourier series. The list above, without any further explanation, feels like a grab-bag of random facts without any real math. Nonetheless, it gives us enough to solve the PDEs that happen to have trigonometric normal modes.

Here's our point: you can look up a few key facts about a function you've never even heard of and then use that function to solve a PDE. Later you can do more research to better understand the solutions you have found.

A Differential Equation and its Solutions

Bessel's Equation and Its Solutions

The ordinary differential equation

$$x^2y'' + xy' + (k^2x^2 - p^2)y = 0 \tag{11.6.1}$$

has two real, linearly independent solutions, which are called "Bessel functions" and designated $J_p(kx)$ and $Y_p(kx)$. Because the equation is linear and homogeneous, its general solution is $AJ_p(kx) + BY_p(kx)$.

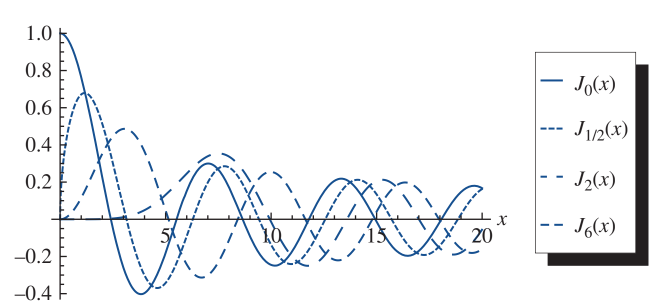
It's important to note that A and B are arbitrary constants (determined by initial conditions), while p and k are specific numbers that appear in the differential equation. p determines what functions solve the equation, and k stretches or compresses those functions. For instance:

- The ordinary differential equation $x^2y'' + xy' + (x^2 - 9)y = 0$ has two real, linearly independent solutions called $J_3(x)$ and $Y_3(x)$. Because the equation is linear and homogeneous, its general solution is $AJ_3(x) + BY_3(x)$.
- The equation $x^2y'' + xy' + \left(x^2 - 1/4\right)y = 0$ has general solution $AJ_{1/2}(x) + BY_{1/2}(x)$. The $J_{1/2}(x)$ that solves this equation is a completely different function from the $J_3(x)$ that solved the previous.
- The equation $x^2y'' + xy' + \left(25x^2 - 9\right)y = 0$ has solutions $J_3(5x)$ and $Y_3(5x)$. Of course $J_3(5x)$ is the same function as $J_3(x)$ but compressed horizontally.

We are not concerned here with negative values of p, although we may encounter fractional values.

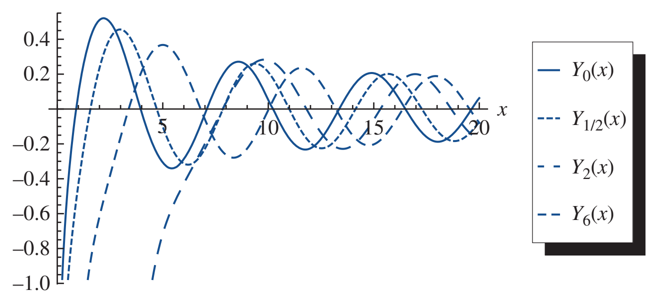
A Few Key Properties

The functions $J_p(x)$ are called "Bessel functions of the first kind," or sometimes just "Bessel functions." Here is a graph and a few of their key properties.



- $J_0(0) = 1$. For all non-zero p-values, $J_p(0) = 0$.
- There's no easy way to figure out the actual values of $J_p(x)$ by hand: you look them up in a table or use a mathematical software program, most of which have built-in routines for calculating Bessel functions. (The same statement can be made for sines and cosines. Do you know a better way to find sin 2?)
- All the $J_p(x)$ functions have an infinite number of zeros. We shall have more to say about these below.

The functions $Y_p(x)$ are called "Bessel functions of the second kind." Here is a graph and a few of their key properties.



- For all p-values, $\lim_{x\to 0^+} Y_p(x) = -\infty$. Therefore $Y_p(x)$ solutions are discarded whenever boundary conditions require a finite answer at $x = 0$.
- All the $Y_p(x)$ functions have an infinite number of zeros.
- Some textbooks refer to these as "Neumann functions," and still others as "Weber functions." Some use the notation $N_p(x)$ instead of $Y_p(x)$. (This may not technically qualify as a "key property" but we had to mention it somewhere.)

More about Those Zeros (Brought to You by the Letter "alpha")

The zeros of the Bessel functions are difficult to calculate, but they are listed in Bessel function tables and can be generated by mathematical software.

Because there is no simple formula for the zeros of a Bessel function, and because the zeros play an important role in many formulas, they are given their own symbols. By convention, $\alpha_{p,n}$ represents the nth positive zero of the Bessel function J_p. For instance, the first four positive zeros of the function J_3 are roughly 6.4, 9.8, 13.0, and 16.2, so we write $\alpha_{3,1} = 6.4$, $\alpha_{3,2} = 9.8$, and so on.

Fourier-Bessel Series Expansions

We start with a quick reminder of a series expansion you should already be familiar with. Given a function $f(x)$ defined on a finite interval $[0, a]$ with $f(0) = f(a) = 0$, you can write the function as a Fourier sine series.

$$f(x) = b_1 \sin\left(\frac{\pi}{a}x\right) + b_2 \sin\left(\frac{2\pi}{a}x\right) + b_3 \sin\left(\frac{3\pi}{a}x\right) + \dots$$

The terms in that series are all the function $\sin(kx)$ where the restriction $k = n\pi/a$ comes from the condition $f(a) = 0$ (since $\sin(n\pi) = 0$ for all integer n). The only other thing you need to know is the formula for the coefficients, $b_n = (2/a)\int_0^a f(x)\sin(n\pi x/a)dx$.

Moving to the less familiar, you can instead choose to represent the same function $f(x)$ as a series of (for instance) $J_7(kx)$ terms.

$$f(x) = A_1 J_7\left(\frac{\alpha_{7,1}}{a}x\right) + A_2 J_7\left(\frac{\alpha_{7,2}}{a}x\right) + A_3 J_7\left(\frac{\alpha_{7,3}}{a}x\right) + \dots$$

The terms in that series are all the function $J_7(kx)$ where the restriction $k = \alpha_{7,n}/a$ comes from the condition $f(a) = 0$ (since $J_7(\alpha_{7,n}) = 0$ for all n by definition). The only other thing you need is the formula for the coefficients.

Of course, there's nothing special about the number 7. You could build the same function as a series of $J_{5/3}(kx)$ functions if you wanted to. The allowable values of k would be different and the coefficients would be different; it would be a completely different series that adds up to the same function.

Fourier-Bessel Series

On the interval $0 \le x \le a$, we can expand a function into a series of Bessel functions by writing

$$f(x) = \sum_{n=1}^{\infty} A_n J_p\left(\frac{\alpha_{p,n}}{a}x\right)$$

- J_p is one of the Bessel functions. For instance, if $p = 3$, you are expanding $f(x)$ in terms of the Bessel function J_3. This creates a "Fourier-Bessel series expansion of order 3." We could expand the same function $f(x)$ into terms of J_5, which would give us a different series with different coefficients.
- $\alpha_{p,n}$ is defined in the section on "zeros" above.[7]
- The formula for the coefficients A_n is given in Appendix J (in the section on Bessel functions).

Do you see the importance of all this? Separation of variables previously led us to $d^2y/dx^2 + k^2y = 0$, which gave us the normal modes sin(kx) and cos(kx). We used the known zeros of those functions to match boundary conditions. And because we know the formula for the coefficients of a Fourier series, we were able to build arbitrary initial conditions as *series* of sines and cosines.

Now suppose that separation of variables on some new PDE gives us $J_5(kx)$ for our normal modes. We will need to match boundary conditions, which requires knowing the zeros: we now know them, or at least we have names for them ($\alpha_{5,1}$, $\alpha_{5,2}$ and so on) and can look them up. Then we will need to write the initial condition as a sum of normal modes: we now know the formula for the coefficients of a Fourier-Bessel expansion of order 5. Later in this section you'll see examples where we solve PDEs in this way.

Appendix J lists just the key facts given above—the ODE, the function that solves the ODE, the zeros of that function, and the coefficients of its series expansion—for Bessel functions and other functions that are normal modes of common PDEs. With those facts in hand, you can solve PDEs with a wide variety of normal modes. But, as we warned in the beginning, nothing in that appendix justifies those mathematical facts in any way. For the mathematical background and derivations, see Chapter 12.

Not-Quite-Bessel Functions

Many equations look a lot like Equation 11.6.1 but don't match it perfectly. There are two approaches you might try in such a case. The first is to let $u = x^2$ or $u = \sqrt{x}$ or $u = \cos x$ or some such and see if you end up with a perfect match. (Such substitutions are discussed in Chapter 10.) The second is to peak into Appendix J and see if you have a known variation of Bessel functions.

[7] Some textbooks define $\lambda_{p,n} = \alpha_{p,n}/a$ to make these equations look simpler.

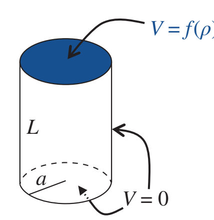
As an example, consider the following.

$$x^2y'' + xy' - (k^2x^2 + p^2)y = 0 \tag{11.6.2}$$

This is just like Equation 11.6.1 except for the sign of the k^2 term. You can turn this into Equation 11.6.1 with a change of variables, but you have to introduce complex numbers to do so. The end result, however, is a pair of functions that give you real answers for any real input x. The two linearly independent solutions to this equation are called "modified Bessel function," generally written $I_p(kx)$ and $K_p(kx)$. The important facts you need to know about these functions when you're solving PDEs are that $K_p(x)$ diverges at $x = 0$ (just as $Y_p(x)$ does), and that none of the modified Bessel functions have any zeroes at any point except $x = 0$. Those facts are all listed in Appendix J.

11.6.2 Discovery Exercise: Separation of Variables—Polar Coordinates and Bessel Functions

A vertical cylinder of radius a and length L has no charge inside. The cylinder wall and bottom are grounded: that is, held at zero potential. The potential at the top is given by the function $f(\rho)$. Find the potential inside the cylinder.



The potential in such a situation will follow Laplace's equation $\nabla^2 V = 0$. The system geometry suggests that we write this equation in cylindrical coordinates. The lack of any ϕ-dependence in the boundary conditions lends the problem azimuthal symmetry: that is, $\partial V/\partial\phi = 0$. We can therefore write Laplace's equation as:

$$\frac{\partial^2 V}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial V}{\partial \rho} + \frac{\partial^2 V}{\partial z^2} = 0 \tag{11.6.3}$$

with the boundary conditions:

$$V(\rho, 0) = 0, \quad V(a, z) = 0, \quad V(\rho, L) = f(\rho)$$

1. To solve this, assume a solution of the form $V(\rho, z) = R(\rho)Z(z)$. Plug this guess into Equation 11.6.3.
2. Separate the variables in the resulting equation so that all the ρ-dependence is on the left and the z-dependence on the right.

   See Check Yourself #75 in Appendix L

3. Let the separation constant equal zero.
   (a) Verify that the function $R(\rho) = A\ln(\rho) + B$ is the general solution for the resulting equation for $R(\rho)$. (You can verify this by simply plugging our solution into the ODE, of course. Alternatively, you can find the solution for yourself by letting $S(\rho) = R'(\rho)$ and solving the resulting first-order separable ODE for $S(\rho)$, and then integrating to find $R(\rho)$.)
   (b) We now note that within the cylinder, the potential V must always be finite. (This type of restriction is sometimes called an "implicit" boundary condition, and will be discussed further in the Explanation, Section 11.6.3.) What does this restriction say about our constants A and B?
   (c) Match your solution to the boundary condition $R(a) = 0$, and show that this reduces your solution to the trivial case $R(\rho) = 0$ everywhere. We conclude that, for any boundary condition at the top other than $f(\rho) = 0$, the separation constant is not zero.

4. Returning to your equation in Part 2, let the separation constant equal k^2.
   (a) Solve for $R(\rho)$ by matching with either Equation 11.6.1 or 11.6.2 for the proper choice of p.
   (b) Use the implicit boundary condition that $R(0)$ is finite to eliminate one of the two solutions.
   (c) Explain why the remaining solution cannot match the boundary condition $R(a) = 0$. You can do this by looking up the properties of the function in your solution or you can plot the remaining solution, choosing arbitrary positive values for k and the arbitrary constant, and explain from this plot why the solution cannot match $R(a) = 0$.
5. Returning to your equation in Part 2, let the separation constant equal $-k^2$.
   (a) Solve for $R(\rho)$ by matching with either Equation 11.6.1 or 11.6.2 for the proper choice of p.
   (b) Use the implicit boundary condition that $R(0)$ is finite to eliminate one of the two solutions.
   (c) Use the boundary condition $R(a) = 0$ to restrict the possible values of k.
6. Based on your results from Parts 3–5 you should have concluded that the separation constant must be negative to match the boundary conditions on $R(\rho)$. Calling the separation constant $-k^2$, solve the equation for $Z(z)$, using the boundary condition $Z(0) = 0$.
7. Write $V(\rho, z)$ as an infinite series.

See Check Yourself #76 in Appendix L

The process you have just gone through is the same as in the previous section, but this ODE led to Bessel functions instead of sines and cosines. Bessel functions, like sines and cosines, form a "complete basis"—you can use them in series to build almost any function by the right choice of coefficients. In the problems you will choose appropriate coefficients to match a variety of boundary conditions for the upper surface in this exercise. You will also see how Bessel functions come up in a variety of different physical situations.

11.6.3 Explanation: Separation of Variables—Polar Coordinates and Bessel Functions

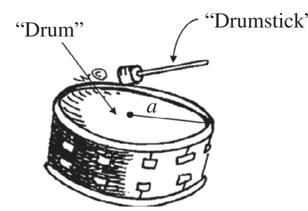
We're going to solve the wave equation by using separation of variables again. Like our example in Section 11.5, we will have three independent variables instead of two: this will require separating variables twice, resulting in a double series. Nothing new so far.

However, because we begin this time in polar coordinates, the ordinary differential equation that results from separating variables will *not* be $d^2y/dx^2 + k^2y = 0$. A different equation will lead us to a different solution: Bessel functions, rather than sines and cosines, will be our normal modes.

Despite this important difference, it's important to note how much this problem has in common with the previous problems we have worked through. Matching boundary conditions, summing up the remaining solutions, and then matching initial conditions all follow the same basic pattern. And once again we can sum our normal modes to represent any initial state of the system. Once you are comfortable with this general outline, you can solve a surprisingly wide variety of partial differential equations, even though the details—especially the forms of the normal modes—may vary from one problem to the next.

The Problem

Consider a simple drum: a thin rubber sheet stretched along a circular frame of radius a. If the sheet is struck with some object—let's call it a "drumstick"—it will vibrate according to the wave equation.



The wave equation can be written as $\nabla^2 z = (1/v^2)(\partial^2 z/\partial t^2)$, where z is vertical displacement. In rectangular coordinates this equation becomes $\partial^2 z/\partial x^2 + \partial^2 z/\partial y^2 = (1/v^2)(\partial^2 z/\partial t^2)$, but using x and y to specify boundary conditions on a circle is an ugly business. Instead we will use the Laplacian in polar coordinates (Appendix F), which makes our wave equation:

$$\frac{\partial^2 z}{\partial t^2} = v^2 \left(\frac{\partial^2 z}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial z}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 z}{\partial \phi^2} \right) \tag{11.6.4}$$

You can probably figure out for yourself that the first boundary condition, the one that led us to polar coordinates, is $z(a, \phi, t) = 0$. This is an "explicit" boundary condition, meaning it comes from a restriction that is explicitly stated in the problem. (The drumhead is attached to the frame.) There are also two "implicit" boundary conditions, restrictions that are imposed by the geometry of the situation.

- In many cases, certainly including this one, z has to be finite throughout the region. This includes the center of the circle, so z cannot blow up at $\rho = 0$. (This is going to be more important than it may sound.)
- The point $\rho = 2, \phi = \pi/3$ is the *same physical place* as $\rho = 2, \phi = 7\pi/3$, so it must give rise to the same z-value. More generally, the function must be periodic such that $z(\rho, \phi, t)$ always equals $z(\rho, \phi + 2\pi, t)$.

These conditions apply because we are using polar coordinates to solve a problem on a disk around the origin, not because of the details of this particular scenario. If you find that your initial and boundary conditions aren't sufficient to determine your arbitrary constants, you should always check whether you missed some implicit boundary conditions.

The problem also needs to specify as initial conditions both $z(\rho, \phi, 0)$ and $\partial z/\partial t(\rho, \phi, 0)$. After we solve the problem with its boundary conditions, we'll look at a couple of possible initial conditions.

Separation of Variables

We begin by guessing a solution of the form

$$z(\rho, \phi, t) = R(\rho)\Phi(\phi)T(t)$$

Substituting this expression directly into the polar wave equation (11.6.4) yields:

$$R(\rho)\Phi(\phi)T''(t) = v^2 \left(R''(\rho)\Phi(\phi)T(t) + \frac{1}{\rho}R'(\rho)\Phi(\phi)T(t) + \frac{1}{\rho^2}R(\rho)\Phi''(\phi)T(t) \right)$$

As always, we need to start by isolating a variable with homogeneous boundary conditions. ρ has the obviously homogeneous condition $R(a) = 0$. But the boundary condition for ϕ, periodicity, is also homogeneous. (You may think of "homogeneous" as a fancy way of saying "something is zero," but that isn't quite true: a homogeneous equation or condition is one for which *any linear combination of solutions is itself a solution.* A sum of functions with the same period is itself periodic, so $z(\rho, \phi, t) = z(\rho, \phi + 2\pi, t)$ is a homogeneous boundary condition.) We'll isolate $\Phi(\phi)$ because its differential equation is simpler.

$$\frac{\rho^2}{v^2}\frac{T''(t)}{T(t)} - \rho^2\frac{R''(\rho)}{R(\rho)} - \rho\frac{R'(\rho)}{R(\rho)} = \frac{\Phi''(\phi)}{\Phi(\phi)}$$

The left side of the equation depends only on ρ and t, and the right side depends only on ϕ, so both sides must equal a constant. A positive constant would lead to an exponential $\Phi(\phi)$ function, which couldn't be periodic, so we call the constant $-p^2$ which must be zero or negative. (Our choice of the letter p is motivated by a sneaky foreknowledge of where this particular constant is going to show up in Bessel's equation, but of course any other letter would serve just as well.)

$$\frac{\rho^2}{v^2}\frac{T''(t)}{T(t)} - \rho^2\frac{R''(\rho)}{R(\rho)} - \rho\frac{R'(\rho)}{R(\rho)} = -p^2$$

$$\frac{\Phi''(\phi)}{\Phi(\phi)} = -p^2 \tag{11.6.5}$$

The ϕ equation is now fully separated, and we will return to it later, but first we want to finish separating variables. This turns the remaining equation into:

$$\frac{1}{v^2}\frac{T''(t)}{T(t)} = \frac{R''(\rho)}{R(\rho)} + \frac{1}{\rho}\frac{R'(\rho)}{R(\rho)} - \frac{p^2}{\rho^2}$$

Both sides must equal a constant, and this new constant is not related to p. We logically expect $T(t)$ to oscillate, rather than growing exponentially or linearly—this is a vibrating drumhead, after all!—so we will choose a negative separation constant again and call it $-k^2$. (If we missed this physical argument we would find that a positive constant would be unable to match the boundary conditions.)

$$\frac{1}{v^2}\frac{T''(t)}{T(t)} = -k^2 \tag{11.6.6}$$

$$\frac{R''(\rho)}{R(\rho)} + \frac{1}{\rho}\frac{R'(\rho)}{R(\rho)} - \frac{p^2}{\rho^2} = -k^2 \tag{11.6.7}$$

One partial differential equation has been replaced by three ordinary differential equations: two familiar, and one possibly less so.

The Variables with Homogeneous Boundary Conditions

The solution to Equation 11.6.5 is:

$$\Phi(\phi) = A\sin(p\phi) + B\cos(p\phi) \tag{11.6.8}$$

The condition that $\Phi(\phi)$ must repeat itself every 2π requires that p be an integer. Negative values of p would be redundant, so we need only consider $p = 0, 1, 2, \ldots$.

Moving on to $R(\rho)$, Equation 11.6.7 can be written as:

$$\rho^2 R''(\rho) + \rho R'(\rho) + \left(k^2\rho^2 - p^2\right) R(\rho) = 0 \tag{11.6.9}$$

You can plug that equation into a computer or look it up in Appendix J. (The latter is a wonderful resource if we do say so ourselves, and we hope you will familiarize yourself with it.) Instead we will note that it matches Equation 11.6.1 in Section 11.6.1 and write down the answer we gave there.

$$R(\rho) = AJ_p(k\rho) + BY_p(k\rho) \tag{11.6.10}$$

Next we apply the boundary conditions on $R(\rho)$. We begin by recalling that all functions $Y_p(x)$ blow up as $x \to 0$. Because our function must be finite at the center of the drum we discard such solutions, leaving only the J_p functions.

Our other boundary condition on ρ was that $R(a) = 0$. Using $\alpha_{p,n}$ to represent the nth zero of J_p we see that this condition is satisfied if $ka = \alpha_{p,n}$.

$$R(\rho) = AJ_p\left(\frac{\alpha_{p,n}}{a}\rho\right)$$

Remember that $J_1(x)$ is one specific function—you could look it up, or graph it with a computer. $J_2(x)$ is a different function, and so on. Each of these functions has an infinite number of zeros. When we write $\alpha_{3,7}$ we mean "the seventh zero of the function $J_3(x)$." The index n, which identifies one of these zeros by count, can be any positive integer.

The Last Equation, and a Series

The equation $(1/v^2)(T''(t)/T(t)) = -k^2$ becomes $T''(t) = -k^2v^2T(t)$ which we can solve by inspection:

$$T(t) = C\sin(kvt) + D\cos(kvt)$$

When we introduced p and k they could each have been, so far as we knew, any real number. The boundary conditions have left us with possibilities that are still infinite, but *discrete: p* must be an integer, and ka must be a zero of J_p. So we can write the solution as a series over all possible n- and p-values. As we do so we add the subscript pn to our arbitrary constants, indicating that they may be different for each term. We also absorb the constant A into our other arbitrary constants.

$$z = \sum_{p=0}^{\infty}\sum_{n=1}^{\infty} J_p\left(\frac{\alpha_{p,n}}{a}\rho\right)\left[C_{pn}\sin\left(\frac{\alpha_{p,n}}{a}vt\right) + D_{pn}\cos\left(\frac{\alpha_{p,n}}{a}vt\right)\right]\left[E_{pn}\sin(p\phi) + F_{pn}\cos(p\phi)\right] \tag{11.6.11}$$

If you aren't paying extremely close attention, those subscripts are going to throw you. $\alpha_{p,n}$ is the nth zero of the function J_p: you can look up $\alpha_{3,5}$ in a table of Bessel function zeros. (It's roughly 19.4.) On the other hand, C_{pn} is one of our arbitrary constants, which we will choose to meet the initial conditions of our specific problem. C_{35} is the constant that we will be multiplying specifically by $\sin\left(\alpha_{3,5}vt/a\right)$.

Can you see why p starts at 0 and n starts at 1? Plug $p = 0$ into Equation 11.6.8 and you get a valid solution: $\Phi(\phi)$ equals a constant. But the first zero of a Bessel function is conventionally labeled $n = 1$, so only positive values of n make sense.

Initial Conditions

We haven't said much about specific initial conditions up to this point, because our goal is to show how you can match *any* reasonable initial conditions in such a situation.

The initial conditions would be some given functions $z_0(\rho, \phi)$ and $\dot{z}_0(\rho, \phi)$. From the solution above:

$$z_0 = \sum_{p=0}^{\infty}\sum_{n=1}^{\infty} D_{pn}J_p\left(\frac{\alpha_{p,n}}{a}\rho\right)\left[E_{pn}\sin(p\phi) + F_{pn}\cos(p\phi)\right]$$

$$\dot{z}_0 = \sum_{p=0}^{\infty}\sum_{n=1}^{\infty}\left(\frac{\alpha_{p,n}}{a}v\right) C_{pn}J_p\left(\frac{\alpha_{p,n}}{a}\rho\right)\left[E_{pn}\sin(p\phi) + F_{pn}\cos(p\phi)\right]$$

We can consider the possible initial conditions in four categories.

1. If our initial conditions specify that z_0 and $\dot{z}_0$ are both zero, then we are left with the "trivial" solution $z = 0$ and the drumhead never moves.

2. If $z_0 = 0$ and $\dot{z}_0 \neq 0$, then $D_{pn} = 0$. Since D_{pn} was the coefficient of $\cos\left(\alpha_{p,n} vt/a\right)$ in our solution, we are left with only the sinusoidal time oscillation, $C_{pn} \sin\left(\alpha_{p,n} vt/a\right)$.
3. If $z_0 \neq 0$ and $\dot{z}_0 = 0$, then $C_{pn} = 0$. This has the opposite effect on our solution, leaving only the cosine term for the time oscillation.
4. The most complicated case, $z_0 \neq 0$ and $\dot{z}_0 \neq 0$, requires a special trick. We find one solution where $z_0 = 0$ and $\dot{z}_0 \neq 0$ (case two above), and another solution where $z_0 \neq 0$ and $\dot{z}_0 = 0$ (case three above). When we *add* the two solutions, the result still solves our differential equation (because it was homogeneous), and also has the correct z_0 and the correct $\dot{z}_0$.

In *all* cases we rely on the "completeness" of our normal modes—which is a fancy way of saying, we rely on the ability of trig functions and Bessel functions to combine in series to create any desired initial function that we wish to match.

Below we solve two different problems of the second type, where $z_0 = 0$ and $\dot{z}_0 \neq 0$. In the problems you'll work examples of the third type. You'll also try your hand at a few problems of the fourth type, but we will take those up more generally in Section 11.8.

First Sample Initial Condition: A Symmetric Blow

Solve for the motion of a circular drumhead that is struck in the center with a circular mallet, so that

$$z_0 = 0, \quad \dot{z}_0 = \begin{cases} s & \rho < \rho_0 \\ 0 & \rho_0 < \rho < a \end{cases}$$

As noted above, we can immediately say that because z_0 is zero, $D_{pn} = 0$.

Next we note that both the problem and the initial conditions have "azimuthal symmetry": the solution will have no ϕ dependence. How can a function that is multiplied by $E_{pn} \sin(p\phi) + F_{pn} \cos(p\phi)$ possibly wind up independent of ϕ? The answer is, if $E_{pn} = 0$ for all values of p and n, and $F_{pn} = 0$ for all values except $p = 0$. Then $\sum_{p=0}^{\infty} E_{pn} \sin(p\phi) + F_{pn} \cos(p\phi)$ simply becomes the constant F_{0n}.

We can now absorb that constant term into the constant C_{0n}. Since the only value of p left in our sum is $p = 0$ we replace p with zero everywhere in the equation and drop it from our subscripts:

$$\dot{z}_0 = \sum_{n=1}^{\infty} \left(\frac{\alpha_{0,n}}{a} v\right) C_n J_0\left(\frac{\alpha_{0,n}}{a}\rho\right)$$

As always, we now build up our initial condition from the series by choosing the appropriate coefficients. If our normal modes were sines and cosines, we would be building a Fourier series. In this case we are building a Fourier-Bessel series, but the principle is the same, and the formula in Appendix J allows us to simply plug in and find the answer:

$$\left(\frac{\alpha_{0,n}}{a} v\right) C_n = \frac{2}{a^2 J_1^2(\alpha_{0,n})} \int_0^a \dot{z}_0(\rho) J_0\left(\frac{\alpha_{0,n}}{a}\rho\right) \rho\, d\rho = \frac{2s}{a^2 J_1^2(\alpha_{0,n})} \int_0^{\rho_0} J_0\left(\frac{\alpha_{0,n}}{a}\rho\right) \rho\, d\rho$$

$$C_n = \frac{2s}{a v \alpha_{0,n} J_1^2(\alpha_{0,n})} \int_0^{\rho_0} J_0\left(\frac{\alpha_{0,n}}{a}\rho\right) \rho\, d\rho$$

This integral can be looked up in a table or plugged into a computer, with the result

$$C_n = \frac{2s}{a v \alpha_{0,n} J_1^2(\alpha_{0,n})} \frac{\rho_0 a J_1\left(\frac{\alpha_{0,n}}{a}\rho_0\right)}{\alpha_{0,n}} = \frac{2s\rho_0}{v\alpha_{0,n}^2} \frac{J_1\left(\frac{\alpha_{0,n}}{a}\rho_0\right)}{J_1^2(\alpha_{0,n})}$$

Plugging this into the series expansion for z gives the final answer we've been looking for:

$$z = \frac{2s\rho_0}{v} \sum_{n=1}^{\infty} \frac{J_1\left(\frac{\alpha_{0,n}}{a}\rho_0\right)}{\alpha_{0,n}^2 J_1^2(\alpha_{0,n})} J_0\left(\frac{\alpha_{0,n}}{a}\rho\right) \sin\left(\frac{\alpha_{0,n}}{a}vt\right) \tag{11.6.12}$$

That hideous-looking answer actually tells us a great deal about the behavior of the drumhead. The constants outside the sum set the overall scale of the vibrations. The big fraction inside the sum is a number for each value of n that tells us the relative importance of each normal mode. For $\rho_0 = a/3$ the first few coefficients are roughly 0.23, 0.16, and 0.069, and they only get smaller from there. So only the first two modes contribute significantly. The first and most important term represents the shape on the left of Figure 11.5 oscillating up and down. The second term is a slightly more complicated shape. Higher order terms will have more wiggles (i.e. more critical points between $\rho = 0$ and $\rho = a$).

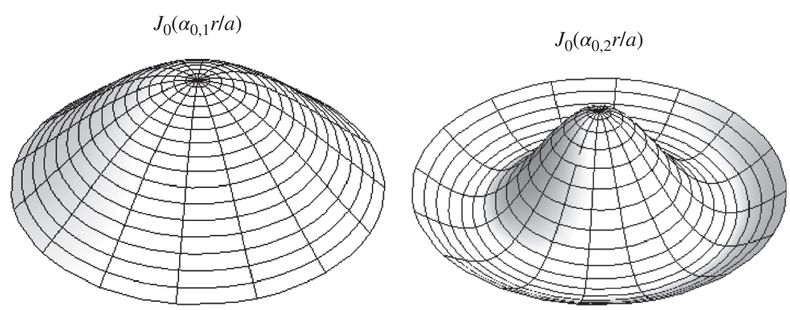
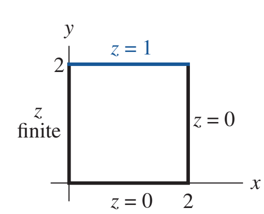


Figure 11.5

EXAMPLE Partial Differential Equation with Bessel Function Normal Modes

Problem:
Solve the equation:



$$4x\frac{\partial^2 z}{\partial x^2} + 4\frac{\partial z}{\partial x} + \frac{\partial^2 z}{\partial y^2} - \frac{9}{x}z = 0$$

on the domain $0 \le x \le 2$, $0 \le y \le 2$ subject to the boundary conditions $z(2, y) = z(x, 0) = 0$ and $z(x, 2) = 1$ and the requirement that $z(0, y)$ is finite.

Solution:
Writing $z(x, y) = X(x)Y(y)$ leads us to:

$$4xX''(x)Y(y) + 4X'(x)Y(y) + X(x)Y''(y) - \frac{9}{x}X(x)Y(y) = 0$$

Divide both sides by $X(x)Y(y)$ and separate the variables:

$$4x\frac{X''(x)}{X(x)} + 4\frac{X'(x)}{X(x)} - \frac{9}{x} = -\frac{Y''(y)}{Y(y)}$$

In the problems you will show that we cannot match the boundary conditions with a positive or zero separation constant, so we call the constant $-k^2$ and we get:

$$4xX''(x) + 4X'(x) + \left(k^2 - \frac{9}{x}\right)X(x) = 0 \tag{11.6.13}$$

$$Y''(y) = k^2 Y(y) \tag{11.6.14}$$

We begin with $X(x)$ because its boundary conditions are homogenous. Equation 11.6.13 is unfamiliar enough that you may just want to pop it into a computer. (If you want to solve it by hand, look for the variable substitution that turns it into Equation 11.6.1.)

$$X(x) = AJ_3\left(k\sqrt{x}\right) + BY_3\left(k\sqrt{x}\right)$$

As before, we discard the Y_p solutions because they blow up at $x = 0$. The second boundary condition, $X(2) = 0$, means that $k = \alpha_{3,n}/\sqrt{2}$.

Turning to the other equation we get $Y(y) = Ce^{ky} + De^{-ky}$. The boundary condition $Y(0) = 0$ gives $C = -D$ so $Y(y) = C\left(e^{ky} - e^{-ky}\right)$ (or equivalently $Y(y) = E\sinh(ky)$). Putting the solutions together, and absorbing one arbitrary constant into another, we get

$$z(x,y) = \sum_{n=1}^{\infty} A_n\left(e^{\alpha_{3,n}y/\sqrt{2}} - e^{-\alpha_{3,n}y/\sqrt{2}}\right)J_3\left(\frac{\alpha_{3,n}}{\sqrt{2}}\sqrt{x}\right)$$

Finally, the inhomogeneous boundary condition $z(x,2) = 1$ gives

$$1 = \sum_{n=1}^{\infty} A_n\left(e^{\alpha_{3,n}\sqrt{2}} - e^{-\alpha_{3,n}\sqrt{2}}\right)J_3\left(\frac{\alpha_{3,n}}{\sqrt{2}}\sqrt{x}\right)$$

This is a Fourier-Bessel series, but the argument of the Bessel function is $\sqrt{x}$ instead of x. So we define $u = \sqrt{x}$ and write this.

$$1 = \sum_{n=1}^{\infty} A_n\left(e^{\alpha_{3,n}\sqrt{2}} - e^{-\alpha_{3,n}\sqrt{2}}\right)J_3\left(\frac{\alpha_{3,n}}{\sqrt{2}}u\right)$$

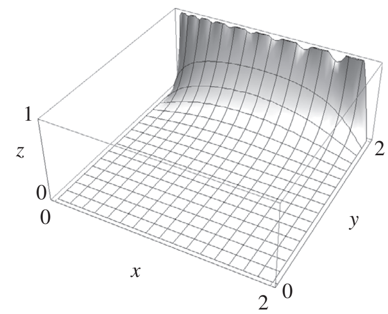
Then we look up the equation for Fourier-Bessel coefficients in Appendix J.

$$A_n\left(e^{\alpha_{3,n}\sqrt{2}} - e^{-\alpha_{3,n}\sqrt{2}}\right) = \frac{1}{J_4^2(\alpha_{3,n})}\int_0^{\sqrt{2}} J_3\left(\frac{\alpha_{3,n}}{\sqrt{2}}u\right)u\,du$$

The analytic solution to that integral is no more enlightening to look at than the integral itself, so we simply leave it as is:

$$z(x,y) = \sum_{n=1}^{\infty}\frac{1}{J_4^2(\alpha_{3,n})}\left(\int_0^{\sqrt{2}} J_3\left(\frac{\alpha_{3,n}}{\sqrt{2}}u\right)u\,du\right)\frac{e^{\alpha_{3,n}y/\sqrt{2}} - e^{-\alpha_{3,n}y/\sqrt{2}}}{e^{\alpha_{3,n}\sqrt{2}} - e^{-\alpha_{3,n}\sqrt{2}}}J_3\left(\frac{\alpha_{3,n}}{\sqrt{2}}\sqrt{x}\right)$$

You can plug that whole mess into a computer, as is, and plot the 20^{th} partial sum of the series. It matches the homogeneous boundary conditions perfectly, matches the inhomogeneous one well, and smoothly interpolates between them everywhere else.



Second Sample Initial Condition: An Asymmetric Blow

You might want to skip this part, work some problems with PDEs that lead to Bessel functions, and then come back here. This will show you how to handle multivariate series with different normal modes: in this example, trig functions of one variable and Bessel functions of another.

Solve for the motion of a circular drumhead that is struck by a mallet at an angle, so it only hits on one side.

$$z_0 = 0, \quad \dot{z}_0 = \begin{cases} s & \rho < \rho_0, 0 < \phi < \pi \\ 0 & otherwise \end{cases}$$

Once again the condition $z_0 = 0$ leads to $D_{pn} = 0$. This time, however, there is ϕ dependence, so we can't throw out as much of our equation.

$$\dot{z}_0(\rho,\phi) = \sum_{p=0}^{\infty}\sum_{n=1}^{\infty}\left(\frac{\alpha_{p,n}}{a}v\right)J_p\left(\frac{\alpha_{p,n}}{a}\rho\right)\left[E_{pn}\sin(p\phi)+F_{pn}\cos(p\phi)\right]$$

(We've absorbed C_{pn} into the other constants of integration.) In Chapter 9 we found double series expansions based entirely on sines and cosines. The principle is the same here, but we are now expanding over both trig and Bessel functions. First note that you can rewrite this as:

$$\begin{aligned}\dot{z}_0(\rho,\phi) = &\sum_{n=1}^{\infty}\left(\frac{\alpha_{0,n}}{a}v\right)J_0\left(\frac{\alpha_{0,n}}{a}\rho\right)F_{0n} + \sum_{p=1}^{\infty}\left\{\sum_{n=1}^{\infty}\left(\frac{\alpha_{p,n}}{a}v\right)J_p\left(\frac{\alpha_{p,n}}{a}\rho\right)F_{pn}\right\}\cos(p\phi)\\ &+\sum_{p=1}^{\infty}\left\{\sum_{n=1}^{\infty}\left(\frac{\alpha_{p,n}}{a}v\right)J_p\left(\frac{\alpha_{p,n}}{a}\rho\right)E_{pn}\right\}\sin(p\phi)\end{aligned}$$

This is now in the form of a Fourier series, where:

$$\sum_{n=1}^{\infty}\left(\frac{\alpha_{0,n}}{a}v\right)J_0\left(\frac{\alpha_{0,n}}{a}\rho\right)F_{0n} = a_0 = \frac{1}{2\pi}\int_0^{2\pi}\dot{z}_0(\rho,\phi)d\phi = \begin{cases}\frac{s}{2} & \rho<\rho_0\\ 0 & \rho>\rho_0\end{cases}$$

$$\sum_{n=1}^{\infty}\left(\frac{\alpha_{p,n}}{a}v\right)J_p\left(\frac{\alpha_{p,n}}{a}\rho\right)F_{pn} = a_p = \frac{1}{\pi}\int_0^{2\pi}\dot{z}_0(\rho,\phi)\cos\left(p\phi\right)d\phi = 0$$

$$\sum_{n=1}^{\infty}\left(\frac{\alpha_{p,n}}{a}v\right)J_p\left(\frac{\alpha_{p,n}}{a}\rho\right)E_{pn} = b_p = \frac{1}{\pi}\int_0^{2\pi}\dot{z}_0(\rho,\phi)\sin\left(p\phi\right)d\phi = \begin{cases}\frac{2s}{p} & \rho<\rho_0,\ p\ odd\\ 0 & otherwise\end{cases}$$

Now we can turn this around, and view the left side of each of these equations as a Fourier-Bessel series expansion of the function on the right side. We see that $F_{pn} = 0$ for all $p > 0$ and $E_{pn} = 0$ for even values of p. To find the other coefficients we return to Appendix J for the coefficients of a Fourier-Bessel series expansion.

$$\left(\frac{\alpha_{0,n}}{a}v\right)F_{0n} = \frac{2}{a^2 J_1^2(\alpha_{0,n})}\int_0^a \left\{\begin{array}{ll} \frac{s}{2} & \rho < \rho_0 \\ 0 & \rho > \rho_0 \end{array}\right\} J_0\left(\frac{\alpha_{0,n}}{a}\rho\right)\rho\, d\rho = \frac{\rho_0 s}{a\alpha_{0,n}J_1^2(\alpha_{0,n})}J_1\left(\frac{\alpha_{0,n}}{a}\rho_0\right)$$

$$\rightarrow F_{0n} = \frac{\rho_0 s}{v\alpha_{0,n}^2 J_1^2(\alpha_{0,n})}J_1\left(\frac{\alpha_{0,n}}{a}\rho_0\right)$$

$$\left(\frac{\alpha_{p,n}}{a}v\right)E_{pn} = \frac{2}{a^2 J_{p+1}^2(\alpha_{p,n})}\int_0^a \left\{\begin{array}{ll} \frac{2s}{p} & \rho < \rho_0 \\ 0 & \rho > \rho_0 \end{array}\right\} J_p\left(\frac{\alpha_{p,n}}{a}\rho\right)\rho\, d\rho = \frac{4s}{pa^2 J_1^2(\alpha_{p,n})}\int_0^{\rho_0} J_p\left(\frac{\alpha_{p,n}}{a}\rho\right)\rho\, d\rho$$

$$\rightarrow E_{pn} = \frac{4s}{pav\alpha_{p,n}J_1^2(\alpha_{p,n})}\int_0^{\rho_0} J_p\left(\frac{\alpha_{p,n}}{a}\rho\right)\rho\, d\rho \quad \text{odd } p \text{ only}$$

The integral in the last equation can be evaluated for a general p, but the result is an ugly expression involving hypergeometric functions, so it's better to leave it as is and evaluate it for specific values of p when you are calculating partial sums. Putting all of this together, the solution is:

$$z = \sum_{n=1}^{\infty}\left\{\frac{\rho_0 s}{v\alpha_{o,n}^2 J_1^2(\alpha_{0,n})}J_1\left(\frac{\alpha_{0,n}}{a}\rho\right)J_0\left(\frac{\alpha_{0,n}}{a}\rho\right)\sin\left(\frac{\alpha_{0,n}}{a}vt\right)\right. \tag{11.6.15}$$
$$\left. + \sum_{p=1}^{\infty}\left[\frac{4s}{pav\alpha_{p,n}J_1^2(\alpha_{p,n})}\left(\int_0^{\rho_0} J_p\left(\frac{\alpha_{p,n}}{a}\rho\right)\rho\, d\rho\right)J_p\left(\frac{\alpha_{p,n}}{a}\rho\right)\sin\left(\frac{\alpha_{p,n}}{a}vt\right)\sin(p\phi)\right] \quad \text{odd } p \text{ only}\right\}$$

Even if you're still reading at this point, you're probably skimming past the equations thinking "I'll take their word for it." But we'd like to draw your attention back to that last equation, because it's not as bad as it looks at first blush. The first term in curly braces is just the solution we found in the previous example. The dominant mode is shown on the left in Figure 11.5, the next mode is on the right in that figure, and higher order terms with more wiggles have much smaller amplitude. The new feature in this solution is the sum over p. Each of these terms is again a Bessel function that oscillates in time, but now multiplied by $\sin(p\phi)$, so it oscillates as you move around the disk. One such mode is shown in Figure 11.6.

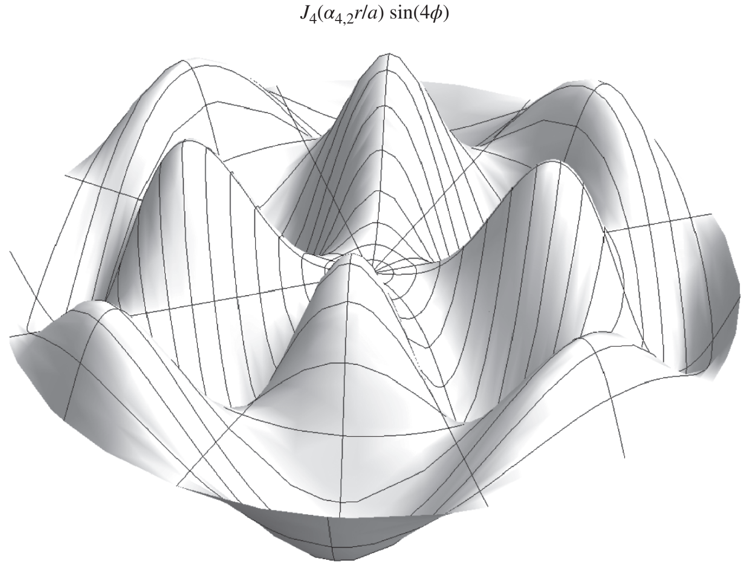


Figure 11.6

The higher the values of n and p the more wiggles a mode will have, but the smaller its amplitude will be. It's precisely because it can support so many different oscillatory modes, each with its own frequency, that a drum can make such a rich sound. At the same time, the frequency of the dominant mode determines the pitch you associate with a particular drum.

Stepping Back

When you read through a long, ugly derivation—and in this book, they don't get much longer and uglier than the "circular drum" above—it's easy to get lost in the details and miss the big picture. Here is everything we went through.

1. We were given the differential equation (the wave equation), the boundary conditions (the tacked-down edge of the drum), and a couple of sample initial conditions (first the symmetric blow, and then the asymmetric). In addition to the explicit boundary conditions, we recognized that the geometry of the problem led to some *implicit* boundary conditions. Without them, we would have found in step 3 that we didn't have enough conditions to determine our arbitrary constants.
2. Recognizing the circular nature of the boundary conditions, we wrote the wave equation in polar coordinates.
3. We separated variables, writing the solution as $z(\rho, \phi, t) = R(\rho)\Phi(\phi)T(t)$, but otherwise proceeding exactly as before: plug in, separate (twice, because there are three variables), solve the ODEs, apply homogeneous boundary conditions, build a series. Not all of this is easy, but it should mostly be familiar by now; the only new part was that one of the ODEs led us to Bessel functions.
4. To match initial conditions, we used a Fourier-Bessel series expansion. This step takes advantage of the fact that Bessel functions, like sines and cosines, form a "complete basis": you can sum them to build up almost any function.
5. Finally—possibly the most intimidating step, but also possibly the most important!—we *interpreted* our answer. We looked at the coefficients to see how quickly they were dropping, so we could focus on only the first few terms of the series. We used computers to draw the normal modes: in the problems you will go farther with computers, and actually view the motion. (Remember that it is almost never possible to explicitly sum an infinite series, but—assuming it converges—you can always approximate it with partial sums.)

It's important to get used to working with a wide variety of normal modes; trig and Bessel functions are just the beginning, but the process always remains the same. It's also important to get used to big, ugly equations that you can't solve or visualize without the aid of a computer. Mathematical software is as indispensable to the modern scientist or engineer as slide rules were to a previous generation.

11.6.4 Problems: Separation of Variables—Polar Coordinates and Bessel Functions

The differential equations in Problems 11.119–11.121 can be converted to the form of Equation 11.6.1 by an appropriate change of variables. For each one do the variable substitution to get the right equation, solve, and then substitute back to get the answer in terms of x. In the first problem the correct variable substitution is given for you; in the others you may have to do some trial and error.

11.119 $x^2(d^2y/dx^2) + x(dy/dx) + \left(4x^4 - 16\right) y = 0$. Use the substitution $u = x^2$.

11.120 $x^2(d^2y/dx^2) + x(dy/dx) + \left(x^6 - 9\right) y = 0$.

11.121 $d^2y/dx^2 + e^{2x}y = 0$. (*Hint*: to properly match Bessel's Equation, you need to end up with u^2 in front of the y term.)

11.122 Plot the first five Bessel functions $J_0(x) - J_4(x)$ from $x = 0$ to $x = 10$.

11.123 The Bessel functions are defined as

$$J_p(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(p+m+1)} \left(\frac{x}{2}\right)^{2m+p}$$

Plot $J_1(x)$ and the first 10 partial sums of this series expansion from $x = 0$ to $x = 10$ and show that the partial sums are converging to the Bessel function. (You may need to look up the gamma function $\Gamma(x)$ in the online help for your mathematical software to see how to enter it.)

11.124 Repeat Problem 11.123 for $J_{1/2}(x)$.

For Problems 11.125–11.127 expand the given function in a Fourier-Bessel series expansion of the given order. You should use a computer to evaluate the integrals, either analytically or numerically. Then plot the first ten partial sums of the expansion on the same plot as the original function so you can see the partial sums converging to the function. Be sure to clearly mark the plot of the function in some way (a different color, a thicker line) so it's clear on the plot which one it is.

11.125 $f(x) = 1$, $0 < x < 1$, $p = 1$

11.126 $f(x) = x$, $0 < x < \pi$, $p = 0$

11.127 $f(x) = \sin x$, $0 < x < \pi$, $p = 1/2$

Problems 11.128–11.131 refer to a vibrating drumhead (a circle of radius a) with a fixed edge: $z(a, \phi, t) = 0$. In the Explanation (Section 11.6.3) we showed that the solution to the wave equation for this system is Equation 11.6.11. The arbitrary constants in that general solution are determined by the initial conditions. For each set of initial conditions write the complete solution $z(\rho, \phi, t)$. In some cases your answers will contain integrals that cannot be evaluated analytically.

11.128 $z(\rho, \phi, 0) = a - \rho$, $\dot{z}(\rho, \phi, 0) = 0$. *This represents pulling the drumhead up by a string attached to its center and then letting go.*

11.129 $z(\rho, \phi, 0) = 0$, $\dot{z}(\rho, \phi, 0) = c \sin(3\pi\rho/a)$

11.130 $z(\rho, \phi, 0) = 0$,
$$\dot{z}(\rho, \phi, 0) = \begin{cases} c & \rho < a/2 \\ 0 & a/2 < \rho < a \end{cases}$$

11.131 $z(\rho, \phi, 0) = (a - \rho)\sin(\phi)$, $\dot{z}(\rho, \phi, 0) = 0$

11.132 In the Explanation (Section 11.6.3), we found the Solution 11.6.12 for an oscillating drumhead that was given a sudden blow in a region in the middle. This might represent a drumhead hit by a drumstick. The solution we got, however, was a fairly complicated looking sum. In this problem you will make plots of this solution; you will need to choose some values for all of the constants in the problem.

(a) Have a computer plot the initial function $\dot{z}(r, 0)$ for the first 20 partial sums of this series. As you add terms you should see the partial sums converging towards the shape of the initial conditions that you were solving for.

(b) Take the first three terms in the series (the individual terms, not the partial sums) and for each one use a computer to make an animation of the shape of the drumhead (z, not $\dot{z}$) evolving over time. You should see periodic behavior. Describe how the behavior of these three normal modes is different from each other.

(c) Now make an animation (or just a sequence of plots at different times) of the 20th partial sum of the solutions. Describe the behavior. How is it similar to or different from the behavior of the individual terms you saw in the previous part? If you've done Problem 11.115 from Section 11.5 compare these results to the animation you got for the "square drumhead." Can you see any qualitative differences?

11.133 Equation 11.6.15 gives the solution for the vibrations of a drumhead struck with an asymmetric blow. Make an animation of this solution showing that motion using the partial sum that goes up to $n = 10$, $p = 11$.

11.134 In the Explanation (Section 11.6.3), we found the Solution 11.6.12 for an oscillating drumhead that was given a sudden blow in a region in the middle and we discussed how the different modes oscillating at different frequencies correspond to pitches produced by the drum. To find the frequency of a mode you have to look at the time dependence and recall that frequency is one over period.

(a) For a rubber drum with a sound speed of $v = 100$m/s and a radius of $a = 0.3$m find the frequency of the dominant mode. Look up what note this corresponds to. Then find the notes corresponding to the next two modes. (The modes other than the dominant one are called "overtones.")

(b) Find the dominant pitch for a rubber drum with sound speed $v =$ 100m/s and a radius of 1.0m.

(c) By stretching the drum more tightly you can increase the sound speed. Find the dominant pitch for a more tightly stretched rubber drum with $v = 200$m/s and a radius of $a = 0.3$m.

(d) Explain in your own words why these different drums sound so different. How would you design a drum if you wanted it to make a low, booming sound?

11.135 Walk-Through: Differential Equation with Bessel Normal Modes. In this problem you will solve the partial differential equation $\partial y/\partial t - x^{3/2}(\partial^2 y/\partial x^2) - x^{1/2}(\partial y/\partial x) = 0$ subject to the boundary condition $y(1, t) = 0$ and the requirement that $y(0, t)$ be finite.

(a) Begin by guessing a separable solution $y = X(x)T(t)$. Plug this guess into the differential equation. Then divide both sides by $X(x)T(t)$ and separate variables.

(b) Find the general solution to the resulting ODE for $X(x)$ three times: with a positive separation constant k^2, a negative separation constant $-k^2$, and a zero separation constant. For each case you can solve the ODE by hand using the variable substitution $u = x^{1/4}$ or you can use a computer. Show which one of your solutions can match the boundary conditions without requiring $X(x) = 0$.

(c) Apply the condition that $y(0, t)$ is finite to show that one of the arbitrary constants in your solution from Part (b) must be 0.

(d) Apply the boundary condition $y(1, t) = 0$ to find all the possible values for k. There will be an infinite number of them, but you should be able to write them in terms of a new constant n that can be any positive integer. Writing k in terms of n will involve $\alpha_{p,n}$, the zeros of the Bessel functions.

(e) Solve the ODE for $T(t)$, expressing your answer in terms of n.

(f) Multiply $X(x)$ times $T(t)$ to find the normal modes of this system. You should be able to combine your two arbitrary constants into one. Write the general solution $y(x, t)$ as a sum over these normal modes. Your arbitrary constant should include a subscript n to indicate that they can take different values for each value of n.

(g) Use the initial condition $y(x, 0) = \sin(\pi x)$ to find the arbitrary constants in your solution, using the equations for a Fourier-Bessel series in Appendix J. Your answer will be in the form of an integral that you will not be able to evaluate. *Hint*: You may have to define a new variable to get the resulting equation to look like the usual form of a Fourier-Bessel series.

11.136 *[This problem depends on Problem 11.135.]*

(a) Have a computer calculate the 10^{th} partial sum of your solution to Problem 11.135 Part (g). Describe how the function $y(x)$ is evolving over time.

(b) You should have found that the function at $x = 0$ starts at zero, then rises slightly, and then asymptotically approaches zero again. Explain why it does that. *Hint*: think about why all the normal modes cancel out at that point initially, and what is happening to each of them over time.

In Problems 11.137–11.140 you will be given a PDE and a set of boundary and initial conditions.

(a) Solve the PDE with the given boundary conditions using separation of variables. You may solve the ODEs you get by hand or with a computer. The solution to the PDE should be an infinite series with undetermined coefficients.

(b) Plug in the given initial condition. The result should be a Fourier-Bessel series. Write an equation for the coefficients in your series using the equations in Appendix J. This equation will involve an integral that you may not be able to evaluate.

(c) Have a computer evaluate the integrals in Part (b) either analytically or numerically to calculate the 20^{th} partial sum of your series solution and either plot the result at several times or make a 3D plot of $y(x, t)$. Describe how the function is evolving over time.

It may help to first work through Problem 11.135 as a model.

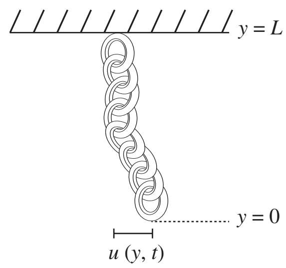
11.137 $\dfrac{\partial y}{\partial t} = \dfrac{\partial^2 y}{\partial x^2} + \dfrac{1}{x}\dfrac{\partial y}{\partial x} - \dfrac{y}{x^2}$, $y(3, t) = 0$, $y(x, 0) = \begin{cases} 1 & 1 \le x \le 2 \\ 0 & \textit{elsewhere} \end{cases}$. Assume y is finite for $0 \le x \le 3$.

11.138 $\dfrac{\partial^2 y}{\partial t^2} = \dfrac{\partial^2 y}{\partial x^2} + \dfrac{1}{x}\dfrac{\partial y}{\partial x} - \dfrac{y}{x^2}$, $y(3,t) = 0$, $y(x,0) = 0$, $\dfrac{\partial y}{\partial t}(x,0) = \begin{cases} 1 & 1 \le x \le 2 \\ 0 & \textit{elsewhere} \end{cases}$. Assume y is finite for $0 \le x \le 3$.

11.139 $\partial z/\partial t - \partial^2 z/\partial x^2 - (1/x)(\partial z/\partial x) - (1/x^2)(\partial^2 z/\partial y^2) = 0$, $z(1,y,t) = z(x,0,t) = z(x,1,t) = 0$, $z(x,y,0) = J_{2\pi}(\alpha_{2\pi,3}x)\sin(2\pi y)$. Assume z is finite throughout $0 \le x \le 1$.

11.140 $\sin^2 x \cos^2 x(\partial z/\partial t) - \sin^2 x(\partial^2 z/\partial x^2) - \tan x(\partial z/\partial x) + 4(\cos^2 x)z = 0$, $z(\pi/2,t) = 0$, $z(x,0) = \cos x$. Assume z is finite throughout $0 \le x \le \pi/2$. After you separate variables you will use the substitution $u = \sin x$ to turn an unfamiliar equation into Bessel's equation. (If your separation constant is the wrong sign you will get modified Bessel functions, which cannot meet the boundary conditions.)

11.141 Daniel Bernoulli first discovered Bessel functions in 1732 while working on solutions to the "hanging chain problem." A chain is suspended at $y = L$ and hangs freely with the bottom just reaching the point $y = 0$. The sideways motions of the chain $u(y,t)$ are described by the equation $\partial^2 u/\partial t^2 = g\left(y(\partial^2 u/\partial y^2) + \partial u/\partial y\right)$.



(a) Separate variables and solve the equation for $T(t)$. Choose the sign of the separation constant that gives you oscillatory solutions.

(b) The equation for $Y(y)$ can be turned into Bessel's equation with a substitution. You can start with $u = cy^q$ and find what values of c and q work, but we'll save you some algebra and tell you the correct substitution is $u = cy^{1/2}$. Plug that in and find the value of c needed to turn your $Y(y)$ equation into Bessel's equation.

(c) Solve the equation for $Y(u)$ and plug the substitution you found back in to get a solution for $Y(y)$ subject to the boundary condition $u(L,t) = 0$ and the condition that Y remain finite in the range $0 \le y \le L$. You should find that your solutions are Bessel functions and that you can restrict the possible values of the separation constant.

(d) Write the solution $u(y,t)$ as an infinite series and use the initial conditions $u(y,0) = f(y)$, $\dot{u}(y,0) = h(y)$ to find the coefficients in this series.

11.142 *[This problem depends on Problem 11.141.]* In this problem you'll use the solution you derived in Problem 11.141 to model the motion of a hanging chain. For this problem you can take $g = 9.8\ \text{m/s}^2$, $L = 1\text{m}$.

(a) Calculate the first five coefficients of the series you derived for $u(y,t)$ in Problem 11.141 using the initial conditions $u(y,0) = d - (d/L)y$, $\dot{u}(y,0) = 0$ where $d = .5\text{m}$ You can do this analytically by hand, use a computer to find it analytically, or use a computer to do it numerically. However you do it, though, you should get numbers for the five coefficients.

(b) Using the fifth partial sum to approximate $u(y,t)$ make an animation showing $u(y)$ at different times or a 3D plot showing $u(y,t)$ at times ranging from $t = 0$ to $t = 5$. Does the behavior look reasonable for a hanging chain?

11.143 Solve the heat equation (11.2.3) on a circular disk of radius a with the temperature of the edge held at zero and initial condition $u(\rho,\phi,0) = T_0 J_2(\alpha_{2,3}\rho/a)\sin(2\phi)$. (You will need to use the formula for the Laplacian in polar coordinates. Because α_{mn} is going to show up in the solution you should use D instead of α in the heat equation.) How long will it take for the point $\rho = a/2$, $\phi = \pi/2$ to drop to half its original temperature?

11.144 In the Explanation (Section 11.6.3), we solved for a vibrating drumhead with generic initial conditions, and then plugged in specific initial conditions with and without azimuthal symmetry. If you know from the beginning of your problem that you have azimuthal symmetry you can eliminate the ϕ dependence from the differential equation before solving. In that case the drumhead will obey the equation

$\partial^2 z/\partial t^2 = v^2\left(\partial^2 z/\partial \rho^2 + (1/\rho)(\partial z/\partial \rho)\right)$. Solve this equation to find $z(\rho, t)$ with a fixed edge $z(a, t) = 0$ and arbitrary initial conditions $z(\rho, 0) = f(\rho)$, $\dot{z}(\rho, 0) = g(\rho)$.

11.145 *[This problem depends on Problem 11.144.]* Solve for the displacement $z(\rho, t)$ of a drum with initial displacement $z(r, 0) = a - \rho$ and initial velocity $\dot{z}(\rho, 0) = c$.

11.7 Separation of Variables—Spherical Coordinates and Legendre Polynomials

Just as we saw that some differential equations in polar coordinates led to normal modes in the form of Bessel functions, we will see here that some differential equations in spherical coordinates lead to normal modes in the form of Legendre polynomials. Once again we will emphasize that you do not have to learn a new process or become an expert on a new function; the process is the same, and you can look up the information you need about the functions as they come up.

We will also discuss another function—spherical harmonics—that can be used as a shorthand for functions of θ and ϕ.

11.7.1 Explanation: Separation of Variables—Spherical Coordinates and Legendre Polynomials

We can summarize almost everything we have done so far as a four-step process.

1. Separate variables to turn one *partial* differential equation into several *ordinary* differential equations.
2. Solve the ordinary differential equations. The product of these solutions is one solution to the original equation: a special solution called a *normal mode*.
3. Match any homogeneous boundary conditions.
4. Sum all the solutions and match the inhomogeneous boundary or initial conditions. This requires that the normal modes form a "complete basis" so you can sum them to meet any given conditions.

Step 1—and, to a large extent, step 4—are much the same from one problem to the next. The two middle steps, on the other hand, depend on the differential equations you end up with. Mathematicians have been studying and cataloguing solutions to ordinary differential equations for centuries. We have seen solutions in the forms of trig functions and Bessel functions. In this section we will see another important form, Legendre polynomials. There are many more.

So how do you solve problems when each new differential equation might require a function you've never seen? One approach is to look each differential equation up in a table such as Appendix J, being ready with a variable substitution or two if the equations don't quite match. Equivalently, you can type your differential equation into the computer and see what it comes up with. We used the "table" approach in Section 11.6 and we will use a computer here. Our point is not to suggest that polar coordinates require a by-hand approach and spherical coordinates are somehow more suitable to a computer; we could just as easily have done it the other way. Our real point is that you need to be ready to use either approach as the situation demands. And in either case, the real skill demanded of you is working with an unfamiliar function once you get it.

Mathematically, both approaches may leave you unsatisfied. Both tell you that the general solution to $x^2y'' + xy' + (49x^2 - 36)y = 0$ is $AJ_6(7x) + BY_6(7x)$, but neither one tells you where this solution comes from. We will address that important question in Chapter 12 for Bessel functions, Legendre polynomials, and some other important examples. Our focus in this chapter is *using* those ODE solutions to solve PDEs.

The Problem

A spherical shell of radius a surrounds a region of space with no charge; therefore, inside the sphere, the potential obeys Laplace's equation 11.2.5. On the surface of the shell, the given boundary condition is a potential function $V(a,\theta) = V_0(\theta)$. Find the potential inside the sphere.

Laplace's equation in spherical coordinates is derived in Chapter 8:

$$\frac{\partial^2 V}{\partial r^2} + \frac{2}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\left(\frac{\partial^2 V}{\partial \theta^2} + \frac{\cos(\theta)}{\sin(\theta)}\frac{\partial V}{\partial \theta} + \frac{1}{\sin^2(\theta)}\frac{\partial^2 V}{\partial \phi^2}\right) = 0$$

Many problems, including this one, have "azimuthal symmetry": the answer will not depend on the angle ϕ. If V has no ϕ-dependency then $\partial V/\partial \phi = 0$, reducing the equation to:

$$\frac{\partial^2 V}{\partial r^2} + \frac{2}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\left(\frac{\partial^2 V}{\partial \theta^2} + \frac{\cos(\theta)}{\sin(\theta)}\frac{\partial V}{\partial \theta}\right) = 0$$

where $0 \le r \le a$ and $0 \le \theta \le \pi$. (We will solve this problem without azimuthal symmetry in the next section.)

As we have done before we notice a key "implicit" boundary condition, which is that the potential function must not blow up anywhere inside the sphere.

Separating the Variables

We begin by assuming a function of the form $V(r,\theta) = R(r)\Theta(\theta)$. We've left the next steps to you (Problem 11.157) but choosing k as the separation constant you should end up with:

$$r^2 R''(r) + 2rR'(r) - kR(r) = 0 \tag{11.7.1}$$

$$\Theta''(\theta) + \frac{\cos(\theta)}{\sin(\theta)}\Theta'(\theta) + k\Theta(\theta) = 0 \tag{11.7.2}$$

Solving for $\Theta(\theta)$

Equation 11.7.2 does not readily evoke any of our standard differential equations. We could find a convenient variable substitution to make the equation look like one of the standard forms in Appendix J as we did in the previous section. Instead, just to highlight another important approach, we're going to pop the equation into Mathematica. (You could just as easily use another program like Matlab or Maple.)

```
In[1]:= DSolve[f''[θ] + (Cos[θ]/Sin[θ]) f'[θ] + k f[θ] == 0, f[θ], θ]
Out[1]:= {{f[θ] → C[1] LegendreP[1/2 (-1 + √(1 + 4k)), Cos[θ]] +
          C[2] LegendreQ[1/2 (-1 + √(1 + 4k)), Cos[θ]]}}
```

Oh no, it's a whole new kind of function that we haven't encountered yet in this chapter! Don't panic: the main point of this section, and one of the main points of this whole chapter, is that you can attack this kind of problem the same way no matter what function you find for the normal modes.

So you look up “LegendreP” and “LegendreQ” in the Mathematica online help, and you find that these are “Legendre polynomials of the first and second kind,” respectively. If Mathematica writes “LegendreP(3,x),” the online help tells you, standard notation would be $P_3(x)$. So the above solution can be written as:

$$\Theta(\theta) = AP_l(\cos\theta) + BQ_l(\cos\theta) \text{ where } l = \frac{1}{2}\left(-1 + \sqrt{1+4k}\right)$$

What else do you need? You need to know a few properties of Legendre polynomials in order to match the boundary conditions (implicit and explicit), and later you'll need to use a Legendre polynomial expansion to match initial conditions, just as we did earlier with trig functions and Bessel functions. So take a moment to look up Legrendre polynomials in Appendix J: all the information we need for this problem is right there.

In this case, the only boundary condition on θ is that the function should be bounded everywhere. Since the argument of P_l and Q_l in this solution is $\cos\theta$, the Legendre polynomials need to be bounded in the domain $[-1, 1]$. That's not true for any of the Q_l functions, and it's only true for P_l when l is a non-negative integer, so we can write the complete set of solutions as

$$\Theta(\theta) = AP_l(\cos\theta), \quad l = 0, 1, 2, 3, \ldots$$

From the definition of l above we can solve for k to find $k = l(l+1)$. In other words, the only values of k for which this equation has a bounded solution are 0, 2, 6, 12, 20, etc.

Solving for $R(r)$

Remember that when you separate variables each side equals the same constant, so we can substitute $k = l(l+1)$ into Equation 11.7.1 to give

$$r^2R''(r) + 2rR'(r) - l(l+1)R(r) = 0$$

You'll solve this equation (sometimes called a “Cauchy–Euler Equation”) in the problems, both by hand and with a computer. The result is

$$R(r) = Br^l + Cr^{-l-1}$$

Since r^{-l-1} blows up at $r = 0$ for all non-negative integers l, we discard this solution based on our implicit boundary condition. Combining our two arbitrary constants, we find that:

$$V(r,\theta) = R(r)\Theta(\theta) = Ar^lP_l(\cos\theta)(l = 0, 1, 2\ldots)$$

As always, since our original equation was linear and homogeneous, we write a general solution as a sum of all the particular solutions, each with its own arbitrary constant:

$$V(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) \tag{11.7.3}$$

We must now meet the boundary condition $V(a,\theta) = V_0(\theta)$.

$$V_0(\theta) = \sum_{l=0}^{\infty} A_l a^l P_l(\cos\theta)$$

We are guaranteed that we can meet this condition, because Legendre polynomials—like the trig functions and Bessel functions that we have seen before—constitute a *complete* set of functions on the interval $-1 \le x \le 1$. Appendix J gives the coefficients necessary to build an arbitrary function as a Legendre series expansion:

$$A_l a^l = \frac{2l+1}{2} \int_0^\pi V_0(\theta) P_l(\cos\theta) \sin\theta \, d\theta$$

So the complete solution is

$$V(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) \tag{11.7.4}$$

where

$$A_l = \frac{2l+1}{2a^l} \int_0^\pi V_0(\theta) P_l(\cos\theta) \sin\theta \, d\theta \tag{11.7.5}$$

11.7.2 Explanation: Spherical Harmonics

In Section 11.7.1 we found the potential distribution inside a spherical shell held at a known potential $V_0(\theta)$. Because the problem had no ϕ-dependence we had to separate variables only once, solve two ordinary differential equations for r and θ, and write the solution as a single series. If we had to work with r, θ, and ϕ as three independent variables we would in general have to separate variables twice, solve three ordinary differential equations, and write the solution as a double series, as we did in Section 11.5.

However, there is a shortcut in spherical coordinates: separate variables only once, giving you one ordinary differential equation for the independent variable r and one partial differential equation for the two variables θ and ϕ. The solution to this partial differential equation will often be a special function called a "spherical harmonic."

The Problem, and the Solution

A spherical shell of radius a contains no charge within the shell; therefore, within the shell, the potential obeys Laplace's equation (11.2.5). On the surface of the shell the potential $V(a,\theta,\phi)$ is given by $V_0(\theta,\phi)$. Find the potential inside the sphere.

Note as always the "implicit" boundary conditions on this problem. The potential must be finite in the domain $0 \le r \le a, 0 \le \theta \le \pi$. The solution must also be 2π-periodic in ϕ.

Laplace's equation in spherical coordinates is derived in Chapter 8:

$$\frac{\partial^2 V}{\partial r^2} + \frac{2}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\left(\frac{\partial^2 V}{\partial \theta^2} + \frac{\cos\theta}{\sin\theta}\frac{\partial V}{\partial \theta} + \frac{1}{\sin^2\theta}\frac{\partial^2 V}{\partial \phi^2}\right) = 0 \tag{11.7.6}$$

Anticipating that we're only going to separate variables once, we look for a solution of the form:

$$V(r,\theta,\phi) = R(r)\Omega(\theta,\phi)$$

We plug this into the original differential equation, multiply through by $r^2/[R(r)\Omega(\theta,\phi)]$, separate out the r-dependent terms, and use the separation constant k to arrive at $r^2R''(r) + 2rR'(r) - kR(r) = 0$ and

$$\frac{\partial^2 \Omega}{\partial \theta^2} + \frac{\cos\theta}{\sin\theta}\frac{\partial \Omega}{\partial \theta} + \frac{1}{\sin^2\theta}\frac{\partial^2 \Omega}{\partial \phi^2} + k\Omega = 0 \tag{11.7.7}$$

In Problem 11.158 you'll solve Equation 11.7.7 using separation of variables. The solutions will of course be products of functions of ϕ with functions of θ. This PDE comes up so often in spherical coordinates, however, that its solution has a name. Looking up Equation 11.7.7 in Appendix J we find that the solutions are the "spherical harmonics" $Y_l^m(\theta,\phi)$, where l is a non-negative integer such that $k = l(l+1)$ and m is an integer satisfying $|m| \le l$. (The requirement that l and m be integers comes from the implicit boundary conditions on θ and ϕ.)

The equation for $R(r)$ is a Cauchy–Euler equation, just like in the last section. If you plug in $k = l(l+1)$ and use the requirement that it be bounded at $r = 0$ the solution is $R(r) = Cr^l$. The solution to Equation 11.7.6 is thus

$$V(r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} C_{lm} r^l Y_l^m(\theta,\phi)$$

Finally, we match the inhomogeneous boundary condition $V(a,\theta,\phi) = V_0(\theta,\phi)$.

$$V_0(\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} C_{lm} a^l Y_l^m(\theta,\phi)$$

From the formula in Appendix J for the coefficients of a spherical harmonic expansion we can write

$$C_{lm} = \frac{1}{a^l}\int_0^{2\pi}\int_0^{\pi} V_0(\theta,\phi)\left[Y_l^m(\theta,\phi)\right]^* \sin\theta \, d\theta \, d\phi$$

where $\left[Y_l^m(\theta,\phi)\right]^*$ designates the complex conjugate of the spherical harmonic function $Y_l^m(\theta,\phi)$. This completes our solution for the potential in a hollow sphere.

Spherical Harmonics

Spherical harmonics are a computational convenience, not a new method of solving differential equations. We could have solved the problem above by separating all three variables: we would have found a complex exponential function in ϕ and an associated Legendre polynomial in θ and then multiplied the two. ("Associated Legendre polynomials" are yet another special function, different from "Legendre polynomials." As you might guess, they are in Appendix J.) All we did here was skip a step by *defining* a spherical harmonic as precisely that product: a complex exponential in ϕ, multiplied by an associated Legendre polynomial in θ.

Complex exponentials form a complete basis (Fourier series), and so do associated Legendre polynomials. By multiplying them we create a complete basis for functions of θ and ϕ. Essentially any function that depends only on direction, and not on distance, can be expanded as a sum of spherical harmonics. That makes them useful for solving partial differential equations in spherical coordinates, but also for a wide variety of other problems ranging from analyzing radiation coming to us from space to characterizing lesions in multiple sclerosis patients.[8]

11.7.3 Stepping Back: An Overview of Separation of Variables

The last four sections of this chapter have all been on solving partial differential equations by separation of variables. The topic deserves that much space: partial differential equations come up in almost every aspect of physics and engineering, and separation of variables is the most common way of handling them.

[8]Goldberg-Zimring, Daniel, et. al., "Application of spherical harmonics derived space rotation invariant indices to the analysis of multiple sclerosis lesions' geometry by MRI," Magnetic Resonance Imaging, Volume 22, Issue 6, July 2004, Pages 815-825.

On the other hand there's a real danger that, after going through four sections on one technique, you will feel like there is a mountain of trivia to master. We started this section by listing the four steps of every separation of variables problem. Let's return to that outline, but fill in all the "gotchas" and forks in the road. You may be surprised at how few there are.

1. **Separate variables to turn one *partial* differential equation into several *ordinary* differential equations.**

   In this step you "guess" a solution that is separated into functions of one variable each. (Recall that such a solution is a *normal mode*.) Then you do a bit of algebra to isolate the dependency: for instance, one side depends only on θ, the other does not depend at all on θ. Finally you set both sides of the equation equal to a constant—the same constant for both sides!—often using k^2 if you can determine up front that the constant is positive, or $-k^2$ if negative.

   If separating the variables turns out to be impossible (as it will for almost any inhomogeneous equation for instance), you can't use this technique. The last few sections of this chapter will present alternative techniques you can use in such cases.

   If there are three variables you go through this process twice, introducing two constants…and so on for higher numbers of variables. (Spherical harmonics represent an exception to this rule in one specific but important special case.)

2. **Solve the ordinary differential equations. The product of these solutions is one solution to the original equation.**

   Sometimes the "solving" step can be done by inspection. When the solution is less obvious you may use a variable substitution and table lookup, or you may use a computer.

   The number of possible functions you might get is the most daunting part of the process. Trig functions (regular and hyperbolic), exponential functions (real and complex), Bessel functions, Legendre polynomials…no matter how many we show you in this chapter, you may encounter a new one next week. But as long as you can look up the function's general behavior, its zeros, and how to use it in series to build up other functions, you can work with it to find a solution. That being said, we have not chosen arbitrarily which functions to showcase: trig functions, Bessel functions, Legendre polynomials, and spherical harmonics are the most important examples.

3. **Match any homogeneous boundary conditions.**

   Matching the homogeneous boundary conditions sometimes tells you the value of one or more arbitrary constants, and sometimes limits the possible values of the constants of separation. If you have homogeneous *initial* conditions, this method generally won't work.

4. **Sum up all solutions into a series, and then match the inhomogeneous boundary and initial conditions.**

   First you write a series solution by summing up all the solutions you have previously found. (The original differential equation must be linear and homogeneous, so a linear combination of solutions—a series—is itself a solution.) Then you set that series equal to your inhomogeneous boundary or initial condition. (Your normal modes must form a "complete basis," so you can add them up to meet any arbitrary conditions.)

   If you had three independent variables—and therefore two separation constants—the result is a double series. This trend continues upward as the number of variables climbs.

   If you have more than one inhomogeneous condition, you create subproblems with one inhomogeneous condition each: this will be the subject of the next section.

The rest of the chapter discusses additional techniques that can be used instead of, or sometimes alongside, separation of variables for different types of problems. Appendix I gives a flow chart for deciding which techniques to use for which problems. If you go through that process and decide that separation of variables is the right technique to use, you might find it helpful to refer to the list above as a reminder of the key steps in the process.

11.7.4 Problems: Separation of Variables—Spherical Coordinates and Legendre Polynomials

In the Explanation (Section 11.7.1) we found that the potential $V(r,\theta)$ inside a hollow sphere is given by Equation 11.7.3, with the coefficients A_l given by Equation 11.7.5. In Problems 11.146–11.148 you will be given a particular potential $V_0(\theta)$ on the surface of the sphere to plug into our general solution.

11.146 $V_0(\theta) = c$ on the upper half of the sphere and 0 on the lower half.

(a) Write the expression for A_l. Have a computer evaluate the integral and plug it in to get an expression for $V(r,\theta)$ as an infinite sum.

(b) Evaluate your solution at $r = a$ and plot the 20^{th} partial sum of the resulting function $V(a,\theta)$ on the same plot as the given boundary condition $V_0(\theta)$. If they don't match well go back and figure out where you made a mistake.

11.147 $V_0(\theta) = \sin\theta$

(a) Write the expression for A_l. Leave the integral unevaluated.

(b) Evaluate your solution at $r = a$ by numerically integrating for the necessary coefficients. Plot the 20^{th} partial sum of the resulting function $V(a,\theta)$ on the same plot as the given boundary condition $V_0(\theta)$. If they don't match well go back and figure out where you made a mistake.

11.148 $V_0(\theta) = P_3(\cos\theta)$. Write the expression for A_l. Use the orthogonality of the Legendre polynomials and the fact that $\int_{-1}^{1} P_l^2(u)\,du = 2/(2l+1)$ to evaluate the integral and write a closed-form solution for $V(r,\theta)$.

11.149 Walk-Through: Differential Equation with Legendre Normal Modes. In this problem you will solve the partial differential equation $\partial y/\partial t - \left(1 - x^2\right)\left(\partial^2 y/\partial x^2\right) + 2x(\partial y/\partial x) = 0$ on the domain $-1 \le x \le 1$. Surprisingly, the only boundary condition you need for this problem is that y is finite on the interval $[-1, 1]$.

(a) Begin by guessing a separable solution $y = X(x)T(t)$. Plug this guess into the differential equation. Then divide both sides by $X(x)T(t)$ and separate variables.

(b) Find the general solution to the resulting ODE for $X(x)$. Using the requirement that y is finite on the domain $[-1, 1]$ show that one of your arbitrary constants must be zero and give the possible values for the separation constant. There will be an infinite number of them, but you should be able to write them in terms of a new constant l that can be any non-negative integer.

(c) Solve the ODE for $T(t)$, expressing your answer in terms of l.

(d) Multiply $X(x)$ times $T(t)$ to find the normal modes of this system. You should be able to combine your two arbitrary constants into one. Write the general solution $y(x,t)$ as a sum over these normal modes. Your arbitrary constants should include a subscript l to indicate that they can take different values for each value of l.

For the rest of the problem you will plug the initial condition $y(x,0) = x$ into the solution you found.

(e) Plugging $t = 0$ into your solution to Problem 11.149 should give you a Fourier-Legendre series for the function $y(x,0)$. Setting this equal to x, use the equation for A_l from Appendix J to find the coefficients in the form of an integral.

(f) Here are two facts about Legendre polynomials: $P_1(x) = x$, and the Legendre polynomials are *orthogonal*, meaning that $\int_{-1}^{1} P_l(x)P_m(x)\,dx = 0$ if $l \ne m$. Using those facts, you can analytically integrate your answer from Part 11.149(e). (For most initial conditions you wouldn't be able to evaluate this integral explicitly.) Use your result for this integral to write the solution $y(x,t)$ in closed form.

(g) Demonstrate that your solution satisfies the original differential equation and the initial condition.

In Problems 11.150–11.153 you will be given a PDE, a domain, and a set of initial conditions. You should assume in each case that the function is finite in the given domain.

(a) Solve the PDE using separation of variables. You may solve the ODEs you get by hand or with a computer. The solution to the PDE should be an infinite series with undetermined coefficients.

(b) Plug in the given initial condition. The result should be a Fourier-Legendre series. Use this series to write an equation for the coefficients in your solution. In most cases, unlike the problem above, you will not be able to evaluate this integral analytically: your solution will be in the form $\sum_{l=1}^{\infty} A_l$<*something*> where A_l is defined as an integral. See Equations 11.7.4–11.7.5 for an example.

(c) Have a computer evaluate the integral in Part (b) either analytically or numerically to calculate the 20^{th} partial sum of your series solution and plot the result at several times. Describe how the function is evolving over time.

11.150 $\partial y/\partial t - \left(1-x^2\right)\left(\partial^2 y/\partial x^2\right) + 2x\left(\partial y/\partial x\right) = 0$, $-1 \le x \le 1$, $y(x,0) = 1$, and $y(x)$ is finite everywhere in that domain

11.151 $\partial y/\partial t = \left(9-x^2\right)(\partial^2 y/\partial x^2) - 2x(\partial y/\partial x)$, $-3 \le x \le 3$, $y(x,0) = \sin(\pi x/3)$

11.152 $\partial y/\partial t = \partial^2 y/\partial\theta^2 + \cot\theta(\partial y/\partial\theta)$, $0 \le \theta \le \pi$, $y(\theta,0) = \sin^2\theta$

11.153 $\partial^2 y/\partial t^2 = \left(9-x^2\right)(\partial^2 y/\partial x^2) - 2x(\partial y/\partial x)$, $-3 \le x \le 3$, $y(x,0) = 4x$, $(\partial y/\partial t)(x,0) = 0$. Using the fact that $P_1(x) = x$ you should be able to get a closed-form solution (no sum) and explicitly check that it solves the original PDE. You don't need to do the computer part for this problem.

In Problems 11.154–11.156 solve the given PDE using the domain, boundary conditions, and initial conditions given in the problem. You should assume in each case that the function is finite in the given domain and that it is periodic in ϕ. Your solutions will be in the form of series involving spherical harmonics. You should write integral expressions for the coefficients. (*Hint*: You may need a variable substitution to get the spherical harmonic equation. If you can't find one that works try separating out all the variables and you should get solutions involving complex exponentials and associated Legendre polynomials that you can recombine into spherical harmonics.)

11.154 $\partial y/\partial t = \partial^2 y/\partial\theta^2 + \cot\theta(\partial y/\partial\theta) + \csc^2\theta(\partial^2 y/\partial\phi^2)$, $0 \le \theta \le \pi$, $0 \le \phi \le 2\pi$, $y(\theta,\phi,0) = \theta(\pi-\theta)\cos(\phi)$

11.155 $\dfrac{\partial y}{\partial t} = \left(1-x^2\right)\dfrac{\partial^2 y}{\partial x^2} - 2x\dfrac{\partial y}{\partial x} + \dfrac{1}{1-x^2}\dfrac{\partial^2 y}{\partial\phi^2}$, $-1 \le x \le 1, 0 \le \phi \le 2\pi$, $y(x,\phi,0) = \left(1-x^2\right)\cos\phi$

11.156 $\dfrac{\partial z}{\partial t} = \left(1-x^2\right)\dfrac{\partial^2 z}{\partial x^2} - 2x\dfrac{\partial z}{\partial x} + \dfrac{1}{1-x^2}\left(2\dfrac{\partial z}{\partial\phi} + 4\phi\dfrac{\partial^2 z}{\partial\phi^2}\right)$, $-1 \le x \le 1$, $0 \le \phi \le 4\pi^2$, $y(x,\phi,0) = \left(1-x^2\right)\cos\left(\sqrt{\phi}\right)$

11.157 In the Explanation (Section 11.7.1) we encountered the differential equation $r^2R''(r) + 2rR'(r) - l(l+1)R(r) = 0$, sometimes called a "Cauchy–Euler Equation."

(a) To solve this ordinary differential equation assume a solution of the form $R(r) = r^p$ where the constant p may depend on l but not on r. Plug this solution into the differential equation, solve for p, and write the general solution $R(r)$. Make sure your answer has two arbitrary constants!

(b) Find the same solution by plugging the differential equation into a computer. (You may need to tell the computer program that l is a positive integer to get it to simplify the answer. You may also in that process discover that some differential equations are easier to solve by hand than with a computer.)

11.158 In the Explanation (Section 11.7.2) we derived a PDE for $\Omega(\theta,\phi)$, which we said led to spherical harmonics. Use separation of variables to find the normal modes of this PDE, using Appendix J for any necessary ODE solutions. Your solution should include using the implicit boundary conditions to limit l and m. Use complex exponentials instead of trig functions. Your final answer should be a formula for $Y_l^m(\theta,\phi)$ in terms of other functions.

11.159 In the Explanation (Section 11.7.2) we derived the formula for the gravitational potential in the interior of a thin spherical shell of radius a with potential $V_0(\theta,\phi)$ on the surface of the shell.

(a) Use the formula we derived to find the gravitational potential inside a spherical shell that is held at a constant potential V_0. Simplify your answer as much as possible. *Hint*: the integral $\int_0^{2\pi}\int_0^{\pi}\left[Y_l^m(\theta,\phi)\right]^*\sin\theta\, d\theta\, d\phi$ equals $2\sqrt{\pi}$ for $l = m = 0$ and 0 for all other l and m.

(b) The gravitational field is given by $\vec{g} = -\vec{\nabla}V$. Use your result from Part (a) to find the gravitational field in the interior of the sphere.

11.160 Solve the equation

$$\frac{\partial^2 y}{\partial t^2} = \left(4 - x^2\right)\frac{\partial^2 y}{\partial x^2} - 2x\frac{\partial y}{\partial x} - \frac{4}{4 - x^2}y$$

on the domain $-2 \le x \le 2$ with initial conditions $y(x,0) = 4 - x^2$, $\partial y/\partial t(x,0) = 0$.

11.161 The equation $\nabla^2 u = -\lambda^2 u$ is known as the "Helmholtz equation." For this problem you will solve it in a sphere of radius a with boundary condition $u(a,\theta,\phi) = 0$. You should also use the implicit boundary condition that the function u is finite everywhere inside the sphere. Note that you will need to use the formula for the Laplacian in spherical coordinates.

(a) Separate variables, putting $R(r)$ on one side and $\Omega(\theta,\phi)$ on the other. Solve the equation for $\Omega(\theta,\phi)$ *without* separating a second time. You should find that this equation only has non-trivial solutions for some values of the separation constant.

(b) Using the values of the separation constant you found in the last part, solve the equation for $R(r)$, applying the boundary condition. You should find that only some values of λ allow you to satisfy the boundary condition. Clearly indicate the allowed values of λ.

(c) Combine your answers to write the general solution to the Helmholtz equation in spherical coordinates. You should find that the boundary conditions given in the problem were not sufficient to specify the solution. Instead the general solution will be a double sum over two indices with arbitrary coefficients in front of each term in the sum.

11.162 A quantum mechanical particle moving freely inside a spherical container of radius a can be described by a wavefunction Ψ that obeys Schrödinger's equation (11.2.6) with $V(\vec{x}) = 0$ and boundary condition $\Psi(a,\theta,\phi,t) = 0$. In this problem you will find the possible energies for such a particle.

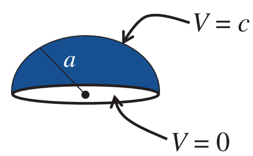
(a) Write Schrödinger's equation with $V(\vec{x}) = 0$ in spherical coordinates. Plug in a trial solution $\Psi(r,\theta,\phi,t) = T(t)\psi(r,\theta,\phi)$ and separate variables to get an equation for $T(t)$. Verify that $T(t) = Ae^{-iEt/\hbar}$ is the solution to your equation for $T(t)$, where E is the separation constant. That constant represents the energy of the atom.

(b) Plug $\psi(r,\theta,\phi) = R(r)\Omega(\theta,\phi)$ into the remaining equation, separate, and solve for $\Omega(\theta,\phi)$. Using the implicit boundary conditions that Ψ must be finite everywhere inside the sphere and periodic in ϕ, show that the separation constant must be $l(l+1)$ where l is an integer. (*Hint*: If you get a different set of allowed values for the separation constant you should think about how you can simplify the equation and which side the constant E should go on to get the answer to come out this way.)

(c) Solve the remaining equation for $R(r)$. If you write the separation constant in the form $l(l+1)$ you should be able to recognize the equation as being similar to one of the ones in Appendix J. You can get it in the right form with a variable substitution or solve it on a computer. You should find that the implicit boundary condition that Ψ is finite allows you to set one arbitrary constant to zero. The explicit boundary condition $\Psi(a,\theta,\phi,t) = 0$ should allow you to specify the allowed values of E. These are the possible energies for a quantum particle confined to a sphere.

When a large star collapses at the end of its life it becomes a dense sphere of particles known as a neutron star. Knowing the possible energies of a particle in a sphere allows astronomers to predict the behavior of these objects.

11.163 Exploration: Potential Inside a Hemisphere In this problem you will find the electric potential inside a hollow hemisphere of radius a with a constant potential $V = c$ on the curved upper surface of the hemisphere and $V = 0$ on the flat lower surface.



Because there is no charge inside the hemisphere, the potential will obey Laplace's equation. Because the problem has "azimuthal symmetry" (no ϕ-dependence) Laplace's equation can be written as

$$\frac{\partial^2 V}{\partial r^2} + \frac{2}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\left(\frac{\partial^2 V}{\partial \theta^2} + \frac{\cos\theta}{\sin\theta}\frac{\partial V}{\partial \theta}\right) = 0$$

Since the PDE is the same as the one we solved in the Explanation (Section 11.7.1), separation of variables will go the same way, leading to Equation 11.7.3. But from that point the problem will be different because of the different domain and boundary conditions.

(a) Write the domains for r and θ and boundary conditions for this problem, including all implicit boundary conditions.

(b) We now define a new variable $u = \cos\theta$. Rewrite your answers to Part (a) (both domain and boundary conditions) in terms of r and u instead of r and θ.

(c) We found in the Explanation that the general solution to our differential equation is $V(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$ or, with our new variables, $V(r,u) = \sum_{l=0}^{\infty} A_l r^l P_l(u)$. Plug into this equation the condition "$V = c$ everywhere on the curved upper surface of the hemisphere."

Our next step is to find the constants A_l by treating your answer to Part (c) as a Fourier-Legendre expansion in u. But u goes from 1 to 0, and Fourier-Legendre expansions are for functions with domain $[-1, 1]$. To get around this problem, we take the boundary condition "$V = c$ as u goes from 0 to 1" and create an odd extension of it, called $W(u)$. This means that the Fourier-Legendre series for $W(u)$ will converge to c on the interval $[0, 1]$.

(d) Write the function $W(u)$.

(e) Now that you can write a Fourier-Legendre expansion of W, use it to find the constants A_l. Your answer will be in the form of an integral.

(f) For even values of l, the Legendre polynomials are even functions. Explain why we can use this fact to discard all even powers of l.

(g) For odd values of l, the Legendre polynomials are odd functions. Use this fact to rewrite your answer to Part (e) in a way that uses c instead of W for odd values of l. (You can leave your result in integral form, or evaluate the resulting integral pretty easily with a table or computer lookup.)

(h) Plug these constants into Equation 11.7.3 to get the solution $V(r,\theta)$.

(i) Show that this solution matches the boundary condition that $V = 0$ on the flat lower surface of the hemisphere.

(j) Have a computer plot the fifth partial sum as a function of r and θ. You will need to choose values for a and c. You should be able to see that it roughly matches the boundary conditions.

11.8 Inhomogeneous Boundary Conditions

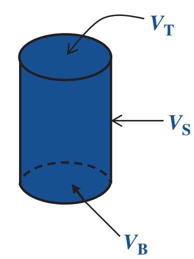
Recall that we solved inhomogeneous ODEs by finding a "particular" solution (that could not meet the initial conditions) and a "complementary" solution (to a different differential equation). When we added these two solutions, we found the general function that solved the original differential equation and could meet the initial conditions.

In this section, we will show how you can use the same approach to solve PDEs with inhomogeneous boundary conditions. You can also use this method on inhomogeneous PDEs; we will guide you through that process in Problem 11.176.

11.8.1 Discovery Exercise: Inhomogeneous Boundary Conditions

1. In each of the problems we have worked so far, there has been only one inhomogeneous boundary *or* initial condition. Explain why the technique of Separation of Variables, as we have described it, relies on this limitation.

Now consider a problem with three inhomogeneous boundary conditions. The cylinder to the right has no charge inside. The potential therefore obeys Laplace's equation, which in cylindrical coordinates is:



$$\frac{\partial^2 V}{\partial \rho^2}+\frac{1}{\rho}\frac{\partial V}{\partial \rho}+\frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2}+\frac{\partial^2 V}{\partial z^2}=0$$

The potential on top is given by V_T, which could be a constant or a function of ρ or even a function of ρ and ϕ. (For our present purpose it doesn't really matter.) The potential on the bottom is given by V_B. The potential on the side is V_S which could be a function of both z and ϕ.

2. The approach to such a problem is to begin by solving three *different* problems. Each problem is the same differential equation as the original, but each has only one inhomogeneous boundary condition. In the first such subproblem, we take $V = V_T$ on top, but $V = 0$ on the side and bottom. What are the other two subproblems?
3. If you solved all three subproblems and added the solutions, would the resulting function solve the original differential equation? Would it solve all the original boundary conditions?

11.8.2 Explanation: Inhomogeneous Boundary Conditions

Every example we have worked so far we has either been an initial value problem with homogeneous boundary conditions or a boundary value problem with one inhomogeneous boundary condition. We applied the homogeneous conditions before we summed; after we summed we could find the coefficients of our series solution to match the inhomogeneous condition at the one remaining boundary.

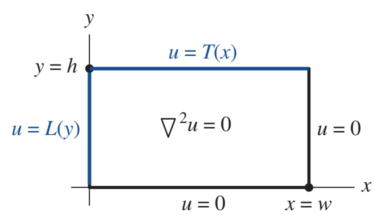
If we have multiple inhomogeneous conditions, we bring back an old friend from Chapter 1. To solve inhomogeneous ordinary differential equations we found two different solutions—a "particular" solution and a "complementary" solution—that summed to the general solution we were looking for. The same technique applies to the world of partial differential equations. In Problem 11.176 you'll apply this to solve an inhomogeneous PDE just as you did with ODEs. In this section, however, we'll show you how to use the same method to solve *linear, homogeneous differential equations* with *multiple inhomogeneous boundary conditions.*

The two examples below are different in several ways. In both cases, however, our approach will involve finding two different functions that we can add to solve the original problem we were given.

The First Problem: Multiple Inhomogeneous Boundary Conditions

If a two-dimensional surface is allowed to come to a steady-state temperature distribution, that distribution will obey Laplace's equation:

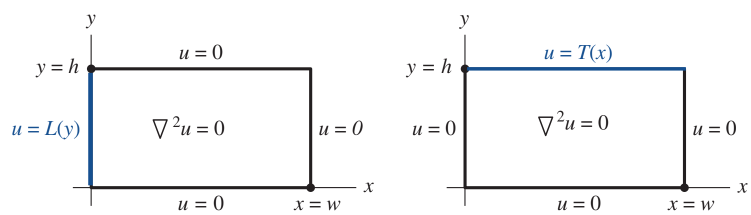
$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$$



Consider a rectangle with length w and height h. For convenience we place the origin of our coordinate system at the lower-left-hand corner of the rectangle, so the upper-right-hand corner is at (w, h). The boundary conditions are $u(x, 0) = u(w, y) = 0$, $u(0, y) = L(y)$, $u(x, h) = T(x)$.

The Approach

We're going to solve two different problems, neither of which is exactly the problem we were given. The first problem is to find a solution such that $u_L(x, y)$ is $L(y)$ on the left side and zero on all three other sides. The second problem is to find a function such that $u_T(x, y)$ is $T(x)$ on the top and zero on all three other sides.



$u_L(x, y)$ $u_T(x, y)$

Each subproblem has only one inhomogeneous condition, so we can take our familiar approach: separate variables and apply the homogeneous conditions, then create a series, and finally match the inhomogeneous condition. Of course, neither u_L nor u_T is a solution to our original problem! However, their sum $\left(u_L + u_T\right)(x, y)$ will satisfy Laplace's equation and fit the boundary conditions on all four sides.

Solving the Individual Problems

We begin by finding the first function, $u_L(x, y)$. Separation of variables leads us quickly to:

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)}$$

Since we wish $u_L(x, y)$ to go to zero at $y = 0$ and $y = h$, the separation constant must be positive, so we shall call it k^2. Solving the second equation and applying the boundary conditions:

$$Y(y) = A\sin(ky)$$

where $k = n\pi/h$. The first equation comes out quite differently. The positive separation constant leads to an exponential solution:

$$X(x) = Ce^{kx} + De^{-kx}$$

The requirement $X(w) = 0$ means $Ce^{kw} + De^{-kw} = 0$, which leads to the unpleasant-looking:

$$X(x) = C\left(e^{kx} - e^{k(2w-x)}\right)$$

Having plugged in our three homogeneous conditions, we now combine the two functions, absorb C into A, and write a sum:

$$u_L(x, y) = \sum_{n=1}^{\infty} A_n \sin(ky)\left(e^{kx} - e^{k(2w-x)}\right)$$

where $k = n\pi/h$.

Our last and inhomogeneous condition, $u_L(0, y) = L(y)$, becomes:

$$\sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{h}y\right)\left(1 - e^{2kw}\right) = L(y)$$

This is a Fourier sine series. You can find the formula for the coefficients in Appendix G.

$$A_n\left(1 - e^{2kw}\right) = \frac{2}{h}\int_0^h L(y)\sin\left(\frac{n\pi}{h}y\right)dy$$

For the second function $u_T(x, y)$ we start the process over, separating the variables and using a negative separation constant $-p^2$ this time. You will go through this process in Problem 11.164 and arrive at:

$$u_T(x, y) = \sum_{m=1}^{\infty} B_m \sin(px)\left(e^{py} - e^{-py}\right) \tag{11.8.1}$$

where $p = m\pi/w$. Since $u_T(x, y) = T(x)$, we have:

$$\sum_{m=1}^{\infty} B_m \sin\left(\frac{m\pi}{w}x\right)\left(e^{py} - e^{-py}\right) = T(x)$$

So the coefficients B_m are determined from

$$B_m\left(e^{ph} - e^{-ph}\right) = \frac{2}{w}\int_0^w T(x)\sin\left(\frac{m\pi}{w}x\right)dx$$

Solving the Actual Problem

None of the "separation of variables" math above is new, and we skipped a lot of hopefully familiar steps along the way. Only the final step is new: having found the two functions $u_L(x, y)$ and $u_T(x, y)$, we add them, and declare the sum $\left(u_L + u_T\right)(x, y)$ to be the solution to our original problem:

$$u(x, y) = \sum_{n=1}^{\infty} A_n \sin(ky)\left(e^{kx} - e^{k(2w-x)}\right) + \sum_{m=1}^{\infty} B_m \sin(px)\left(e^{py} - e^{-py}\right)$$

where $k = n\pi/h$, $p = m\pi/w$, $A_n\left(1 - e^{2kw}\right) = (2/h)\int_0^h L(y)\sin\left(n\pi y/h\right)dy$, and $B_m\left(e^{ph} - e^{-ph}\right) = (2/w)\int_0^w T(x)\sin(m\pi x/w)\,dx$.

That is the step you need to think about! *Assuming* that u_L and u_T solve the individual problems they were designed to solve…

- Can you convince yourself that $u_L + u_T$ is still a solution to the differential equation $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = 0$? (This would not work for *all* differential equations: why must it for this one?)
- Can you convince yourself that $u_L + u_T$ meets all the proper boundary conditions? (These are different from the boundary conditions that either function meets alone!)

As a final thought, before we move on to the next example: suppose all four boundary conditions had been inhomogeneous instead of only two. Can you outline the approach we would use to find a general solution? What individual problems would we have to solve first?

The Second Problem: Initial-Value Problem with Inhomogeneous Boundary

A rod stretches from $x = 0$ to $x = L$ with each end held at a fixed non-zero temperature: $u(0, t) = T_L$ and $u(L, t) = T_R$. The initial temperature distribution of the rod is represented by the function $u(x, 0) = T_0(x)$. Thereafter the temperature in the rod obeys the heat equation:

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2} \; (\alpha > 0)$$

Find the temperature distribution in the rod as a function of time.

The Approach

Once again we are going to break the initial problem into two subproblems that will sum to our solution. In this case we will find a "particular" solution and a "complementary" solution, just as we did for ordinary differential equations.

The "particular" solution $u_P(x, t)$ will solve our differential equation with the original boundary conditions. It will be a very simple, very specific solution with no arbitrary constants.

The "complementary" solution will solve our differential equation with homogeneous boundary conditions $u_C(0, t) = u_C(L, t) = 0$. It will have the arbitrary constants of a general solution: therefore, after we add it to our particular solution, we will be able to match initial conditions.

The Particular Solution

Our goal for a particular solution is to find *any* function $u_P(x, t)$ that solves our original differential equation and matches our boundary conditions. To find a simple solution, we make a simple assumption: for our particular solution, $\partial u_P/\partial t$ will equal zero.

Where does that leave us? $\partial^2 u_P/\partial x^2 = 0$ is easy to solve by inspection; the solution is any line $u_P = mx + b$. Our boundary conditions $u(0, t) = T_L$ and $u(L, t) = T_R$ allow us to solve quickly for the slope and y-intercept, bringing us to:

$$u_P = \frac{T_R - T_L}{L} x + T_L$$

Before we move on, stop to consider what that function represents. It certainly solves $\partial u/\partial t = \alpha(\partial^2 u/\partial x^2)$, since it makes both sides of the differential equation zero. It also matches the boundary conditions properly. On the other hand, we can't possibly make this function solve the *initial* conditions: there are no arbitrary constants to play with, and in fact no time dependence at all. That's where the complementary solution comes in.

The Complementary Solution

For the complementary solution we're going to solve the original differential equation, but with boundary conditions $u_C(0, t) = u_C(L, t) = 0$. With homogeneous boundary conditions and an inhomogeneous initial condition we have the perfect candidate for separation of variables. So we set $u_C(x, t) = X(x)T(t)$, plug in, simplify, and end up with:

$$\frac{T'(t)}{T(t)} = \alpha \frac{X''(x)}{X(x)}$$

The process at this point is familiar. A positive separation constant would make $X(x)$ exponential, and a zero separation constant would make $X(x)$ linear. Neither solution could reach zero at both ends with non-zero values in the middle, so we choose a negative separation constant $-k^2$. In Problem 11.170 you'll solve for $X(x)$ and apply the boundary conditions $X(0) = X(L) = 0$, then solve for $T(t)$, and end up here.

$$u_C(x,t) = Ae^{-k^2 t} \sin\left(\frac{k}{\sqrt{\alpha}}x\right) \text{ where } k = (n\pi/L)\sqrt{\alpha},\ n = 1, 2, 3, \dots \tag{11.8.2}$$

Since u_C solves a homogeneous differential equation with homogeneous boundary conditions, any sum of solutions is itself a solution, so the general solution is:

$$u_C(x,t) = \sum_{n=1}^{\infty} A_n e^{-\alpha\left(n^2\pi^2/L^2\right)t} \sin\left(\frac{n\pi}{L}x\right)$$

The Actual Solution

The functions u_P and u_C both solve the differential equation $\partial u/\partial t = \alpha(\partial^2 u/\partial x^2)$. They do *not* meet the same boundary conditions: u_P meets the original boundary conditions, and u_C goes to zero on both ends. When we add them, we have a solution to both the original differential equation and the original boundary conditions:

$$u(x,t) = \frac{T_R - T_L}{L}x + T_L + \sum_{n=1}^{\infty} A_n e^{-\alpha\left(n^2\pi^2/L^2\right)t} \sin\left(\frac{n\pi}{L}x\right)$$

As a final step, of course, we must make this *entire function* match the initial condition $u(x,0) = T_0(x)$. Plugging $t = 0$ into our function gives us:

$$u(x,0) = \frac{T_R - T_L}{L}x + T_L + \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{L}x\right) = T_0(x)$$

As usual we appeal to the power of a Fourier sine series to represent any function on a finite domain. In this case we have to choose our coefficients A_n so that:

$$\sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{L}x\right) = T_0(x) - \frac{T_R - T_L}{L}x - T_L$$

The solution is straightforward to write down in general, even though it may be intimidating to calculate for a given function T_0:

$$A_n = \frac{2}{L}\int_0^L \left(T_0(x) - \frac{T_R - T_L}{L}x - T_L\right) \sin\left(\frac{n\pi}{L}x\right) dx$$

A Physical Look at the "Particular" Solution

We presented the process above as nothing more than a mathematical trick. We made a simple mathematical assumption—in this case, $\partial u/\partial t = 0$—so we could find a function that would match both our differential equation and our boundary conditions. If we had been solving a different PDE we might have used a different assumption for the same purpose.

But $\partial u/\partial t = 0$ is not just any assumption: it says "I want to find the solution to this equation that will *never change*," the "steady-state" solution. If the heat in the rod ever happens to assume the distribution $u = [(T_R - T_L)/L]x + T_L$, it will stay that way forever.

And what of the complementary solution? Because we were forced by the second-order equation for $X(x)$ to choose a negative separation constant, the first-order equation for $T(t)$ led to a decaying exponential function $e^{-k^2 t}$. No matter what the initial conditions, the complementary solution will gradually die down toward zero; the total solution $u_P + u_C$ will approach the solution represented by u_P. In the language of physics, u_P represents a *stable* solution: the system will always trend toward that state.

In some situations you will find a steady-state solution that the system does not approach over time, analogous to an unstable equilibrium point in mechanics. In many cases such as this one, though, the steady-state solution represents the late-time behavior of the system and the complementary solution represents the transient response to the initial conditions.

The fact that this physical system approaches the steady-state solution is not surprising if you think about it. If the ends of the rod are held at constant temperature for a long time, the rod will move toward the simplest possible temperature distribution, which is a linear change from the left temperature to the right. We mention this to remind you of where we began the very first chapter of the book. *Solving* equations is a valuable skill, but computer solutions are becoming faster and more accurate all the time. *Understanding the solutions,* on the other hand, will require human intervention for the foreseeable future. To put it bluntly, interpreting solutions is the part that someone might pay you to do.

11.8.3 Problems: Inhomogeneous Boundary Conditions

11.164 Derive Equation 11.8.1 in the Explanation (Section 11.8.2) for $u_T(x, y)$. The process will be very similar to the one shown for deriving u_L.

11.165 **Walk-Through: Inhomogeneous Boundary Conditions.** In this problem you will solve the equation $2\partial^2 f/\partial x^2 - \partial^2 f/\partial y^2 = 0$ on a square extending from the origin to the point (π, π) with boundary conditions $f(x, 0) = f(0, y) = 0$, $f(x, \pi) = 4\sin(5x)$, and $f(\pi, y) = 3\sin(7y)$.

(a) Use separation of variables to find a function $f_1(x, y)$ that satisfies this PDE with boundary conditions $f_1(x, 0) = f_1(x, \pi) = f_1(0, y) = 0$, and $f_1(\pi, y) = 3\sin(7y)$.

(b) Use separation of variables to find a function $f_2(x, y)$ that satisfies this PDE with boundary conditions $f_1(x, 0) = f_2(0, y) = f_2(\pi, y) = 0$ and $f_2(x, \pi) = 4\sin(5x)$.

(c) Demonstrate that the function $f(x, y) = f_1(x, y) + f_2(x, y)$ satisfies the original PDE and the original boundary conditions.

For Problems 11.166–11.169 solve the given PDEs subject to the given boundary conditions.

11.166 $\partial^2 V/\partial x^2 + \partial^2 V/\partial y^2 + \partial^2 V/\partial z^2 = 0$, $V(0, y, z) = V(x, 0, z) = V(x, y, 0) = V(x, y, L) = 0$, $V(L, y, z) = V_0$, $V(x, L, z) = 2V_0$

11.167 $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 - \gamma^2 u = 0$, $u(0, y) = u(x, L) = 0$, $u(L, y) = u_0$, $u(x, 0) = u_0 \sin(2\pi x/L)$

11.168 $y^2\left(\partial^2 u/\partial x^2 - \partial^2 u/\partial y^2\right) - y(\partial u/\partial y) + u = 0$, $u(0, y) = u(x, 0) = 0$, $u(L, y) = \sin(\pi y/H)$, $u(x, H) = u_0$. Your answer should be an infinite sum. The formula for the coefficients will include an integral that you will not be able to evaluate analytically, so you should simply leave it as an integral.

11.169 $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 - \partial u/\partial y = 0$ in the rectangular region $0 \le x \le 1$, $0 \le y \le 3$ subject to the boundary conditions $u(0, y) = u(1, y) = 0$, $u(x, 0) = 3\sin(2\pi x)$, $u(x, 3) = \sin(\pi x)$. (*Warning*: The answer will be somewhat messy. You should, however, be able to get a solution with no sums or integrals in it.)

11.170 Derive Equation 11.8.2 in the Explanation (Section 11.8.2) for $u_C(x, t)$.

11.171 **Walk-Through: Particular and Complementary Solutions.** In this problem you'll solve the equation $\partial u/\partial t = \partial^2 u/\partial x^2 - u$ with the boundary conditions $u(0, t) = 0$, $u(1, t) = 1$ and the initial condition $u(x, 0) = x$.

(a) Find the steady-state solution $u_{ss}(x)$ by solving $\partial^2 u_{SS}/\partial x^2 - u_{SS} = 0$ subject to the boundary conditions above. (You can solve this ODE by inspection, and plug in the boundary conditions to find the arbitrary constants.) The solution $u_{SS}(x)$ should solve our original PDE and boundary conditions, but since it has no arbitrary constants it cannot be made to satisfy the initial condition.

(b) Find a complementary solution to the equation $\partial u_C/\partial t = \partial^2 u_C/\partial x^2 - u_C$ subject to the homogeneous boundary conditions $u(0,t) = u(1,t) = 0$. (You can solve this PDE by separation of variables.) This complementary solution should have an arbitrary constant, but it will not satisfy the original boundary conditions.

(c) Add these two solutions to get the general solution to the original PDE with the original boundary conditions.

(d) Apply the initial conditions to this general solution and solve for the arbitrary constants to find the complete solution $u(x,t)$ to this problem.

(e) What function is your solution approaching as $t \to \infty$?

For Problems 11.172–11.174 solve the given PDEs subject to the given boundary and initial conditions. For each one you will need to find a steady-state solution and a complementary solution, add the two, and then apply the initial conditions. For each problem, will the general solution approach the steady-state solution or not? Explain. It may help to first work through Problem 11.171 as a model.

11.172 $\partial^2 u/\partial t^2 = v^2(\partial^2 u/\partial x^2)$, $u(0,t) = 0$, $u(1,t) = 2$, $u(x,0) = \partial u/\partial t(x,0) = 0$

11.173 $\partial u/\partial t = \partial^2 u/\partial \theta^2 + u$, $u(0,t) = 0$, $u(\pi/2,t) = \alpha$, $u(\theta,0) = \alpha\,[\sin(\theta) + \sin(4\theta)]$.

11.174 $\partial u/\partial t = \partial^2 u/\partial x^2 + (1/x)(\partial u/\partial x) - (1/x^2)u$, $u(0,t) = 0$, $u(1,t) = 1$, $u(x,0) = 0$

11.175 The left end of a rod at $x = 0$ is immersed in ice water that holds the end at 0° C and the right edge at $x = 1$ is immersed in boiling water that keeps it at 100° C. (You may assume throughout the problem that all temperatures are measured in degrees Celsius and all distances are measured in meters.)

(a) What is the steady-state temperature $u_{SS}(x)$ of the rod at late times?

(b) Solve the heat equation (11.2.3) with these boundary conditions and with initial condition $u(x,0) = 100x(2-x)$ to find the temperature $u(x,t)$. You can use your steady-state solution as a "particular" solution.

(c) The value of α for a copper rod is about $\alpha = 10^{-4}$ m^2/s. If a 1 meter copper rod started with the initial temperature given in this problem how long it would take for the point $x = 1/2$ to get within 1% of its steady-state temperature? Although your solution for $u(x,t)$ is an infinite series, you should answer this question by neglecting all of the terms except u_{SS} and the first non-zero term of u_C.

11.176 **Exploration: An Inhomogeneous PDE**
We have used the technique of finding a particular and a complementary solution in two different contexts: inhomogeneous ODEs, and homogeneous PDEs with multiple inhomogeneous conditions. We can use the same technique for inhomogeneous PDEs. Consider as an example the equation $\partial u/\partial t - \partial^2 u/\partial x^2 = \kappa$ subject to the boundary conditions $u(0,t) = 0$, $u(1,t) = 1$ and the initial condition $u(x,0) = (1+\kappa/2)\,x - (\kappa/2)x^2 + \sin(3\pi x)$.

(a) Find a steady-state solution u_{SS} that solves $-\partial^2 u_{SS}/\partial x^2 = \kappa$ subject to the boundary conditions given above. Because this will be a *particular* solution you do not need any arbitrary constants.

(b) Find the general complementary solution that solves $\partial u_C/\partial t - \partial^2 u_C/\partial x^2 = 0$ subject to the boundary conditions $u(0,t) = u(1,t) = 0$. You don't need to apply the initial conditions yet.

(c) Add the two to find the general solution to the original PDE subject to the original boundary conditions. Apply the initial condition and solve for the arbitrary constants to get the complete solution.

(d) Verify by direct substitution that this solution satisfies the PDE, the boundary conditions, and the initial condition.

11.9 The Method of Eigenfunction Expansion

The rest of the chapter will be devoted to three different techniques: "eigenfunction expansion," "the method of Fourier transforms," and "the method of Laplace transforms." All three can solve some PDEs that cannot be solved by separation of variables.

All three techniques start by writing the given differential equation in a different form. The specific transformation is chosen to turn a derivative into a multiplication, thereby turning a PDE into an ODE.

As you read through this section, watch how a derivative turns into a multiplication, and how the resulting differential equation can be solved to find the function we were looking for—in the form of a Fourier series. After you follow the details, step back and consider what it was that made a Fourier series helpful for this particular differential equation, and you will be well set for the other transformations we will discuss.

11.9.1 Discovery Exercise: The Method of Eigenfunction Expansion

The temperature in a bar obeys the heat equation, with its ends fixed at zero temperature:

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2} \qquad (11.9.1)$$

$$u(0,t) = u(L,t) = 0$$

We previously found the temperature $u(x,t)$ of such a bar by using separation of variables; you are now going to solve the same problem (and hopefully get the same answer!) using a different technique, "eigenfunction expansion." Later we will see that eigenfunction expansion can be used in some situations where separation of variables cannot—most notably in solving some inhomogeneous equations.

To begin with, replace the unknown function $u(x,t)$ with its unknown Fourier sine expansion in x.

$$u(x,t) = \sum_{n=1}^{\infty} b_n(t) \sin\left(\frac{n\pi}{L}x\right) \qquad (11.9.2)$$

1. Does this particular choice (a Fourier series with no cosines) guarantee that you meet your boundary conditions? If so, explain why. If not, explain what further steps will be taken later to meet them.
2. What is the second derivative with respect to x of $b_n(t)\sin(n\pi x/L)$?
3. What is the first derivative with respect to t of $b_n(t)\sin(n\pi x/L)$?
4. Replacing $u(x,t)$ with its Fourier sine series as shown in Equation 11.9.2, rewrite Equation 11.9.1.

See Check Yourself #77 in Appendix L

Now we use one of the key mathematical facts that makes this technique work: if two Fourier sine series with the same frequencies are equal to each other, then the coefficients must equal each other. For instance, the coefficient of $\sin(3x)$ in the first series must equal the coefficient of $\sin(3x)$ in the second series, and so on.

5. Set the nth coefficient on the left side of your answer to Part 4 equal to the nth coefficient on the right. The result should be an *ordinary* differential equation for the function $b_n(t)$.
6. Solve your equation to find the function $b_n(t)$.
7. Write the function $u(x,t)$ as a Fourier sine series, with the coefficients properly filled in.
8. How will your temperature function behave after a long time? Answer this question based on your answer to Part 7; then explain why this answer makes sense in light of the physical situation.

11.9.2 Explanation: The Method of Eigenfunction Expansions

Separation of variables is a powerful approach to solving partial differential equations, but it is not a universal one. Its most obvious limitation is that you may find it algebraically impossible to separate the variables: for example, separation of variables can never solve an *inhomogeneous* partial differential equation (although it may still prove useful in finding the "complementary" solution). Another limitation that you may recall from Section 11.4 is that you generally cannot apply separation of variables to a problem with homogeneous initial conditions.

The "method of eigenfunction expansions" may succeed in some cases where separation of variables fails.

The Problem

Let us return to the first problem we solved with separation of variables: a string fixed at the points $(0,0)$ and $(L,0)$ but free to vibrate between those points. The initial position $y(x,0)=f(x)$ and initial velocity $(dy/dt)(x,0)=g(x)$ are specified as before.

In this case, however, there is a "driving function"—which is to say, the string is subjected to an external force that pushes it up or down. This could be a very simple force such as gravity (pulling down equally at all times and places), or it could be something much more complicated such as shifting air pressure around the string. Our differential equation is now inhomogeneous:

$$\frac{\partial^2 y}{\partial x^2}-\frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}=q(x,t) \tag{11.9.3}$$

where $q(x,t)$ is proportional to the external force on the string. In Problem 11.177 you will show that separation of variables doesn't work on this problem. Below we demonstrate a method that does.

Overview of the Method

Here's the plan. We're going to replace all the functions in this problem—$y(x,t)$, $q(x,t)$ and even the initial functions $f(x)$ and $g(x)$—with their respective Fourier sine series. (A physicist would say we are translating the problem into "Fourier space.") The resulting differential equation will give us the coefficients of the Fourier expansion of $y(x,t)$.

That may sound pointlessly roundabout, so it's worth discussing a few obvious questions before we jump into the math.

- **Can you really do that?** What makes this technique valid is that Fourier series are unique: when you find a Fourier sine series for a given function, you have found the *only* Fourier sine series for that function with those frequencies. Put another way, if two different Fourier sine series with the same frequencies equal each other for all values of x,

  $$a_1\sin(x)+a_2\sin(2x)+a_3\sin(3x)+\ldots=b_1\sin(x)+b_2\sin(2x)+b_3\sin(3x)+\ldots$$

  then a_1 must equal b_1, and so on. So after we turn both sides of our equation into Fourier series, we will confidently assert that the corresponding coefficients must be equal, and solve for them.
- **How does this make the problem easier?** Suppose some function $f(x)$ is expressed as a Fourier sine series:

  $$f(x)=b_1\sin(x)+b_2\sin(2x)+b_3\sin(3x)+\ldots$$

  What happens, term by term, when you take the second derivative? $b_1\sin(x)$ is just multiplied by -1. $b_2\sin(2x)$ is multiplied by -4, the next term by -9, and so on. In general, for any function $b_n \sin(nx)$, *taking a second derivative is the same as multiplying by* $-n^2$.

So, what happens to a differential equation when you replace a derivative with a multiplication? If you started with an ordinary differential equation, you are left with an algebra equation. In our example, starting with a two-variable partial differential equation, we will be left with an ordinary differential equation. If we had started with a three-variable PDE, we would be down to two…and so on.

This will all (hopefully) become clear as you look through the example, but now you know what you're looking for.

- **Do you always use a Fourier sine series?** No. Below we discuss why that type of series is the best choice for this particular problem.

A note about notation: There is no standard notation for representing different Fourier series in the same problem, but we can't just use b_3 to mean "the coefficient of $\sin(3x)$" when we have four different series with different coefficients. So we're going to use b_{y3} for "the coefficient of $\sin(3x)$ in the Fourier series for $y(x, t)$," b_{q3} for the third coefficient in the expansion of $q(x, t)$, and similarly for $f(x)$ and $g(x)$.

The Expansion

The first step is to replace $y(x, t)$ with its Fourier expansion in x. Recalling that $y(x, t)$ is defined on the interval $0 \le x \le L$, we have three options: a sine-and-cosine series with period L, a sine-only expansion with period $2L$ based on an odd extension on the negative side, or a cosine-only expansion with period $2L$ based on an even extension on the negative side. Chapter 9 discussed these alternatives and why the boundary conditions $y(0, t) = y(L, t) = 0$ lend themselves to a sine expansion. We therefore choose to create an odd extension of our function, and write:

$$y(x, t) = \sum_{n=1}^{\infty} b_{yn}(t) \sin\left(\frac{n\pi}{L}x\right) \tag{11.9.4}$$

with the frequency $n\pi/L$ chosen to provide the necessary period $2L$.

This differs from Chapter 9 because we are expanding a multivariate function (x and t) in one variable only (x). You can think about it this way: when $t = 2$, Equation 11.9.4 represents a specific function $y(x)$ being expanded into a Fourier series with certain (constant) coefficients. When $t = 3$ a different $y(x)$ is being expanded, with different coefficients…and so on, for all relevant t-values. So the coefficients b_{yn} are constants with respect to x, but vary with respect to time.

We will similarly expand all the other functions in the problem:

$$q(x, t) = \sum_{n=1}^{\infty} b_{qn}(t) \sin\left(\frac{n\pi}{L}x\right), \quad f(x) = \sum_{n=1}^{\infty} b_{fn} \sin\left(\frac{n\pi}{L}x\right), \quad g(x) = \sum_{n=1}^{\infty} b_{gn} \sin\left(\frac{n\pi}{L}x\right)$$

Since the initial functions $f(x)$ and $g(x)$ have no time dependence, b_{fn} and b_{gn} are constants.

Plugging In and Solving

We now plug Equation 11.9.4 into Equation 11.9.3 and get:

$$\frac{\partial^2}{\partial x^2}\left[\sum_{n=1}^{\infty} b_{yn}(t) \sin\left(\frac{n\pi}{L}x\right)\right] - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\left[\sum_{n=1}^{\infty} b_{yn}(t) \sin\left(\frac{n\pi}{L}x\right)\right] = \sum_{n=1}^{\infty} b_{qn}(t) \sin\left(\frac{n\pi}{L}x\right)$$

The next step—rearranging the terms—may look suspicious if you know the properties of infinite series, and we will discuss later why it is valid in this case. But if you put aside your doubts, the algebra is straightforward.

$$\sum_{n=1}^{\infty}\left[\frac{\partial^2}{\partial x^2}\left(b_{yn}(t)\sin\left(\frac{n\pi}{L}x\right)\right)-\frac{1}{v^2}\frac{\partial^2}{\partial t^2}\left(b_{yn}(t)\sin\left(\frac{n\pi}{L}x\right)\right)\right]=\sum_{n=1}^{\infty}b_{qn}(t)\sin\left(\frac{n\pi}{L}x\right)$$

Now we take those derivatives. Pay particular attention, because this is the step where the equation becomes easier to work with.

$$\sum_{n=1}^{\infty}\left[-\frac{n^2\pi^2}{L^2}b_{yn}(t)\sin\left(\frac{n\pi}{L}x\right)-\frac{1}{v^2}\frac{d^2b_{yn}(t)}{dt^2}\sin\left(\frac{n\pi}{L}x\right)\right]=\sum_{n=1}^{\infty}b_{qn}(t)\sin\left(\frac{n\pi}{L}x\right)$$

As we discussed earlier, the $\partial^2/\partial x^2$ operator has been replaced by a multiplied constant. At the same time, the partial derivative with respect to time has been replaced by an *ordinary* derivative, since b_{yn} has no x-dependence.

We can now rewrite our original differential equation 11.9.3 with both sides Fourier expanded:

$$\sum_{n=1}^{\infty}\left[-\frac{n^2\pi^2}{L^2}b_{yn}(t)-\frac{1}{v^2}\frac{d^2b_{yn}(t)}{dt^2}\right]\sin\left(\frac{n\pi}{L}x\right)=\sum_{n=1}^{\infty}b_{qn}(t)\sin\left(\frac{n\pi}{L}x\right)$$

This is where the uniqueness of Fourier series comes into play: if the Fourier sine series on the left equals the Fourier sine series on the right, then their corresponding coefficients must be the same.

$$-\frac{n^2\pi^2}{L^2}b_{yn}(t)-\frac{1}{v^2}\frac{d^2b_{yn}(t)}{dt^2}=b_{qn}(t) \tag{11.9.5}$$

It doesn't look pretty, but consider what we are now being asked to do. For any given $q(x,t)$ function, we find its Fourier sine series: that is, we find the coefficients $b_{qn}(t)$. That leaves us with an *ordinary* second-order differential equation for the coefficients $b_{yn}(t)$. We may be able to solve that equation by hand, or we may hand it over to a computer.[9] Either way we will have the original function $y(x,t)$ we were looking for—but we will have it in the form of a Fourier series.

In Problem 11.178 you will address the simplest possible case, that of $q(x,t)=0$, to show that the solution matches the result we found using separation of variables. Below we show the result of a constant force such as gravity. In Problem 11.198 you will tackle the problem more generally using the technique of variation of parameters.

Of course the solution will contain two arbitrary constants, which bring us to our initial conditions. Our first condition is $y(x,0)=f(x)$. Taking the Fourier expansion of both sides of *that* equation—and remembering once again the uniqueness of Fourier series—we see that $b_{yn}(0)=b_{fn}$. Our other condition, $dy/dt(x,0)=g(x)$, tells us that $\dot{b}_{yn}(0)=b_{gn}$. So we can use our initial conditions on y to find the initial conditions for b: or to put it another way, we translate our initial conditions into Fourier space.

Once we have solved our ordinary differential equation with the proper initial conditions, we have the functions $b_{yn}(t)$. These are the coefficients of the Fourier series for y, so we now have our final solution in the form of a Fourier sine series.

[9]Fortunately, computers—which are still poor at solving partial differential equations—do a great job of finding Fourier coefficients and solving ordinary differential equations.

A Sample Driving Function

As a sample force let's consider the downward pull of gravity, so $q(x, t)$ is a constant Q. The Fourier sine series for a constant on the domain $0 \le x \le L$ is:

$$Q = \sum_{odd\ n} \frac{4Q}{\pi n} \sin\left(\frac{n\pi}{L}x\right)$$

For even values of n, b_{qn} is zero. A common mistake is to ignore those values entirely, but they are not irrelevant; instead, they turn Equation 11.9.5 into

$$\frac{d^2 b_{yn}(t)}{dt^2} = -\frac{v^2 n^2 \pi^2}{L^2} b_{yn}(t) \quad \text{even } n \tag{11.9.6}$$

We can solve this by inspection:

$$b_{yn} = A_n \sin\left(\frac{vn\pi}{L}t\right) + B_n \cos\left(\frac{vn\pi}{L}t\right) \quad \text{even } n \tag{11.9.7}$$

For odd values of n Equation 11.9.5 becomes:

$$\frac{d^2 b_{yn}}{dt^2} + Cb_{yn} = D \text{ where } C = \frac{v^2 n^2 \pi^2}{L^2} \text{ and } D = -\frac{4Qv^2}{\pi n} \quad \text{odd } n$$

This is an inhomogeneous second-order ODE and we've already solved its complementary equation: the question was Equation 11.9.6 and our answer was Equation 11.9.7. That leaves us only to find a particular solution, and $b_{yn,particular} = D/C$ readily presents itself. So we can write:

$$b_{yn} = A_n \sin\left(\frac{vn\pi}{L}t\right) + B_n \cos\left(\frac{vn\pi}{L}t\right) - \left[\frac{4QL^2}{\pi^3 n^3},\ n \text{ odd}\right] \tag{11.9.8}$$

Next we solve for A_n and B_n using the initial conditions $b_{yn}(0) = b_{fn}$ and $db_{yn}/dt(0) = b_{gn}$. You'll do this for different sets of initial conditions in the problems. The solution to our original PDE is

$$y(x, t) = \sum_{n=1}^{\infty} b_{yn}(t) \sin\left(\frac{n\pi}{L}x\right)$$

with b_{yn} defined by Equation 11.9.8.

EXAMPLE **Eigenfunction Expansion**

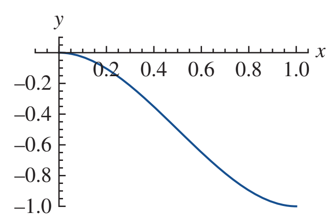
Solve the differential equation

$$-9\frac{\partial^2 y}{\partial x^2} + 4\frac{\partial y}{\partial t} + 5y = x$$

on the domain $0 \le x \le 1$ for all $t \ge 0$ subject to the boundary conditions $\partial y/\partial x(0, t) = \partial y/\partial x(1, t) = 0$ and the initial condition $y(x, 0) = f(x) = 2x^3 - 3x^2$.

Before we start solving this problem, let's think about what kind of solution we expect. The PDE is simplest to interpret if we write it as $4(\partial y/\partial t) = 9(\partial^2 y/\partial x^2) - 5y + x$, which we can read as "the vertical velocity depends on…." The first term says that y tends to increase when the concavity is positive and decrease when it is negative. The second two terms taken together say that y tends to increase if $-5y + x > 0$: that is, it will tend to move up if it is below the line $y = x/5$ and move down if it is above it. Both of these effects are always at work. For instance, if the curve is concave up and below the line $y = x/5$ both effects will push y upward; if it is concave down below the line the two effects will push in opposite directions.

The initial condition is shown below:



Since the initial function is negative over the whole domain, the $-5y + x$ terms will push it upward. Meanwhile, the concavity will push it down on the left and up on the right. You can easily confirm that the concavity will "win" on the left, so y should initially decrease at small values of x and increase at larger values. However, as the values on the left become more negative, they will eventually be pushed back up again.

To actually solve the equation, we note that the boundary conditions can be most easily met with a Fourier cosine series. So we replace y with its Fourier cosine expansion:

$$y(x,t) = \frac{a_{y0}(t)}{2} + \sum_{n=1}^{\infty} a_{yn}(t)\cos(n\pi x)$$

On the right side of our differential equation is the function x. Remember that we only care about this function from $x = 0$ to $x = 1$, but we need to create an even extension to find a cosine expansion. While you can do this entirely by hand (using integration by parts) or entirely with a computer, you may find that the easiest path is a hybrid: you first determine by hand that $a_n = 2\int_0^1 x\cos(n\pi x)dx$ and then hand that integral to a computer, or use one of the integrals in Appendix G. (For the case $n = 0$ in particular, it's easiest to do the integral yourself.) You find that:

$$x = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2\left(-1+(-1)^n\right)}{\pi^2 n^2}\cos(n\pi x)$$

Plugging our expansions for y and x into the original problem yields:

$$2a'_{y0}(t) + \frac{5a_{y0}(t)}{2} + \sum_{n=1}^{\infty}\left[9n^2\pi^2 a_{yn}(t)\cos(n\pi x) + 4a'_{yn}(t)\cos(n\pi x) + 5a_{yn}(t)\cos(n\pi x)\right]$$
$$= \frac{1}{2} + \sum_{n=1}^{\infty}\frac{2\left[-1+(-1)^n\right]}{\pi^2 n^2}\cos(n\pi x)$$

Setting the Fourier coefficients on the left equal to the coefficients on the right and rearranging leads to the differential equations:

$$a'_{y0}(t) = -\frac{5}{4}a_{y0}(t) + \frac{1}{4}$$
$$a'_{yn}(t) = \left(\frac{-9n^2\pi^2 - 5}{4}\right)a_{yn}(t) + \frac{\left[-1+(-1)^n\right]}{2\pi^2 n^2}, \quad n > 0$$

Each of these is just the separable first-order differential equation $df/dx = Af + B$ with uglier-looking constants. The solution is $f = Ce^{Ax} - B/A$, or, in our case,

$$a_{y0}(t) = C_0 e^{-(5/4)t} + \frac{1}{5}$$
$$a_{yn}(t) = C_n e^{\frac{9n^2\pi^2-5}{4}t} + \frac{2[-1+(-1)^n]}{\pi^2 n^2(9n^2\pi^2+5)}, \quad n > 0$$

Note that the numerator of the last fraction simplifies to 0 for even n and -4 for odd n.

We now turn our attention to the initial condition $y(x,0) = f(x) = 2x^3 - 3x^2$. To match this to our Fourier-expanded solution, we need the Fourier expansion of this function. Remember that we need a Fourier cosine expansion that is valid on the domain $0 \le x \le 1$. We leave it to you to confirm the result (which we obtained once again by figuring out by hand what integral to take and then handing that integral over to a computer):

$$a_{f0} = -1, \; a_{fn} = \frac{48}{\pi^4 n^4} \quad \text{(odd } n \text{ only)}$$

Setting $a_{y0}(0) = a_{f0}$ leads to $C_0 = -6/5$ and doing the same for higher values of n gives:

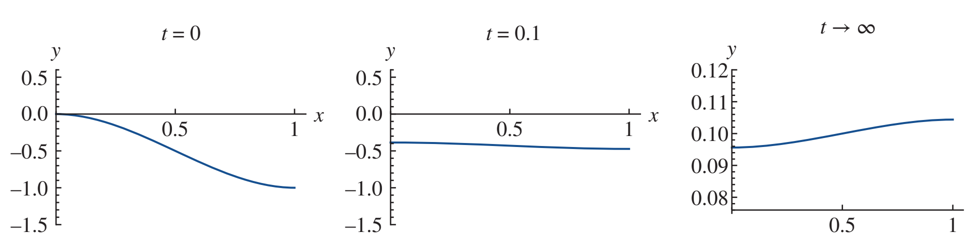
$$C_n = \frac{48}{\pi^4 n^4} + \frac{4}{\pi^2 n^2(9n^2\pi^2+5)} \quad \text{(odd } n \text{ only)}$$

We have now solved the entire problem—initial conditions and all—in Fourier space. That is, we have found the a_n-values which are the coefficients of the Fourier series for $y(x,t)$. So the solution to our original problem is:

$$y(x,t) = \frac{1}{10} - \frac{3}{5}e^{-(5/4)t} + \sum_{odd\ n}\left[C_n e^{-(9n^2\pi^2+5)t/4} - \frac{4}{\pi^2 n^2(9n^2\pi^2+5)}\right]\cos(n\pi x)$$

with the C_n coefficients defined as above.

It's an intimidating-looking formula to make sense of, but we can use a computer to evaluate partial sums and see how it behaves.



The figures above show the solution (using the 10^{th} partial sum) at three times. As predicted the function initially moves down on the left and up on the right, but eventually moves up everywhere until the terms all cancel out to give $\partial y/\partial t = 0$.

Stepping Back

We can break the process above into the following steps.

1. Replace all the functions—for instance $y(x,t)$, $q(x,t)$, $f(x)$, and $g(x)$ in our first example above—with their respective Fourier series.
2. Plug the Fourier series back into the original equation and simplify.
3. You now have an equation of the form *<this Fourier series>=<that Fourier series>*. Setting the corresponding coefficients equal to each other, you get an equation to solve for the coefficients—one that should be easier than the original equation, because some of the derivatives have been replaced with multiplications.
4. Solve the resulting equation to find the Fourier coefficients of the function you are looking for—with arbitrary constants, of course. You find these arbitrary constants based on the Fourier coefficients of the initial conditions. You now have the function you were originally looking for, expressed in the form of a Fourier series.
5. In some cases, you may be able to explicitly sum the resulting Fourier series to find your original function in closed form. More often, you will use computers to plot or analyze the partial sums and/or the dominant terms of the Fourier series.

As always, it is important to understand some of the subtleties.

- **Why did we expand in x instead of in t?** There are two reasons, based on two very general limitations on this technique.

  First, *you can only do an eigenfunction expansion for a variable with homogeneous boundary conditions*. To understand why, think about how we dealt with the boundary conditions and the initial conditions in the problems above. For the initial condition—the condition on t, which we did *not* expand in—we expanded the condition itself into a Fourier series in x and matched that to our equation, coefficient by coefficient. But for the boundary condition—the condition on x, which we *did* expand in—we chose our sine or cosine series so that each individual term would match the boundary condition, knowing that this would make the entire series match the homogeneous condition.

  Second, as you may remember from Chapter 9, *you can only take a Fourier series for a function that is periodic, or defined on a finite domain*. In many problems, including the ones we worked above, the domain of x is finite but the domain of t is unlimited. For non-periodic functions you need to do a *transform* instead of a series expansion. Solving partial differential equations by transforms is the subject of Sections 11.10–11.11.
- **The "suspicious step"—moving a derivative in and out of a series.** In general, $\frac{d}{dx}\left(\sum f_n(x)\right)$ is *not* the same as $\sum\left(df_n/dx\right)$. For instance, you will show in Problem 11.199 that if you take the Fourier series for x, and take the derivative term by term, you do *not* end up with the Fourier series for 1. However, moving a derivative inside a Fourier series—which is vital to this method—*is* valid when the periodic extension of the function is continuous.[10] Since our function was equal on both ends its extension is everywhere continuous, so the step is valid.
- **Which Fourier series?** The choice of a Fourier sine or cosine series is dictated by the boundary conditions. In the "wave equation" problem above the terms in the expansion had to be sines in order to match the boundary conditions $y(0,t) = y(L,t) = 0$. In the example that started on Page 628 we needed cosines to match the condition $\partial y/\partial x(0,t) = \partial y/\partial x(1,t) = 0$.
- **Why a Fourier series at all?** We used a Fourier series because our differential equations were based on second derivatives with respect to x.

[10] See A. E. Taylor's "Differentiation of Fourier Series and Integrals" in *The American Mathematical Monthly*, Vol. 51, No. 1.

A function $f(x)$ is an "eigenfunction" of the operator L if $L\left[f(x)\right] = kf(x)$ for some constant k. That is the property we used in these problems: $b_n \sin(nx)$ and $b_n \cos(nx)$ are eigenfunctions of the second derivative operator. When $y(x,t)$ is expressed as a series of sinusoidal terms, the second derivative acting on each term is replaced with a simple multiplication by $-n^2$.

Different equations will require different expansions. For example, in Problem 11.205 you will show that the eigenfunctions of the Laplacian in polar coordinates involve Bessel functions and then use an expansion in Bessel functions to solve an inhomogeneous wave equation on a disk.

11.9.3 Problems: The Method of Eigenfunction Expansion

11.177 Try to solve the equation $\partial^2 y/\partial x^2 - (1/v^2)(\partial^2 y/\partial t^2) = xt$ using separation of variables. Explain why it doesn't work. (Your answer should not depend on the initial or boundary conditions.)

In the Explanation (Section 11.9.2) we found the general solution to the inhomogeneous wave equation 11.9.3 with boundary conditions $y(0,t) = y(L,t) = 0$. For Problems 11.178–11.183 you should use the given inhomogeneous term $q(x,t)$ and initial conditions $y(x,0) = f(x)$, $\partial y/\partial t(x,0) = g(x)$ to solve Equation 11.9.5 for $b_{yn}(t)$ and plug it into Equation 11.9.4 to get the complete solution $y(x,t)$. Your answers will be in the form of infinite series.

11.178 $q(x,t) = 0$. For this problem you should use generic initial conditions $f(x)$ and $g(x)$ and show that the solution you get from the method of eigenfunction expansion is the same one we got in Section 11.4 from separation of variables.

11.179 $q(x,t) = \kappa \sin(\omega t)$, $f(x) = g(x) = 0$

11.180 $q(x,t) = \kappa \sin\left(p\pi x/L\right), f(x) = g(x) = 0$ (p an integer)

11.181 $q(x,t) = \kappa \sin\left(p\pi x/L\right) e^{-\omega t}$, $f(x) = g(x) = 0$ (p an integer)

11.182 $q(x,t) = \begin{cases} 1 & L/3 < x < 2L/3 \\ 0 & \textit{elsewhere} \end{cases}$,

$f(x) = \sin\left(\dfrac{2\pi}{L}x\right), g(x) = 0$

11.183 $q(x,t) = \kappa e^{-\omega t}$, $f(x) = \begin{cases} x & 0 \le x \le L/2 \\ L-x & L/2 < x \le L \end{cases}$, $g(x) = 0$

11.184 **Walk-Through: Eigenfunction Expansion.** In this problem you will solve the partial differential equation $\partial u/\partial t - \partial^2 u/\partial x^2 = xt$ with boundary conditions $u(0,t) = u(\pi,t) = 0$ using the method of eigenfunction expansions.

(a) In the first step, you replace the function $u(x,t)$ with the series $\sum_{n=1}^{\infty} b_{un}(t)\sin(nx)$. Explain why it's necessary to use a Fourier series with sines only (no cosines).

(b) Write the right side of the PDE as a Fourier sine series in x and find the coefficients $b_{qn}(t)$.

(c) Plug the Fourier sine expansions into both sides of the PDE. The x-derivatives should turn into multiplications. The result should look like:

$$\sum_{n=1}^{\infty}\left(\text{an expression involving } b_{un}(t) \text{ and } \frac{\partial b_{un}}{\partial t}\right)$$

$$\sin(nx) = \sum_{n=1}^{\infty}(\text{a function of } n \text{ and } t)\sin(nx)$$

(d) If two Fourier series are equal to each other then each coefficient of one must equal the corresponding coefficient of the other. This means you can set the expressions in parentheses on left and right in Part (c) equal. The result should be an ODE for $b_{un}(t)$.

(e) Find the general solution to the ODE you wrote for $b_{un}(t)$ in Part (d) and use this to write the solution $u(x,t)$ as an infinite series. The answer should involve an arbitrary coefficient A_n inside the sum.

11.185 *[This problem depends on Problem 11.184.]* In this problem you will plug the initial condition $u(x,0) = \begin{cases} 1 & \pi/3 < x < 2\pi/3 \\ 0 & \textit{elsewhere} \end{cases}$ into the solution you found to Problem 11.184.

(a) Expand the given initial condition into a Fourier sine series.

(b) Plug $t = 0$ into your general solution to Problem 11.184 and set it equal to the Fourier-expanded initial condition you wrote in Part (a). Setting the coefficients on the left equal to the corresponding coefficients on the right, solve to find the coefficients A_n.

(c) Plugging the solution you just found for A_n into the solution you found in Problem 11.184 gives you a series for $u(x, t)$. Plot the 20^{th} partial sum of this series solution at several times and describe its behavior.

Problems 11.186–11.192 are initial value problems that can be solved by the method of eigenfunction expansion. For each problem use the boundary conditions $y(0, t) = y(L, t) = 0$, and assume that time goes from 0 to ∞. When the initial conditions are given as arbitrary functions, write the solution as a series and write expressions for the coefficients in the series. When specific initial conditions are given, solve for the coefficients. The solution may still be in the form of a series. It may help to first work through Problems 11.184–11.185 as a model.

11.186 $\partial y/\partial t - \left(\partial^2 y/\partial x^2\right) + y = \kappa$, $y(x, 0) = 0$

11.187 $\partial^2 y/\partial t^2 - (\partial^2 y/\partial x^2) + y = \sin(\pi x/L)$, $y(x, 0) = \partial y/\partial t(x, 0) = 0$

11.188 $\partial^2 y/\partial t^2 - (\partial^2 y/\partial x^2) + y = \sin(\pi x/L)\cos(\omega t)$, $y(x, 0) = 0$, $\partial y/\partial t(x, 0) = g(x)$

11.189 $\partial y/\partial t - (\partial^2 y/\partial x^2) = e^{-t}$, $y(x, 0) = f(x)$

11.190 $\partial y/\partial t - (\partial^2 y/\partial x^2) = e^{-t}$, $y(x, 0) = \sin(3\pi x/L)$

11.191 $\dfrac{\partial^3 y}{\partial t\, \partial x^2} + y = \kappa$, $y(x, 0) = 0$.

11.192 $\dfrac{5}{4}\dfrac{\partial^2 y}{\partial t^2} - \dfrac{\partial^2 y}{\partial x^2} - 3\dfrac{\partial^3 y}{\partial t\, \partial x^2} = (\sin x)e^{-t}$, $y(x, 0) = 0$, $\dfrac{\partial y}{\partial t}(x, 0) = 0$. Take $L = \pi$. You should be able to express your answer in closed form (with no series).

Problems 11.193–11.197 are boundary problems that can be solved by the method of eigenfunction expansion. In all cases, x goes from 0 to L and y goes from 0 to H.

11.193 $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = y$, $u(0, y) = u(L, y) = u(x, H) = u(x, 0) = 0$

11.194 $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 + \partial u/\partial x = y$, $u(0, y) = u(L, y) = u(x, H) = u(x, 0) = 0$

11.195 $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 - u = \kappa$, $u(0, y) = u(L, y) = u(x, H) = u(x, 0) = 0$

11.196 $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = \sin(3\pi x/L)$, $u(x, 0) = u(0, y) = u(L, y) = 0$, $u(x, H) = \sin(2\pi x/L)$

11.197 $\partial^2 u/\partial x^2 + x(\partial^2 u/\partial y^2) + \partial u/\partial x = 0$, $u(L, y) = u(x, H) = u(x, 0) = 0$, $u(0, y) = \kappa$. Use eigenfunction expansion to reduce this to an ODE and use a computer to solve it with the appropriate boundary conditions. The solution $u(x, y)$ will be an infinite sum whose terms are hideous messes. Verify that it's correct by showing that each term individually obeys the original PDE, and by plotting a large enough partial sum of the series as a function of x and y to show that it matches the boundary conditions. (Making the plot will require choosing specific values for L, H, and κ.)

11.198 In the Explanation (Section 11.9.2) we showed that the problem of a bounded driven string could be solved by solving the ordinary differential equation 11.9.5. For particular driving functions $q(x, t)$ you might use a variety of techniques to solve this equation, but this ODE can be solved for a generic q with the technique *variation of parameters,* which we discussed in Chapter 10.

(a) Begin by solving the complementary homogeneous equation $-(n^2\pi^2/L^2)b_{yn}(t) - (1/v^2)(d^2 b_{yn}(t)/dt^2) = 0$ by inspection. You should end up with two linearly independent solutions $y_1(t)$ and $y_2(t)$. The general solution to the complementary equation is therefore $Ay_1(t) + By_2(t)$.

Our goal is now to find a solution—any particular solution!—to the original equation. We can then add this particular solution to $Ay_1(t) + By_2(t)$ to find the general solution.

(b) Use variation of parameters to find a particular solution to this equation. (You will need to begin by putting Equation 11.9.5 into the correct form for this technique.) Your solution will involve integrals based on the unknown function $b_{qn}(t)$, the Fourier coefficients of $q(x, t)$.

(c) As an example, consider the driving function $q(x, t) = t$. (The force is uniform across the string, but increases over time.) For that given driving force, take a Fourier sine series to find $b_{qn}(t)$.

(d) Plug that $b_{qn}(t)$ into your formulas and integrate to find $u(t)$ and $v(t)$. (You do *not* need an arbitrary constant when you integrate; remember, all we need is one working solution!) Put them together with your complementary solution to find the general solution to this problem.

(e) Demonstrate that your solution correctly solves the differential equation $-(n^2\pi^2/L^2)b_{yn}(t) - (1/v^2)(d^2 b_{yn}(t)/dt^2) = 4t/(n\pi)$.

11.199 **Derivatives of Fourier sine series.** The method of eigenfunction expansion relies on taking the derivative of an infinite series term by term. For example, it assumes that $\frac{d}{dx}\left(\sum b_n \sin x\right) = \sum \frac{d}{dx}\left(b_n \sin x\right)$ (so the derivative of a Fourier sine series is a Fourier cosine series). This step can safely be taken for a function that is continuous on $(-\infty, \infty)$.[11] Otherwise, it can get you into trouble!

Consider, as an example, the function $y = x$ on $0 \le x \le 1$. To find a Fourier sine series for this function, we create an odd extension on the interval $-1 \le x \le 0$ and then extend out periodically. Define $f(x)$ as this odd, extended version of the function $y = x$.

(a) Make a sketch of $f(x)$ from $x = -3$ to $x = 3$. Is it everywhere continuous?

(b) Find the Fourier series of the function $f(x)$ that you drew in Part (a). This is equivalent to taking the Fourier sine series of the original function, $y = x$ on $0 \le x \le 1$.

(c) What is $f'(x)$?

(d) Is $f'(x)$ odd, even, or neither? What does that tell you about its Fourier series?

(e) Find the Fourier series of $f'(x)$. (You can ignore the discontinuities in $f'(x)$: "holes" do not change a Fourier series as long as there is a finite number of them per period. Ignoring them, finding this Fourier series should be trivial and require no calculations.)

(f) Take the term-by-term derivative of the Fourier series for $f(x)$—the series you found in Part (b). Do you get the Fourier series for $f'(x)$, the series you find in Part (e)? Is the term-by-term derivative of the Fourier series for $f(x)$ a convergent series?

(g) Your work above should convince you that the derivative of a Fourier series is not always the Fourier series of the derivative. It is therefore important to know if we are dealing with a continuous function! Fortunately, it isn't hard to tell. In general, for any function $f(x)$ defined on a finite interval $0 \le f(x) \le L$, if you make an odd extension of the function and extend it periodically over the real line, the resulting function will be continuous if and only if $f(x)$ is continuous on the interval $0 < x < L$ and $f(0) = f(L) = 0$. Explain why these conditions are necessary and sufficient for the extension to be continuous.

11.200 *[This problem depends on Problem 11.199.]* **Derivatives of other Fourier series:** Suppose the function $f(x)$ is defined on the domain $0 < x < L$ and is continuous within that domain. Recall that we said you can take the derivative of a Fourier series term by term if the function is continuous on the entire real line.

(a) What are the conditions on $f(x)$ between 0 and L under which you can differentiate its Fourier cosine series term by term to find the Fourier cosine series of $\partial f/\partial t$? (In the notation of this section, what are the conditions under which $a_{(\partial f/\partial t)n} = \partial a_{fn}/\partial t$?) Recall that a Fourier cosine series involves an even extension of the function on the interval $-L < x < 0$. Include a brief explanation of why your answer is correct.

(b) What are the conditions on $f(x)$ between 0 and L under which you can differentiate the regular Fourier series (the one with sines and cosines) term by term to find the Fourier series of $\partial f/\partial t$? Include a brief explanation of why your answer is correct.

11.201 The air inside a flute obeys the wave equation with boundary conditions $\partial s/\partial x(0, t) = \partial s/\partial x(L, t) = 0$. The wave equation in this case is typically inhomogeneous because of someone blowing across an opening, creating a driving force that varies with x (position in the flute). Over a small period of time, it is reasonable to treat this driving function as a constant with respect to time. In this problem you will solve the equation $\partial^2 s/\partial x^2 - (1/c_s^2)(\partial^2 s/\partial t^2) = q(x)$ with the initial conditions $s(x, 0) = \partial s/\partial t(x, 0) = 0$.

(a) Explain why, for this problem, a cosine expansion will be easier to work with than a sine expansion.

[11]Strictly speaking your function must also be "piecewise smooth," meaning the derivative exists and is continuous at all but a finite number of points per period. Most functions you will encounter pass this test with no problem, but being continuous is a more serious issue, as the example in this problem illustrates.

(b) Expanding $s(x, t)$ and $q(x)$ into Fourier cosine series, write a differential equation for $a_{sn}(t)$.

(c) Find the general solution to this differential equation. You should find that the solution for general n doesn't work for $n = 0$ and you'll have to treat that case separately. Remember that $q(x)$ has no time dependence, so each a_{qn} is a constant.

(d) Now plug in the initial conditions to find the arbitrary constants and write the series solution $s(x, t)$.

(e) As a specific example, consider $q(x) = ke^{-(x-L/2)^2/x_0^2}$. This driving term represents a constant force that is strongest at the middle of the tube and rapidly drops off as you move towards the ends. (This is not realistic in several ways, the most obvious of which is that the hole in a concert flute is not at the middle, but it nonetheless gives a qualitative idea of some of the behavior of flutes and other open tube instruments.) Use a computer to find the Fourier cosine expansion of this function and plot the 20^{th} partial sum of $s(x, t)$ as a function of x at several times t (choosing values for the constants). Describe how the function behaves over time. Based on the results you find, explain why this equation could not be a good model of the air in a flute for more than a short time.

11.202 A thin pipe of length L being uniformly heated along its length obeys the inhomogeneous heat equation $\partial u/\partial t = \alpha(\partial^2 u/\partial x^2) + Q$ where Q is a constant. The ends of the pipe are held at zero degrees and the pipe is initially at zero degrees everywhere. (Assume temperatures are in celsius.)

(a) Solve the PDE with these boundary and initial conditions.

(b) Take the limit as $t \to \infty$ of your answer to get the steady-state solution. If you neglect all terms whose amplitude is less than 1% of the amplitude of the first term, how many terms are left in your series? Sketch the shape of the steady-state solution using only those non-negligible terms.

11.203 **Exploration: Poisson's Equation—Part I**
The electric potential in a region with charges obeys Poisson's equation, which in Cartesian coordinates can be written as $\partial^2 V/\partial x^2 + \partial^2 V/\partial y^2 + \partial^2 V/\partial z^2 = -(1/\epsilon_0)\rho(x, y, z)$. In this problem you will solve Poisson's equation in a cube with boundary conditions $V(0, y, z) = V(L, y, z) = V(x, 0, z) = V(x, L, z) = V(x, y, 0) = V(x, y, L) = 0$. The charge distribution is given by $\rho(x, y, z) = \sin(\pi x/L)\sin\left(2\pi y/L\right)\sin(3\pi z/L)$.

(a) Write Poisson's equation with this charge distribution. Next, expand V in a Fourier sine series in x. When you plug this into Poisson's equation you should get a PDE for the Fourier coefficients $b_{vn}(y, z)$. Explain why the solution to this PDE will be $b_{vn} = 0$ for all but one value of n. Write the PDE for $b_{vn}(y, z)$ for that one value.

(b) Do a Fourier sine expansion of your b-variable in y. The notation becomes a bit strained at this point, but you can call the coefficients of this new expansion b_{bvn}. The result of this expansion should be to turn the equation from Part (a) into an ODE for $b_{bvn}(z)$. Once again you should find that the solution is $b_{bvn} = 0$ for all but one value of n. Write the ODE for that one value.

(c) Finally, do a Fourier sine expansion in z and solve the problem to find $V(x, y, z)$. Your answer should be in closed form, not a series. Plug this answer back in to Poisson's equation and show that it is a solution to the PDE and to the boundary conditions.

11.204 **Exploration: Poisson's Equation—Part II**
[This problem depends on Problem 11.203.]
Solve Poisson's equation for the charge distribution $\rho(x, y, z) = \sin(\pi x/L)\sin\left(2\pi y/L\right) z$. The process will be the same as in the last problem, except that for the last sine series you will have to expand both the right and left-hand sides of the equation, and your final answer will be in the form of a series.

11.205 **Exploration: A driven drum**
In Section 11.6 we solved the wave equation on a circular drum of radius a in polar coordinates, and we found that the normal modes were Bessel functions. If the drum is being excited by an external source (imagine such a thing!) then it obeys an inhomogeneous wave equation

$$\frac{\partial^2 z}{\partial t^2} - v^2\left(\frac{\partial^2 z}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial z}{\partial \rho}\right) = \begin{cases} \kappa\cos(\omega t) & 0 \le \rho \le a/2 \\ 0 & \rho > a/2 \end{cases}$$

Since everything in the problem depends only on ρ we eliminated the ϕ-dependence from the Laplacian. The drum is clamped down so $z = 0$ at the outer edge, and the initial conditions are $z(\rho, 0) = \partial z/\partial t(\rho, 0) = 0$.

(a) To use eigenfunction expansion, we need the right eigenfunctions: a set of normal modes $R_n(\rho)$ with the property that $d^2R_n/d\rho^2 + (1/\rho)(dR_n/d\rho) = qR_n(\rho)$ for some proportionality constant q. For positive q-values, this leads to modified (or "hyperbolic") Bessel functions. Explain why these functions cannot be valid solutions for our drum.

(b) We therefore assume a negative proportionality constant: replace q with $-s^2$ and solve the resulting ODE for the functions $R_n(\rho)$. Use the implicit boundary condition to eliminate one arbitrary constant, and the explicit boundary condition to constrain the values of s. You do *not* need to apply the initial conditions at this stage. The Bessel functions you are left with as solutions for $R_n(\rho)$ are the eigenfunctions you will use for the method of eigenfunction expansion.

(c) Now return to the inhomogeneous wave equation. Expand both sides in a Fourier-Bessel expansion using the eigenfunctions you found in Part (b). The result should be an ODE for the coefficients $A_{zn}(t)$.

(d) The resulting differential equation looks much less intimidating if you rewrite the right side in terms of a new constant:

$$\gamma_n = \frac{2}{J_1^2(\alpha_{0,n})}\int_0^{1/2} J_0(\alpha_{0,n}u)u\,du$$

There is no simple analytical answer for this integral, but you can find a numerical value for γ_n for any particular n. Rewrite your answer to Part (c) using γ_n.

(e) Solve this ODE with the initial conditions $A_{zn}(0) = \partial A_{zn}/\partial t(0) = 0$.

(f) Plug your answer for $A_{zn}(t)$ into your Fourier-Bessel expansion for $z(\rho, t)$ to get the solution.

(g) Use the 20th partial sum of your series solution to make a 3D plot of the shape of the drumhead at various times. Describe how it evolves in time.

11.10 The Method of Fourier Transforms

The previous section used a Fourier series to turn a derivative into a multiplication, which turned a PDE into an ODE. For a non-periodic function on an infinite domain, you can accomplish the same thing using a Fourier transform instead of a Fourier series.

11.10.1 Discovery Exercise: The Method of Fourier Transforms

We have seen that the temperature in a bar obeys the heat equation:

$$\frac{\partial u}{\partial t} = \alpha\frac{\partial^2 u}{\partial x^2} \tag{11.10.1}$$

Now consider the temperature $u(x, t)$ of an *infinitely long* bar.

1. When you used the method of eigenfunction expansions to solve this problem for a finite bar (Exercise 11.9.1), you began by expanding the unknown solution $u(x, t)$ in a Fourier series. Explain why you cannot do the same thing in this case.

You can take an approach that is similar to the method of eigenfunction expansion, but in this case you will use a Fourier *transform* instead of a Fourier series. You begin by taking a Fourier transform of both sides of Equation 11.10.1. Using $\mathcal{F}$ to designate a Fourier transform with respect to x, this gives:

$$\mathcal{F}\left[\frac{\partial u}{\partial t}\right] = \mathcal{F}\left[\alpha\frac{\partial^2 u}{\partial x^2}\right] \tag{11.10.2}$$

2. The u you are looking for is a function of x and t. When you solve Equation 11.10.2 you will find a new function $\mathcal{F}[u]$. What will that be a function of? (It's not a function of u. That's the original function you're taking the Fourier transform of.)
3. One property of Fourier transforms is "linearity" which tells us that, in general, $\mathcal{F}[af + bg] = a\mathcal{F}(f) + b\mathcal{F}(g)$. Another property of Fourier transforms is that $\mathcal{F}\left[\partial^2 f/\partial x^2\right] = -p^2\mathcal{F}[f]$. Apply these properties (in order) to the right side of Equation 11.10.2.
4. Another property of Fourier transforms is that, if the Fourier transform is with respect to x and the derivative is with respect to t, you can move the derivative in and out of the transform: $\mathcal{F}\left[\partial f/\partial t\right] = \frac{\partial}{\partial t}\mathcal{F}[f]$. Apply this property to the left side of Equation 11.10.2.

See Check Yourself #78 in Appendix L

5. Solve this first-order differential equation to find $\mathcal{F}[u]$ as a function of p and t. Your solution will involve an arbitrary function $g(p)$.
6. Use the formula for an inverse Fourier transform to write the general solution $u(x, t)$ as an integral. (Do not evaluate the integral.) See Appendix G for the Fourier transform and inverse transform formulas. Your answer should depend on x and t. (Even though p appears in the answer, it only appears inside a definite integral, so the answer is *not* a function of p.)
7. What additional information would you need to solve for the arbitrary function $g(p)$ and thus get a particular solution to this PDE?

11.10.2 Explanation: The Method of Fourier Transforms

Recall from Chapter 9 that a Fourier *series* always represents a periodic function. If a function is defined on a finite domain, you can make a Fourier series for it by periodically extending it over the whole real line. But for a non-periodic function on an infinite domain, a Fourier *transform* is needed instead.

In the last section we expanded partial differential equations in Fourier series; in this section we use the "method of transforms." Watch how the example below parallels the method of eigenfunction expansion, but solves a problem that is defined on an infinite domain.

Notation and Properties of Fourier Transforms

Fourier transforms are discussed in Section 9.6, and the formulas are collected in Appendix G. But we need to raise a few issues that were not mentioned in that chapter. First, we need a bit of new notation. We will use $\hat{f}(p)$ for the Fourier transform of $f(x)$ just as we did in Chapter 9, but we also need a way of representing the Fourier transform of a larger expression. We will use the symbol $\mathcal{F}$.

More substantially, in this section we will be taking Fourier transforms of multivariate functions. These are not really multivariate Fourier transforms; we are taking the Fourier transform with respect to x, treating t as a constant.

$$\mathcal{F}\left[f(x,t)\right] = \hat{f}(p,t)$$

Most importantly, our work here requires a few properties of Fourier transforms. Given that a Fourier transform represents a function as an integral over terms of the form $\hat{f}(p,t)e^{ipx}$, the following two properties are not too surprising:

$$\mathcal{F}\left[\frac{\partial^{(n)} f}{\partial t^{(n)}}\right] = \frac{\partial^{(n)}}{\partial t^{(n)}}\mathcal{F}[f] \tag{11.10.3}$$

$$\mathcal{F}\left[\frac{\partial^{(n)} f}{\partial x^{(n)}}\right] = (ip)^n\mathcal{F}[f] \tag{11.10.4}$$

The second formula turns a derivative with respect to x into a multiplication, and therefore turns a partial differential equation into an ordinary differential equation, just as Fourier *series* did in the previous section.

Finally, we will need the "linearity" property of Fourier transforms:

$$\mathcal{F}[af+bg]=aF(f)+bF(g)$$

The Problem

In order to highlight the similarity between this method and the previous one, we're going to solve essentially the same problem: a wave on a one-dimensional string driven by an arbitrary force function.

$$\frac{\partial^2 y}{\partial x^2}-\frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}=q(x,t) \tag{11.10.5}$$

$$y(x,0)=f(x),\ \frac{\partial y}{\partial t}(x,0)=g(x)$$

However, our new string is infinitely long. This change pushes us from a Fourier *series* to a Fourier *transform.*

We are going to Fourier transform all three of these equations before we're through. Of course, we can only use this method in this way if it is possible to Fourier transform all the relevant functions! At the end of this section we will talk about some of the limitations that restriction imposes.

The Solution

We begin by taking the Fourier transform of both sides of Equation 11.10.5:

$$\mathcal{F}\left[\frac{\partial^2 y}{\partial x^2}-\frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}\right]=\mathcal{F}[q]$$

Applying the linearity property first, we write:

$$\mathcal{F}\left[\frac{\partial^2 y}{\partial x^2}\right]-\frac{1}{v^2}\mathcal{F}\left[\frac{\partial^2 y}{\partial t^2}\right]=\mathcal{F}[q] \tag{11.10.6}$$

Now our derivative properties come into play. Equations 11.10.3 and 11.10.4 turn this differential equation into $-p^2\hat{y}-(1/v^2)(\partial^2\hat{y}/\partial t^2)=\hat{q}$. This is the key step: a second derivative with respect to x has become a multiplication by p^2. This equation can be written more simply as:

$$\frac{\partial^2\hat{y}}{\partial t^2}+v^2p^2\hat{y}=-v^2\hat{q} \tag{11.10.7}$$

It may look like we have just traded our old $y(x,t)$ PDE for a $\hat{y}(p,t)$ PDE. But our new equation has derivatives only with respect to t. The variable p in this equation acts like n in the eigenfunction expansion: for any given value of p, we have an ODE in t that we can solve by hand or by computer. The result will be the function $\hat{y}(p,t)$, the Fourier transform of the function we are looking for.

Just as you can use a series solution by evaluating as many partial sums as needed, you can often use a Fourier transform solution by finding numerical approximations to the inverse

Fourier transform. In some cases you will be able to evaluate the integral explicitly to find a closed-form solution for $y(x,t)$.

A Sample Driving Function

As a sample force, consider the effect of hanging a small weight from our infinitely long string:

$$\frac{\partial^2 y}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} = \begin{cases} Q & -L < x < L \\ 0 & \textit{elsewhere} \end{cases} \tag{11.10.8}$$

The driving force in this case is independent of time. Notice that we are neglecting any forces other than this small weight, so the rest of the string will not fall until it is pulled down by the weighted part. Before starting the problem take a moment to think about what you would expect the solution to look like in the simplest case, where the string starts off perfectly still and horizontal.

Throughout this calculation there will be a number of steps such as taking a Fourier transform or solving an ODE that you could solve either by hand or on a computer. We'll just show the results as needed and in Problem 11.206 you'll fill in the missing calculations.

We take the Fourier transform of the right hand side of Equation 11.10.8 and plug it into Equation 11.10.7.

$$\frac{\partial^2 \hat{y}}{\partial t^2} + v^2 p^2 \hat{y} = -\frac{Qv^2}{\pi p}\sin(Lp) \tag{11.10.9}$$

Since the inhomogeneous part of this equation has no t dependence, it's really just the equation $\hat{y}''(t) + a\hat{y}(t) = b$ where a and b act as constants. (They depend on p but not on t.) You can solve it with guess and check and end up here.

$$\hat{y}(p,t) = A(p)\sin(pvt) + B(p)\cos(pvt) - \frac{Q}{\pi p^3}\sin(Lp) \tag{11.10.10}$$

The arbitrary "constants" A and B are constants with respect to t, but they are functions of p and we have labeled them as such. They will be determined by the initial conditions, just as A_n and B_n were in the series expansions of the previous section. In Problem 11.206 you will solve this for the simplest case $y(x,0) = \partial y/\partial t(x,0) = 0$ and show that $B(p) = (Q/\pi p^3)\sin(Lp)$ and $A(p) = 0$. So $\hat{y}(p,t) = (Q/\pi p^3)\sin(Lp)\left(\cos(pvt) - 1\right)$. This can be simplified with a trig identity to become $\hat{y}(p,t) = -(2Q/\pi p^3)\sin(Lp)\sin^2\left(pvt/2\right)$. The solution $y(x,t)$ is the inverse Fourier transform of $\hat{y}(p,t)$.

$$y(x,t) = -\frac{2Q}{\pi}\int_{-\infty}^{\infty}\frac{\sin(Lp)}{p^3}\sin^2\left(\frac{pvt}{2}\right)e^{ipx}\,dp \tag{11.10.11}$$

This inverse Fourier transform can be calculated analytically, but the result is messy because you get different functions in different domains. With the aid of a computer, however, we can get a clear—and physically unsurprising—picture of the result. At early times the weight pulls the region around it down into a parabola. At later times the weighted part makes a triangle, with the straight lines at the top and edges smoothly connected by small parabolas. In the regions $x > d + vt$ and $x < -d - vt$, the function $y(x,t)$ is zero because the effect of the weight hasn't yet reached the string.

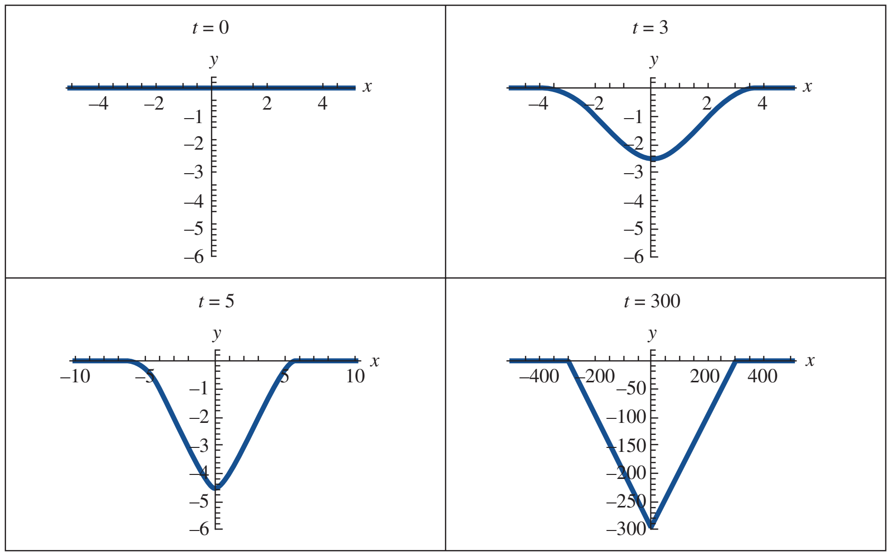


FIGURE 11.7 The solution for an infinite string continually pulled down by a weight near the origin. These figures represent the solution given above with $Q = L = v = 1$.

EXAMPLE Using a Fourier Transform to Solve a PDE

Solve the differential equation:

$$-9\frac{\partial^2 y}{\partial x^2} + 4\frac{\partial y}{\partial t} + 5y = \begin{cases} 1 & -1 \le x \le 1 \\ 0 & |x| > 1 \end{cases}$$

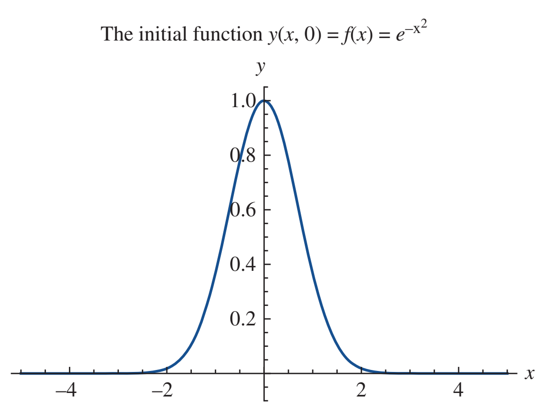
on the domain $-\infty \le x \le \infty$ for all $t \ge 0$ with initial condition $y(x,0) = f(x) = e^{-x^2}$.

Before solving this let's consider what kind of behavior we expect. If we rewrite this as

$$4\frac{\partial y}{\partial t} = 9\frac{\partial^2 y}{\partial x^2} - 5y + \begin{cases} 1 & -1 \le x \le 1 \\ 0 & |x| > 1 \end{cases}$$

then we can see the function will tend to decrease when it is concave down and/or positive. In addition, it will always have a tendency to increase in the range $-1 \le x \le 1$.

The initial position is shown below.



Near $x = 0$, the negative concavity and positive values will push downward more strongly than the driving term pushes upward, so the peak in the middle will decrease until these forces cancel. At larger values of x the upward push from the positive concavity is larger than the downward push from the positive values of y (you can check this), so the function will initially increase there. At very large values of x there is no driving force, and the concavity and y-values are near zero, so there will be little initial movement.

Now let's find the actual solution. We begin by taking the Fourier transform of both sides.

$$\mathcal{F}\left[-9\frac{\partial^2 y}{\partial x^2}+4\frac{\partial y}{\partial t}+5y\right]=\mathcal{F}\left[\left\{\begin{array}{cc}1 & -1\le x\le 1\\ 0 & |x|>1\end{array}\right.\right]$$

Apply the linearity of Fourier transforms on the left.

$$-9\mathcal{F}\left[\frac{\partial^2 y}{\partial x^2}\right]+4\mathcal{F}\left[\frac{\partial y}{\partial t}\right]+5\mathcal{F}[y]=\mathcal{F}\left[\left\{\begin{array}{cc}1 & -1\le x\le 1\\ 0 & |x|>1\end{array}\right.\right]$$

Next come our derivative properties.

$$9p^2\hat{y}+4\frac{\partial \hat{y}}{\partial t}+5\hat{y}=\mathcal{F}\left[\left\{\begin{array}{cc}1 & -1\le x\le 1\\ 0 & |x|>1\end{array}\right.\right]$$

which we can rearrange as:

$$4\frac{\partial \hat{y}}{\partial t}+(5+9p^2)\hat{y}=\mathcal{F}\left[\left\{\begin{array}{cc}1 & -1\le x\le 1\\ 0 & |x|>1\end{array}\right.\right]$$

The right side of this equation is an easy enough Fourier transform to evaluate, and we're skipping the integration steps here.

$$4\frac{\partial \hat{y}}{\partial t}+(5+9p^2)\hat{y}=\frac{\sin p}{\pi p}$$

Although the constants are ugly, this is just a separable first-order ODE. Once again we can solve it by hand or by software.

$$\hat{y}=C(p)e^{-(9p^2+5)t/4}+\frac{\sin p}{\pi p(9p^2+5)} \tag{11.10.12}$$

Next we apply the initial condition to find $C(p)$. You can find the Fourier transform of the initial condition in Appendix G.

$$\hat{f}(p)=\frac{1}{2\sqrt{\pi}}e^{-p^2/4}$$

Setting $\hat{y}(p,0)=\hat{f}(p)$ gives:

$$C(p)=\frac{1}{2\sqrt{\pi}}e^{-p^2/4}-\frac{\sin p}{\pi p(9p^2+5)} \tag{11.10.13}$$

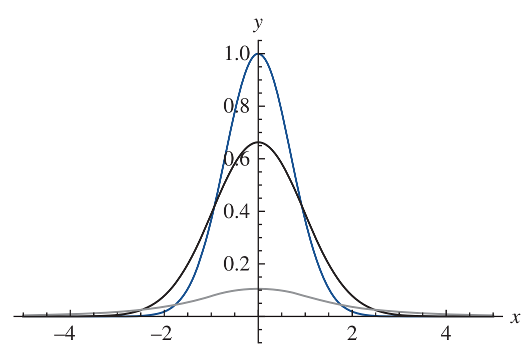
Finally, we can get the solution $y(x,t)$ by plugging Equation 11.10.13 into 11.10.12 and taking the inverse Fourier transform.

$$y(x,t) = \int_{-\infty}^{\infty} \left[\left(\frac{1}{2\sqrt{\pi}} e^{-p^2/4} - \frac{\sin p}{\pi p(9p^2+5)} \right) e^{-(9p^2+5)t/4} + \frac{\sin p}{\pi p(9p^2+5)} \right] e^{ipx} dp$$

There's no simple way to evaluate this integral in the general case, but we can understand its behavior by looking at the time dependence. At $t = 0$ the last two terms cancel and we are left with the inverse Fourier transform of $e^{-p^2/4}/(2\sqrt{\pi})$, which reproduces the initial condition $y(x,0) = e^{-x^2}$. (If the solution didn't reproduce the initial conditions when $t = 0$ we would know we had made a mistake.) At late times the term $e^{-\frac{9p^2+5}{4}t}$ goes to zero and we are left with the inverse Fourier transform of the last term, which can be analytically evaluated on a computer to give

$$\lim_{t\to\infty} y(x,t) = \begin{cases} \frac{1}{10}\left(e^{\sqrt{5}/3} - e^{-\sqrt{5}/3}\right) e^{(\sqrt{5}/3)x} & x < -1 \\ \frac{1}{10}\left[2 - e^{-\sqrt{5}/3}\left(e^{(\sqrt{5}/3)x} + e^{-(\sqrt{5}/3)x}\right)\right] & -1 \le x \le 1 \\ \frac{1}{10}\left(e^{\sqrt{5}/3} - e^{-\sqrt{5}/3}\right) e^{-(\sqrt{5}/3)x} & x > 1 \end{cases} \qquad (11.10.14)$$

As complicated as that looks, it is just some numbers multiplied by some exponential functions.



The solution 11.10.14 at $t = 0$ (blue), $t = .1$ (black), and in the limit $t \to \infty$ (gray).

This picture generally confirms the predictions we made earlier. The peak at $x = 0$ shrinks and the tail at large $|x|$ initially grows. We were not able to predict ahead of time that in some places the function would grow for a while and then come back down some. (Look for example at $x = 2$.) Moreover, we now have an exact function with numerical values for the late time limit of the function.

Stepping Back

The method of transforms boils down to a five-step process.

1. Take the Fourier transform of both sides of the differential equation.
2. Use the rules of Fourier transforms to simplify the resulting equation, which should turn one derivative operation into a multiplication. If you started with a two-variable partial differential equation, you are now effectively left with a one-variable, or ordinary, differential equation.

3. Solve the resulting differential equation.
4. Plug in the initial conditions. (You will need to Fourier transform these as well.)
5. If possible, take an *inverse* Fourier transform to find the function you were originally looking for. If it is not possible to do this analytically, you can still approximate the relevant integral numerically.

If a function is defined on the entire number line, and has no special symmetry, its Fourier transform will be expressed in terms of sines and cosines—or, equivalently, in complex exponentials as we did above. If a function is defined on *half* the number line, from 0 to 8, then you can create an "odd extension" of that function and use a Fourier sine transform, or an "even extension" with a Fourier cosine transform. You'll work through an example of this technique in Problem 11.222.

The method of transforms can be used in situations where separation of variables cannot, such as inhomogeneous equations. It can also be used in situations where series expansions cannot: namely, infinite non-periodic functions. There are, however, two basic requirements that must be met in order to use a Fourier transform with respect to a variable x.

First, x must not appear in any of the coefficients of your equation. We've seen the simple formulas for $\mathcal{F}\left[\partial f/\partial x\right]$ and $\mathcal{F}\left[\partial f/\partial t\right]$. The corresponding formulas for terms like $\mathcal{F}\left[xf\right]$ or $\mathcal{F}\left[x(\partial f/\partial x)\right]$ are not simple, and they are not useful for solving PDEs.

Second, you must be able to take the Fourier transform. In Chapter 9 we discuss the conditions required for $f(x)$ to have a Fourier transform. The most important is that $\int_{-\infty}^{\infty}\left|f(x)\right|dx$ must be defined; that restriction, in turn, means that $f(x)$ must approach zero as x approaches $\pm\infty$. This means that some problems cannot be approached with a Fourier transform. It also means that we almost always do our Fourier transforms in x rather than t, since we are rarely guaranteed that a function will approach zero as $t \to \infty$. To turn time derivatives into multiplications you can often use a "Laplace transform," which is the subject of the next section.

Finally, we should note that a Fourier transform is particularly useful for simplifying equations like the wave equation because the only spatial derivative is second order. When you take the second derivative of e^{ipx} you get $-p^2 e^{ipx}$ which gives you a simple, *real* ODE for $\hat{f}$. For a first-order spatial derivative, the Fourier transform would bring down an imaginary coefficient. You can still use this technique for such equations, but you have to work harder to physically interpret the results: see Problem 11.219.

11.10.3 Problems: The Method of Fourier Transforms

For some of the Fourier transforms in this section, you should be able to evaluate them by hand. (See Appendix G for the formula.) For some you will need a computer. (Such problems are marked with a computer icon.) And for some, the following formulas will be useful. (If you're not familiar with the Dirac delta function, see Appendix K.)

$$\mathcal{F}[1] = \delta(p) \text{ (the Dirac delta function)}$$
$$\mathcal{F}\left[e^{-(x/k)^2}\right] = \frac{k}{2\sqrt{\pi}}e^{-(kp/2)^2} \tag{11.10.15}$$

It will also help to keep in mind that a "constant" depends on what variable you're working with. If you are taking a Fourier transform with respect to x, then $t^2 \sin t$ acts as a constant. If you are taking a derivative with respect to t, then $\delta(x)$ acts as a constant.

Unless otherwise specified, your final answer will be the Fourier transform of the PDE solution. Remember that if your answer has a delta function in it you can simplify it by replacing anything of the form $f(p)\delta(p)$ with $f(0)\delta(p)$.

11.206 In this problem you'll fill in some of the calculations from the Explanation (Section 11.10.2).

(a) To derive Equation 11.10.9 we needed the Fourier transform of the right hand side of Equation 11.10.8. Evaluate this Fourier transform directly using the formula for a Fourier transform in Appendix G.

(b) Verify that Equation 11.10.10 is a solution to Equation 11.10.9.

(c) The initial conditions $y(x,0) = \partial y/\partial t(x,0) = 0$ can trivially be Fourier transformed into $\hat{y}(p,0) = \partial\hat{y}/\partial t(p,0) = 0$. Plug those conditions into Equation 11.10.10 and derive the formulas for $A(p)$ and $B(p)$ given in the Explanation.

11.207 *[This problem depends on Problem 11.206.]* Use a computer to evaluate the inverse Fourier transform 11.10.11. (You can do it by hand if you prefer, but it's a bit of a mess.) Assume $t > 2L/v$ and simplify the expression for $y(x,t)$ in each of the following regions: $0 < x < L$, $L < x < vt - L$, $vt - L < x < vt + L$, $x > vt + L$. In each case you should find a polynomial of degree 2 or less in x. As a check on your answers, reproduce the last frame of Figure 11.7.

11.208 **Walk-Through: The Method of Fourier Transforms.** In this problem you will solve the following partial differential equation on the domain $-\infty < x < \infty$, $0 \le t < \infty$.

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} + u = e^{-x^2}$$

$$u(x,0) = \begin{cases} 1+x & -1 \le x \le 0 \\ 1-x & 0 < x \le 1 \\ 0 & \textit{elsewhere} \end{cases}$$

(a) Take the Fourier transform of both sides of this PDE. On the left side you will use Equations 11.10.3–11.10.4 to get an expression that depends on $\hat{u}(p,t)$ and $\partial\hat{u}/\partial t$. On the right you should get a function of p and/or t. Equations 11.10.15 may be helpful.

(b) Take the Fourier transform of the initial condition to find the initial condition for $\hat{u}(p,t)$.

(c) Solve the differential equation you wrote in Part (a) with the initial condition you found in Part (b) to get the solution $\hat{u}(p,t)$. (*Warning: the answer will be long and messy.*)

(d) Write the solution $u(x,t)$ as an integral over p.

Solve Problems 11.209–11.213 on the domain $-\infty < x < \infty$ using the method of transforms. When the initial conditions are given as arbitrary functions the Fourier transforms of those functions will appear as part of your solution. It may help to first work through Problem 11.208 as a model.

11.209 $\partial y/\partial t - c^2(\partial^2 y/\partial x^2) = e^{-t}$, $y(x,0) = f(x)$

11.210 $\partial y/\partial t - c^2(\partial^2 y/\partial x^2) = e^{-t}$, $y(x,0) = e^{-x^2}$

11.211 $\partial y/\partial t - c^2(\partial^2 y/\partial x^2) + y = \kappa$, $y(x,0) = 0$. (You should be able to inverse Fourier transform your solution and give your answer as a function $y(x,t)$.)

11.212 $$\frac{\partial y}{\partial t} - c^2\frac{\partial^2 y}{\partial x^2} + y = \begin{cases} Q & -1 < x < 1 \\ 0 & \textit{elsewhere} \end{cases}, \quad y(x,0) = 0$$

11.213 $$\frac{\partial^2 y}{\partial t^2} - c^2\frac{\partial^2 y}{\partial x^2} + y = e^{-x^2}\cos(\omega t), \quad y(x,0) = 0,$$
$$\frac{\partial y}{\partial t}(x,0) = \begin{cases} x & 0 \le x \le 1/2 \\ 1-x & 1/2 < x \le 1 \\ 0 & \textit{elsewhere} \end{cases}$$

For Problems 11.214–11.217 solve the equation $\partial^2 y/\partial x^2 - (1/v^2)(\partial^2 y/\partial t^2) = q(x,t)$ with initial conditions $y(x,0) = f(x)$, $\dot{y}(x,0) = g(x)$. Equations 11.10.15 may be needed.

11.214 $q(x,t) = 0$, $f(x) = \begin{cases} F & -1 < x < 1 \\ 0 & \textit{otherwise} \end{cases}$, $g(x) = 0$

11.215 $q(x,t) = \begin{cases} -Q & -1 \le x < 0 \\ Q & 0 \le x \le 1 \\ 0 & \textit{elsewhere} \end{cases}$, $f(x) = g(x) = 0$

11.216 $q(x,t) = \kappa e^{-(x/d)^2}\sin(\omega t)$, $f(x) = 0$, $g(x) = 0$

11.217 $q(x,t) = -Q$, $f(x) = 0$, $g(x) = Ge^{-(x/d)^2}$. The letters Q and G stand for constants. In general any answer with $f(p)\delta(p)$ in it can be simplified by replacing $f(p)$ with the value $f(0)$, since $\delta(p) = 0$ for all $p \ne 0$. In this case, however, you should have terms with $\delta(p)/p^2$, which is undefined at $p = 0$. Expand $\cos(vpt)$ in your answer in a Maclaurin series in p and simplify the result. You should get something where the coefficient in front of $\delta(p)$ is non-singular, and you can replace that coefficient with its value at $p = 0$.

11.218 **(a)** Solve the equation $\partial^2 y/\partial x^2 - (1/v^2)(\partial^2 y/\partial t^2) = 0$ with initial conditions $y(x, 0) = Fe^{-(x/d)^2}$, $dy/dt(x, 0) = 0$. Your answer should be an equation for $\hat{y}(p, t)$ with no arbitrary constants.

(b) You're going to take the inverse Fourier transform of your solution, but to do so it helps to start with the following trick. Rewrite the second half of Equation 11.10.15 in the form $e^{-(x/k)^2}$ =<*an integral*>.

(c) Now take the inverse Fourier transform of your answer from Part (a) to find $y(x, t)$. Start by writing the integral for the Fourier transform. Then do a variable substitution to make it look like the integral you wrote in Part (b). Use that to evaluate the integral and then reverse your substitution to get a final answer in terms of x and t.

(d) Verify that your solution $y(x, t)$ satisfies the differential equation and initial conditions.

11.219 The complex exponential function e^{ipx} is an eigenfunction of $\partial y/\partial x$, but with an imaginary eigenvalue. This makes it harder to interpret the results of the method of transforms when single derivatives are involved. To illustrate this, use the method of transforms to solve the equation

$$\frac{\partial y}{\partial t} - \frac{\partial^2 y}{\partial x^2} - \frac{\partial y}{\partial x} = \begin{cases} 1 & -1 < x < 1 \\ 0 & \textit{elsewhere} \end{cases}$$
$$\text{with initial condition } y(x, 0) = 0$$

Your final result should be the Fourier transform $\hat{y}(p, t)$, which will be a complex function that cannot easily be inverse Fourier transformed and admits no obvious physical interpretation.

11.220 An infinite rod being continually heated by a localized source at the origin obeys the differential equation $\partial u/\partial t - \alpha(\partial^2 u/\partial x^2) = ce^{-(x/d)^2}$ with initial condition $u(x, 0) = 0$.

(a) Solve for the temperature $u(x, t)$. Your answer will be in the form of a Fourier transform $\hat{u}(p, t)$.

(b) Take the inverse Fourier transform of your answer to get the function $u(x, t)$. Plot the temperature distribution at several different times and describe how it is evolving over time.

11.221 The electric potential in a region is given by Poisson's equation $\partial^2 V/\partial x^2 + \partial^2 V/\partial y^2 + \partial^2 V/\partial z^2 = (1/\varepsilon_0)\rho(x, y, z)$. An infinitely long bar, $0 \le x \le 1$, $0 \le y \le 1$, $-\infty < z < \infty$ has charge density $\rho(x, y, z) = \sin(\pi x)\sin(\pi y)e^{-z^2}$. Assume the edges of the bar are grounded so $V(0, y, z) = V(1, y, z) = V(x, 0, z) = V(x, 1, z) = 0$ and assume the potential goes to zero in the limits $z \to \infty$ and $z \to -\infty$.

(a) Write Poisson's equation for this charge distribution and take the Fourier transform of both sides to turn it into a PDE for $\hat{V}(x, y, p)$.

(b) Plug in a guess of the form $\hat{V} = C(p)\sin(\pi x)\sin(\pi y)$ and solve for $C(p)$.

11.222 **Exploration: Using Fourier Sine Transforms** For a variable with an infinite domain $-\infty < x < \infty$ you can take a Fourier transform as described in this section. For a variable with a semi infinite domain $0 < x < \infty$, however, it is often more useful to take a Fourier sine transform. The formulas for a Fourier sine transform and its inverse are in Appendix G. As long as the boundary condition on f is that $f(0, t) = 0$ the rules for Fourier sine transforms of second derivatives are the same as the ones for regular Fourier transforms.

$$F_s\left[\frac{\partial^2 f}{\partial x^2}\right] = -p^2 F_s[f], \quad F_s\left[\frac{\partial^2 f}{\partial t^2}\right] = \frac{\partial^2 F_s[f]}{\partial t^2}$$

In this problem you will use a Fourier sine transform to solve the wave equation $\partial^2 y/\partial x^2 - (1/v^2)(\partial^2 y/\partial t^2) = 0$ on the interval $0 < x < \infty$ with initial conditions

$$y(x, 0) = \begin{cases} x & 0 < x < d \\ 2d - x & d < x < 2d \\ 0 & x > 2d \end{cases}, \quad \dot{y}(x, 0) = 0$$

and boundary condition $y(0, t) = 0$.

(a) Take the Fourier sine transform of the wave equation to rewrite it as an ordinary differential equation for $\hat{y}_s(p, t)$.

(b) Find the Fourier sine transform of the initial conditions. These will be the initial conditions for the ODE you derived in Part (a).

(c) Solve the ODE using these initial conditions and use the result to write the solution $\hat{y}_s(t)$.

11.11 The Method of Laplace Transforms

Laplace transforms are an essential tool in many branches of engineering. When we use them to solve PDEs they act much like Fourier transforms, turning a derivative into a multiplication, but they generally operate on the time variable rather than a spatial variable.

This section completes our chapter on PDEs. There are many techniques we did not discuss, but we believe we have presented you with the most important approaches. Going through this section as you have gone through the ones before, pay attention to the details so you can apply the technique yourself. But after the details, the "Stepping Back" will help you figure out which technique to apply to which problem. This important question is addressed more broadly in Appendix I, so you can look at a new equation and sort through the various methods that you have mastered.

11.11.1 Explanation: The Method of Laplace Transforms

We have seen that when a function is defined on a finite domain, it is sometimes useful to replace that function with a series expansion. We have also seen that when a function is defined on an *infinite* domain, we can use a transform—an integral—instead of a series.

Just as with series, the right transform may involve sines, cosines, and/or complex exponentials (Fourier), but other transforms may be called for in other circumstances. In this section we look at one of the most important examples, the Laplace transform. We also briefly discuss the general topic of finding the right transform for any given problem.

Laplace Transforms

Chapter 10 introduces Laplace transforms and their use in solving ODEs; Appendix H contains a table of Laplace transforms and some table-looking-up techniques. We're not going to review all that information here, so you may want to refresh yourself.

Solving a PDE by Laplace transform is very similar to solving a PDE by Fourier transform, except that we tend to use Laplace transforms on the time variable instead of spatial variables. Based on our work in the previous sections, you can probably see how we are going to put the following rules to use.

$$\mathcal{L}\left[\frac{\partial^{(n)}f(x,t)}{\partial x^{(n)}}\right] = \frac{\partial^{(n)}}{\partial x^{(n)}}\left(F(x,s)\right) \tag{11.11.1}$$

$$\mathcal{L}\left[\frac{\partial^{(n)}f(x,t)}{\partial t^{(n)}}\right] = s^n F(x,s) - s^{n-1}f(x,0) - s^{n-2}\frac{\partial f}{\partial t}(x,0) - s^{n-3}\frac{\partial^2 f}{\partial t^2}(x,0) - \ldots - \frac{\partial^{(n-1)}f}{\partial t^{(n-1)}}(x,0) \tag{11.11.2}$$

The first rule is not surprising: since we are using $\mathcal{L}\,[]$ to indicate a Laplace transform in the time variable, a derivative with respect to x can move in and out of the transform. (When we took Fourier transforms with respect to x, derivatives with respect to t followed a similar rule.)

The second rule is analogous to an eigenfunction relationship. It shows that inside a Laplace transform, taking the nth time derivative is equivalent to multiplying by s^n. So the Laplace transform serves the same purpose as the Fourier transform in the previous section, and the Fourier series in the section before that: by turning a derivative into a multiplication, it turns a PDE into an ODE. In this case, however, there are correction terms on the boundary that bring initial conditions into the calculations. (Similar corrective terms come into play with Fourier transforms on semi-infinite intervals.)

As with the Fourier transform, the final property we need is linearity: $\mathcal{L}[af + bg] = a\mathcal{L}[f] + b\mathcal{L}[g]$ where a and b are constants, f and g functions of time.

Temperature on a Semi-Infinite Bar

A bar extends from $x = 0$ to $x = \infty$. The left side of the bar is kept at temperature $u(0, t) = u_L$, and the rest of the bar starts at $u(x, 0) = 0$. Find the temperature distribution in the bar as a function of time, assuming it obeys the heat equation:

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}$$

As often occurs in problems with infinite domains, there is an "implicit" boundary condition that is never stated in the problem. The entire bar starts at 0°, and the heat applied to the left side must propagate at a finite speed through the bar; therefore, it is safe to assume that $\lim_{x \to \infty} u(x, t) = 0$.

We begin by taking the Laplace transform of both sides of the equation, using the linearity property to leave the constant α outside.

$$\mathcal{L}\left[\frac{\partial u}{\partial t}\right] = \alpha \mathcal{L}\left[\frac{\partial^2 u}{\partial x^2}\right]$$

Next we apply our derivative rules to both sides.

$$sU(x, s) - u(x, 0) = \alpha \frac{\partial^2}{\partial x^2} U(x, s)$$

(Note that the second term is the original function u, not the Laplace transform U.) Plugging in our initial condition $u(x, 0) = 0$, we can rewrite the problem as:

$$\frac{\partial^2 U}{\partial x^2} = \frac{s}{\alpha} U$$

Just as we saw with Fourier transforms in the previous section, a partial differential equation has turned into an effectively *ordinary* differential equation, since the derivative with respect to time has been replaced with a multiplication. We can solve this by inspection.

$$U(x, s) = A(s) e^{\sqrt{s/\alpha}\, x} + B(s) e^{-\sqrt{s/\alpha}\, x}$$

A and B must be constants with respect to x, but may be functions of s.

The "implicit" boundary condition says that in the limit as $x \to \infty$ the function $u(x, t)$—and therefore the transformed function $U(x, s)$—must approach zero. This condition kills the growing exponential term. Before we can apply the explicit boundary condition we must Laplace transform it as well. The Laplace transform of any constant function k is k/s, so $u(0, t) = u_L$ becomes $U(0, s) = u_L/s$, which gives $B(s) = u_L/s$. That allows us to write the answer $U(x, s)$.

$$U(x, s) = \frac{u_L}{s} e^{-\sqrt{s/\alpha}\, x} \tag{11.11.3}$$

Equation 11.11.3 is the Laplace transform of the function $u(x, t)$ that we're looking for. What can you do with *that?*

In some cases you stop there. Engineers who are used to working with Laplace transforms can read a lot into a solution in that form, and we provide some tips for interpreting Laplace transforms in Chapter 10. In some cases you can apply an inverse Laplace transform to find the actual function you're looking for. This process requires integration on the complex plane, so we defer it to Chapter 13. Our approach in this section will be electronic: we asked our computer for the inverse Laplace transform of Equation 11.11.3, and thus got the solution to this problem.

$$u(x,t) = u_L \operatorname{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right)$$

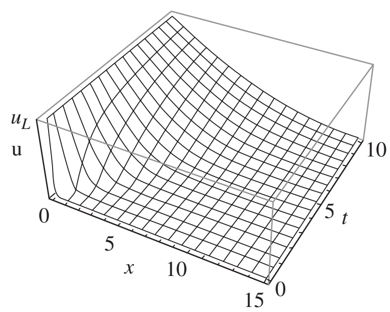


FIGURE 11.8 Temperature on a semi-infinite rod with zero initial temperature and a fixed boundary temperature. The effect of the boundary spreads from left to right, pulling the temperature at each point asymptotically up towards u_L.

As always we urge you not to panic at the sight of an unfamiliar function! Even without knowing anything about the "complementary error function" erfc(x) (which you can look up in Appendix K, or in Chapter 12 for more details), you can describe the temperature in the bar by looking at a computer-generated graph of the solution (Figure 11.8).

- At $x = 0$ the plot shows a temperature of u_L for all t-values. This was our boundary condition: the temperature at the left side of the bar is held constant.
- At $x = 15$ the temperature is uniformly zero. It will rise eventually, but in the domain of our picture ($0 \le t \le 10$) the heat from the left side has not yet had time to propagate that far.
- The in-between values are the most interesting. For instance, at $x = 5$, we see that the temperature stays at zero for a few seconds, until the heat from the left side reaches it. The temperature then starts to rise dramatically. Eventually it will approach u_L, as we can see happening already at lower x-values.

Before we leave this problem we should point out a peculiarity that you may not have noticed: the initial and boundary conditions contradict each other. The problem stipulated that $u(0,t) = u_L$ at all times, and also that $u(x,0) = 0$ at all x-values, but $u(0,0)$ cannot possibly be two different values! Maybe the entire bar was at $u = 0$ when the left side was suddenly brought into contact with an object of temperature u_L; it is only an approximation to say that the temperature at the boundary will rise instantaneously to match. As you have seen, we can solve the problem analytically despite this contradiction. But mathematical software packages sometimes get confused by such conflicts.

EXAMPLE Laplace Transform

Problem:
Solve

$$\frac{\partial v}{\partial t} + \frac{\partial v}{\partial x} = 1 \tag{11.11.4}$$

on the domain $0 \le x < \infty$, $0 \le t < \infty$ with initial condition $v(x,0) = 0$ and boundary condition $v(0,t) = 0$.

Solution:
We begin by taking the Laplace transform of both sides, applying the linearity and derivative properties on the left. The Laplace transform of 1 is $1/s$, so

$$sV(x,s) + \frac{\partial V(x,s)}{\partial x} = \frac{1}{s}$$

You can solve this by finding a complementary and a particular solution, with the result

$$V(x,s) = \frac{1}{s^2} + C(s)e^{-sx}$$

The boundary condition $v(0,t) = 0$ becomes $V(0,s) = 0$, which means

$$V(x,s) = \frac{1-e^{-sx}}{s^2} \tag{11.11.5}$$

You can easily find the inverse Laplace transform of Equation 11.11.5 on a computer:

$$v(x,t) = \begin{cases} t & t < x \\ x & t \geq x \end{cases} \tag{11.11.6}$$

It's an odd-looking solution, isn't it? You'll investigate it in Problem 11.230.

Choosing the Right Transform, Part 1: Remember the Eigenfunction!

We have seen that for a variable that is defined on a finite domain (or that is periodic) we can expand into a series; on an infinite domain we use a "transform," expanding into an integral. But we have also seen that there are different *kinds* of series and transforms, and you have to start by picking the right one.

The most important rule, whether you are doing series or transforms, is to remember the eigenfunction. The Laplace transform is based on a simple exponential, which is an eigenfunction of derivatives of any order. For instance, if you take the Laplace transform of the operator $\alpha(\partial^3 v/\partial t^3) + \beta(\partial^2 v/\partial t^2) + \gamma(\partial v/\partial t) + \delta v$, you get $(-\alpha s^3 + \beta s^2 - \gamma s + \delta)V$ plus some boundary terms that don't depend on t. Can you see the point of that simple exercise? If your problem includes derivatives of any order, multiplied by only constants, then a Laplace transform will turn the entire differential operator into a multiplication.

The Fourier transform is also based on an exponential function, albeit a complex one, and is therefore an eigenfunction of the same operators. So Fourier and Laplace transforms both work on the same differential operators: you choose one or the other based on the boundary conditions, as we will discuss below.

However, not all equations involve just derivatives multiplied by constants, so Fourier and Laplace transforms are not always useful. For instance, consider an equation based on the Laplacian in polar coordinates. We have seen that in the homogeneous case, such an equation leads to Bessel's equation, whose normal modes—found by separating variables and solving an ordinary differential equation—are Bessel functions. In the *inhomogeneous* case, where separation of variables doesn't work, we can expand such a problem in a "Hankel transform": an integral over Bessel functions. The Hankel transform has the property that $H_0\left(f'' + f'/x\right) = s^2 H_0(f)$, which allows it to simplify differential equations based on that operator.

Hankel transforms and many others are discussed in texts on partial differential equations.[12] We have chosen to focus on the two most important transforms, Fourier and Laplace. But if you understand the underlying principles that make these transforms useful, it isn't difficult to make a pretty good guess, given an unfamiliar problem, about what transform will be most likely to help.

[12] One that we got a lot out of is Asmar, Nakhle, *Partial Differential Equations with Fourier Series and Boundary Value Problems* 2^{nd} *edition*, Prentice Hall, 2004.

Choosing the Right Transform, Part 2: Fourier or Laplace?

As discussed above, Fourier and Laplace transforms both apply to the same differential operators. Nonetheless, one of these transforms will often succeed where the other fails, so it's generally important to choose the right one. We'll start with one big rule of thumb: not guaranteed, but easy to apply and generally useful.

We usually take Fourier transforms in space, and Laplace transforms in time.

Now we'll offer some specific limitations of both transforms. Along the way, we'll show how these limitations lead to that rule of thumb. That way, you'll know why the rule works—and when you need to make an exception.

1. **Laplace transforms can be defined on semi-infinite intervals (such as $0 \le x < \infty$) only; Fourier transforms can be defined on semi-infinite or infinite ($-\infty < x < \infty$) intervals.** The domain of time is often semi-infinite, whereas spatial variables are defined on all kinds of domains.
2. **A Fourier transform requires the integral $\int_{-\infty}^{\infty} f(x)dx$ to exist, which in turn requires $f(x)$ to approach zero at infinity.** It is common (although certainly not universal) for a function to approach zero as $x \to \infty$, but it is rare to get a guarantee up front about what will happen as $t \to \infty$. The Laplace transform, by contrast, exists as long as $f(t)$ grows exponentially or slower at late times. This is not very restrictive: it's rare to find a function that grows faster than e^t does!
3. **When you use a Laplace transform for an nth-order derivative, you need to know the function's value and its first $(n-1)$ derivatives at $t = 0$.** That's exactly what you tend to get from initial conditions. For instance, if you are solving the wave equation on a string, your initial conditions are generally the position and velocity at $t = 0$, which is just what you need for a Laplace transform. Boundary conditions don't go that way: you may get y on both ends, or $\partial y/\partial x$ on both ends, but you are less likely to get both variables on the same end.

There are a lot of rules to guide you in choosing the right method for a given differential equation. Our goal has been to present these rules in a common-sense way: you will work with these rules better, and remember them longer, if you understand where they come from. In the summary flow chart in Appendix I we forget all the "why" questions and just list the rules.

11.11.2 Problems: The Method of Laplace Transforms

11.223 **Walk-Through: The Method of Laplace Transforms.** In this problem you will solve the partial differential equation $\partial^2 u/\partial t^2 - \partial^2 u/\partial x^2 = \cos(t)\sin(\pi x)$ in the domain $0 \le x \le 1$, $0 \le t < \infty$ with boundary conditions $u(0,t) = 0$, $u(1,t) = t$ and initial condition $u(x,0) = -\sin(\pi x)$, $\partial u/\partial t(x,0) = 0$ using the method of Laplace transforms.

(a) Take the Laplace transform of both sides of this PDE. On the left side you will use Equations 11.11.1–11.11.2 to get an expression that depends on $U(x,s)$ and its spatial derivatives. On the right you should get a function of x and s.

(b) Take the Laplace transform of the boundary conditions to find the boundary conditions for $U(x,s)$.

(c) Solve the differential equation you wrote in Part (a) with the boundary conditions you found in Part (b) to get the solution $U(x,s)$.

Solve Problems 11.224–11.229 on the domain $0 \le x \le 1$ using the method of transforms. Your answer in most cases will be the Laplace transform $Y(x,s)$ of the solution $y(x,t)$. It may help to first work through Problem 11.223 as a model.

11.224 $\partial y/\partial t - c^2(\partial^2 y/\partial x^2) = e^{-t}$, $y(x,0) = y(0,t) = y(1,t) = 0$

11.225 $\partial^2 y/\partial t^2 - c^2(\partial^2 y/\partial x^2) = \sin t$, $y(x,0) = \partial y/\partial t(x,0) = y(0,t) = y(1,t) = 0$

11.226 $\partial^2 y/\partial t^2 + (\partial^2 y/\partial x^2) = e^{-t}$, $y(x,0) = \sin(\pi x)$, $\partial y/\partial t(x,0) = 0$, $y(0,t) = y(1,t) = 0$

11.227 $\partial y/\partial t - (\partial^2 y/\partial x^2) = 0$, $y(x,0) = 0$, $y(0,t) = t$, $y(1,t) = 0$

11.228 $\partial^2 y/\partial t^2 - \partial y/\partial t - \partial^2 y/\partial x^2 = \sin(t)\sin(\pi x)$, $y(0,t) = te^{-t}$, $y(1,t) = 0$, $y(x,0) = \sin(\pi x)$, $\partial y/\partial t(x,0) = 0$

11.229 $\partial y/\partial t - c^2(\partial^2 y/\partial x^2) = xt$, $y(x,0) = 0$, $y(0,t) = y(1,t) = 0$

11.230 In the Explanation (Section 11.11.1) we solved a differential equation and ended up with Equation 11.11.6. In this problem you'll consider what that solution looks like.

(a) Draw sketches of $v(x)$ on the domain $0 \le x \le 10$ at $t = 5$ and $t = 6$. (This should be pretty quick and easy.)

(b) What is $\partial v/\partial x$ at the point $x = 4$ in both sketches? (You can see the answer by looking.)

(c) How does v at the point $x = 4$ move or change between the two sketches? Based on that, what is $\partial v/\partial t$ at that point?

(d) What is $\partial v/\partial x$ at the point $x = 7$ in both sketches?

(e) How does v at the point $x = 7$ move or change between the two sketches? Based on that, what is $\partial v/\partial t$ at that point?

(f) Explain why this function solves Equation 11.11.4.

(g) Describe the life story of the point $x = x_0$ (where $x_0 > 0$) as t goes from 0 to ∞.

11.231 A string that starts at $x = 0$ and extends infinitely far to the right obeys the wave equation 11.2.2. The string starts at rest with no vertical displacement but the end of it is being shaken so $y(0,t) = y_0 \sin(\omega t)$.

(a) Using words and a few sketches of how the string will look at different times, predict how the string will behave—not by solving any equations (yet), but by physically thinking about the situation.

(b) Convert the wave equation and the initial conditions into an ODE for the Laplace transform $Y(x,s)$.

(c) Find the general solution to this ODE.

(d) It takes time for waves to physically propagate along a string. Since the string starts with zero displacement and velocity and is only being excited at $x = 0$, it should obey the implicit boundary condition $\lim_{x \to \infty} y(x,t) = 0$. Use this boundary condition to solve for one of the arbitrary constants in your solution.

(e) Use the explicit boundary condition at $x = 0$ to solve for the other arbitrary constant and thus find the Laplace transform $Y(x,s)$.

(f) Take the inverse Laplace transform to find the solution $y(x,t)$. Your answer will involve the "Heaviside step function" $H(x)$ (see Appendix K).

(g) Explain in words what the string is doing. Does the solution you found behave like the sketches you made in Part (a)? If not, explain what is different.

11.12 Additional Problems (see felderbooks.com)

CHAPTER 11

Partial Differential Equations (Online)

11.12 Additional Problems

11.232 The temperature along a rod is described by the function $T(x,t) = ae^{-(bx^2+ct^2)}$.

(a) Sketch temperature as a function of position at several times. Your vertical and horizontal axes should include values that depend on (some of) the constants a, b, and c.

(b) Sketch temperature as a function of time at several positions. Your vertical and horizontal axes should include values that depend on (some of) the constants a, b, and c.

(c) Make a three-dimensional plot of temperature as a function of position and time. *For this part and the next you may assume any positive values for the constants in the temperature function.*

(d) Make an animation of temperature along the rod evolving in time.

(e) Describe the behavior of the temperature of the rod.

(f) Does this temperature function satisfy the heat equation 11.2.3?

11.233 The solution to the PDE $4(\partial z/\partial t) - 9(\partial^2 z/\partial x^2) - 5z = 0$ with boundary conditions $z(0,t) = z(6,t) = 0$ is $z(x,t) = \sum_{n=1}^{\infty} D_n \sin(\pi n x/6) e^{(-\pi^2 n^2+20)t/16}$. In this problem you will explore this solution for different initial conditions. If you approach this problem correctly it requires almost no calculations.

(a) From this general solution we can see that one particular solution is $z(x,t) = 2\sin(\pi x/6)e^{(-\pi^2+20)t/16}$. What is the initial condition that corresponds to this solution?

(b) Describe how the solution from Part (a) evolves in time.

(c) If the initial condition is $z(x,0) = \sin(5\pi x/6)$, what is the full solution $z(x,t)$?

(d) Describe how the solution from Part (c) evolves in time.

(e) If the initial condition is $z(x,0) = \sin(\pi x/6) + \sin(5\pi x/6)$, what is the full solution $z(x,t)$?

(f) Two of the solutions above look nearly identical at late times; the third looks completely different. (If you can see how these solutions behave from the equations, you are welcome to do so. If not, it may help to get a computer to make these plots for you.) Which two plots look similar at late times? Explain why these two become nearly identical and the third looks completely different.

(g) If you had a non-sinusoidal function such a triangle or a square wave as your initial condition, you could write it as a sum of sinusoidal normal modes and solve it that way. Based on your previous answers, which of those normal modes would grow and which would decay? Explain why almost any initial condition would end up with the same shape at late times. What would that shape be?

In Problems 11.234–11.236 assume that the string being described obeys the wave equation (11.2.2) and the boundary conditions $y = 0$ at both ends.

11.234 A string of length π begins with zero velocity in the shape $y(x,0) = 5\sin(2x) - \sin(4x)$.

(a) Guess at the function $y(x,t)$ that will describe the string's motion. The constant v from the wave equation will need to be part of your answer.

(b) Demonstrate that your solution satisfies the wave equation and the initial and boundary conditions. (If it doesn't go back and make a better guess!)

11.235 A string of length π is given an initial blow so that it starts out with $y(x,0)=0$ and

$$\frac{\partial y}{\partial t}(x,0)=\begin{cases} 0 & x<\pi/3 \\ s & \pi/3\le x\le 2\pi/3 \\ 0 & 2\pi/3<x \end{cases}$$

(a) Rewrite the initial velocity as a Fourier sine series.

(b) Write the solution $y(x,t)$. Your answer will be expressed as a series.

(c) Let $v=s=1$ and have a computer numerically solve the wave equation with these initial conditions and plot the result at several different times. Then make a plot of this numerical solution and of the 10th partial sum of the series solution at $t=2$ on the same plot. Do they match?

11.236 A string of length L begins with $y(x,0)=0$ and initial velocity $\frac{\partial y}{\partial t}(x,0)=\begin{cases} x & 0\le x\le L/2 \\ L-x & L/2\le x\le L \end{cases}$. Find the solution $y(x,t)$.

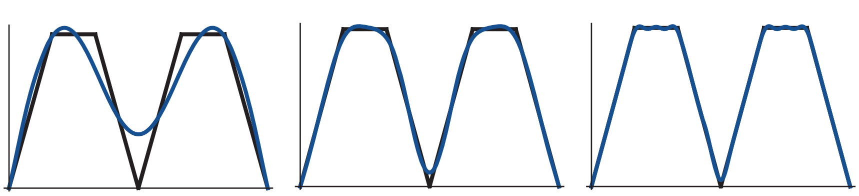
For Problems 11.237–11.239

(a) Solve the given problem using separation of variables. The result will be an infinite series.

(b) Plot the first three non-zero *terms* (not partial sums) of the series at $t=0$ and at least three other times. For each one describe the shape of the function and how it evolves in time.

(c) Plot successive partial sums at $t=0$ until the plot looks like the initial condition for the problem. Examples are shown below of what constitutes a good match.

(d) Having determined how many terms you have to include to get a good match at $t=0$, plot that partial sum at three or more other times and describe the evolution of the function. How is it similar to or different from the evolution of the individual terms in the series?



This is a bad match   Still not good enough   This is a good match

11.237

$$\frac{\partial^2 y}{\partial t^2}-\frac{\partial^2 y}{\partial x^2}+y=0,\ y(0,t)=y(4,t)=0,$$

$$y(x,0)=\begin{cases} 1 & 1<x<2 \\ -1 & 2\le x<3 \\ 0 & \textit{elsewhere} \end{cases},\ \frac{\partial y}{\partial t}(x,0)=0$$

11.238

$$\frac{\partial y}{\partial t}-t^2\frac{\partial^2 y}{\partial x^2}=0,\ y(0,t)=y(3,t)=0,$$

$$y(x,0)=\begin{cases} x & 0\le x<1 \\ 1 & 1\le x\le 2 \\ 3-x & 2x\le 3 \end{cases}$$

11.239 $\partial^2 y/\partial t^2-\partial y/\partial t-\partial^2 y/\partial x^2+y/4=0$, $y(0,t)=y(1,t)=0$, $y(x,0)=x(1-x)$, $\partial y/\partial t(x,0)=x(x-1)$

Appendix I gives a series of questions designed to guide you to the right solution method for a PDE. For Problems 11.240–11.243 answer the questions in that appendix until you get to a point where it tells you what solution method to try, and then solve the PDE using that method. As always, your answer may end up in the form of a series or integral.

As an example, if you were solving the example on Page 648 you would say

1 The equation is linear, so we can move to step 2 and consider separation of variables.

2(a) The equation is not homogeneous, which brings us to...

2(e) We can't find a particular solution because the domain is infinite and anything simple (e.g. a line) would diverge as x goes to infinity. So we can't use separation of variables.

3 The domain is infinite so we can't use eigenfunction expansion.

4 The equation involves a first derivative of x, so we can't use the Fourier transform method.

5 Time has a semi infinite domain $(0\le t<\infty)$ and t appears only in the derivatives, so we can try a Laplace transform.

Having come to this conclusion, you would then finish the problem by solving the PDE using the method of Laplace transforms.

11.240 $\partial y/\partial t - \alpha^2(\partial^2 y/\partial x^2) + \beta^2 y = 0$, $y(0,t) = y(L,t) = 0$, $y(x,0) = \kappa\left(x^3 - Lx^2\right)$

11.241 $\dfrac{\partial y}{\partial t} - \dfrac{\partial^2 y}{\partial x^2} = 1$, $y(0,t) = y(3,t) = 0$, $y(x,0) = \begin{cases} 1 & 1 \le x \le 2 \\ 0 & \textit{elsewhere} \end{cases}$

11.242 $\partial^2 u/\partial t^2 - c^2(\partial^2 u/\partial x^2) = \cos(\omega t)\sin^2(\pi x/L)$, $u(0,t) = u(L,t) = u(x,0) = \dot{u}(x,0) = 0$

11.243 $\partial^2 y/\partial t^2 - c^2(\partial^2 y/\partial x^2) = \alpha^2\cos(\omega t)e^{-\beta^2(x/x_0)^2}$, $y(x,0) = \dot{y}(x,0) = 0$, $\lim_{x\to-\infty} y(x,t) = 0$, $\lim_{x\to\infty} y(x,t) = 0$

In Problems 11.244–11.257 solve the given PDE with the given boundary and initial conditions. The domain of all the spatial variables is implied by the boundary conditions. You should assume t goes from 0 to ∞.

If your answer is a series see if it can be summed explicitly. If your answer is a transform see if you can evaluate the inverse transform. Most of the time you will not be able to, in which case you should simply leave your answer in series or integral form.

11.244 $\partial y/\partial t - 9(\partial^2 y/\partial x^2) = 0$, $y(0,t) = y(3,t) = 0$, $y(x,0) = 2\sin(2\pi x)$

11.245 $\partial y/\partial t - 9(\partial^2 y/\partial x^2) = 0$, $y(0,t) = y(3,t) = 0$, $y(x,0) = x^2(3-x)$

11.246 $\partial y/\partial t - 9\left(\partial^2 y/\partial x^2\right) = 9$, $y(0,t) = y(3,t) = 0$, $y(x,0) = 2\sin(2\pi x)$. *Hint*: After you find the coefficients, take special note of the case $n = 6$.

11.247 $\partial u/\partial t - \partial^2 u/\partial x^2 + \partial u/\partial x = e^{-t}$, $u(x,0) = u(0,t) = u(1,t) = 0$ *Hint*: depending on how you solve this, you may find the algebra simplifies if you use hyperbolic trig functions.

11.248 $\partial u/\partial t - \alpha^2(\partial^2 u/\partial x^2) - \beta^2(\partial^2 u/\partial y^2) = 0$, $u(0,y,t) = u(L,y,t) = u(x,0,t) = u(x,L,t) = 0$, $u(x,y,0) = \sin(\pi x/L)\sin(4\pi x/L)$

11.249 $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 - \partial^2 u/\partial z^2 = 0$, $u(0,y,z) = u(L,y,z) = u(x,0,z) = u(x,L,z) = u(x,y,0) = 0$, $u(x,y,L) = V$

11.250 $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 - \partial^2 u/\partial z^2 = 0$, $u(0,y,z) = u(L,y,z) = u(x,0,z) = u(x,L,z) = 0$, $u(x,y,0) = V_1$, $u(x,y,L) = V_2$ *Warning: the answer is long and ugly looking.*

11.251 $\partial y/\partial t - \partial^2 y/\partial x^2 = t\sin(3\pi x)$, $y(0,t) = y(1,t) = y(x,0) = 0$

11.252 $\partial y/\partial t = \partial^2 y/\partial x^2 + (1/x)(\partial y/\partial x) - y/x^2$, $y(0,t) = y(1,t) = 0$, $y(x,0) = x(1-x)$. You can leave an unevaluated integral in your answer.

11.253 $\dfrac{\partial y}{\partial t} - \alpha^2\dfrac{\partial^2 y}{\partial x^2} + \beta^2 y = \begin{cases} x & 0 < x < 1 \\ 1 & 1 \le x \le 2 \\ 3-x & 2 < x < 3 \end{cases}$, $y(0,t) = y(3,t) = 0$, $y(x,0) = x(3-x)$

11.254 $\dfrac{\partial y}{\partial t} - \dfrac{\partial^2 y}{\partial x^2} = 0$, $y(0,t) = y(3,t) = 0$, $y(x,0) = \begin{cases} 1 & 1 \le x \le 2 \\ 0 & \textit{elsewhere} \end{cases}$

11.255 $\partial y/\partial t - \partial^2 y/\partial x^2 = 0$, $y(0,t) = 0$, $y(1,t) = 1$, $y(x,0) = x^2$

11.256 $\partial^2 y/\partial t^2 - \alpha^2 t(\partial^2 y/\partial x^2) = 0$, $y(0,t) = y(\pi,t) = 0$, $y(x,0) = 5\sin(3x)$, $\partial y/\partial t(x,0) = 0$

11.257 $\partial^2 y/\partial t^2 - v^2(\partial^2 y/\partial x^2) = e^{-(x+t)^2}$, $y(x,0) = \partial y/\partial t(x,0) = 0$, $\lim_{x\to\infty} y(x,t) = \lim_{x\to-\infty} y(x,t) = 0$

11.258 The electric potential in a region without charges obeys Laplace's equation (11.2.5). Solve for the potential on the domain $0 \le x < \infty$, $0 \le y \le L$, $0 \le z \le L$ with boundary conditions $V(0,y,z) = V_0$, $V(x,0,z) = V(x,y,0) = V(x,L,z) = V(x,y,L) = 0$, $\lim_{x\to\infty} V = 0$.

11.259 You are conducting an experiment where you have a thin disk of radius R (perhaps a large Petri dish) with the outer edge held at zero temperature. The chemical reactions in the dish provide a steady, position-dependent source of heat. The steady-state temperature in the disk is described by Poisson's equation in polar coordinates.

$$\rho^2\frac{\partial^2 V}{\partial \rho^2} + \rho\frac{\partial V}{\partial \rho} + \frac{\partial^2 V}{\partial \phi^2} = \frac{\rho}{R}\sin\phi$$

(a) Begin by applying the variable substitution $\rho = Re^{-r}$ to rewrite Poisson's equation.

(b) What is the domain of the new variable r?

(c) Based on the domain you just described, the method of transforms is appropriate here. You are going to use a Fourier sine transform. Explain why this makes more sense for this problem than a Laplace transform or a Fourier cosine transform.

(d) Transform the equation. The formula is in Appendix G. You can evaluate the integral using a formula from that appendix (or just give it to a computer). Then solve the resulting ODE. Your general solution will have two arbitrary functions of p.

(e) The boundary conditions for ϕ are implicit, namely that $\hat{V}_s(\phi)$ and $\hat{V}_s'(\phi)$ must both have period 2π. You should be able to look at your solution and immediately see what values the arbitrary functions must take to lead to periodic behavior.

(f) Take the inverse transform. (You can get the formula from Appendix G and use a computer to take the integral.) Then substitute back to find the solution to the original problem in terms of ρ.

11.260 **Exploration: A Time Dependent Boundary** A string of length 1 obeys the wave equation 11.2.2 with $v = 2$. The string is initially at $y = 1 - x + \sin(\pi x)$ with $\partial y/\partial t(x, 0) = x - 1$. The right side of the string is fixed $(y(1, t) = 0)$, but the left side is gradually lowered: $y(0, t) = e^{-t}$.

(a) First find a particular solution $y_P(x, t)$ that satisfies the boundary conditions, but does not necessarily solve the wave equation or match the initial conditions. To make things as simple as possible your solution should be a linear function of x at each time t.

The complete solution will be $y(x, t) = y_C(x, t) + y_P(x, t)$ where y_P is the solution you found in Part (a) and y_C is a complementary solution, still to be found.

(b) What boundary conditions and initial conditions should y_C satisfy so that y satisfies the boundary and initial conditions given in the problem?

(c) Using the fact that $y = y_C + y_P$ and $y(x, t)$ solves the wave equation, figure out what PDE y_C must solve. The result should be an inhomogeneous differential equation.

(d) Using the method of eigenfunction expansion, solve the PDE for y_C with the boundary and initial conditions you found.

(e) Put your results together to write the total solution $y(x, t)$.

(f) Based on your results, how will the string behave at very late times $(t >> 1)$? Does the particular solution you found represent a steady-state solution? Explain.