CHAPTER 1

Introduction to Ordinary Differential Equations (Online)

1.7 Coupled Equations

When two or more dependent variables depend on each other, their equations are "coupled." The rules we have presented so far can be generalized to such situations.

1.7.1 Discovery Exercise: Coupled Equations

Consider a population of foxes and rabbits. They each reproduce, but their populations are both limited by the fact that foxes eat rabbits. For this exercise we'll adopt a simplified model of this relationship.

1. Write a differential equation for the rabbit population $R(t)$ that expresses the sentence: "Each year every rabbit produces 5 babies on average, but 10 rabbits are killed by each fox." Use $F(t)$ for the number of foxes.
See Check Yourself #7 in Appendix L
2. Explain why you cannot solve this equation for $R(t)$ with the information given. What more information would you need? (*Hint*: The answer is not the initial number $R(0)$. Not knowing that just means there would be an arbitrary constant in your answer.)
3. In Question 1 we gave you a verbal description of the rabbit population and asked you for the differential equation. Here we will do the opposite for the foxes. The fox population is described by the differential equation $dF/dt = R/2 - F$. Give the verbal description that explains where this equation comes from.
4. If $dR/dt = dF/dt = 0$ then both populations remain constant. What would have to be true about the values of R and F for this condition to hold? (They would not both have to be zero, although that is one way to get this result.)
5. Which of the following pairs of functions solve the equations for R and F? (More than one answer may be correct. Indicate all of the correct solutions.)
(a) $R(t) = 2000$, $F(t) = 1000$.
(b) $R(t) = 2\cos t$, $F(t) = 2\cos t$
(c) $R(t) = 1000e^{4t} + 200$, $F(t) = 100e^{4t} + 100$
(d) $R(t) = e^{4t}$, $F(t) = 2e^{4t}$

1.7.2 Explanation: Coupled Equations

A differential equation in the form $dy/dt =$ "some function of y and t" says that the change in y depends on the current values of y (the dependent variable) and t (the independent variable). In many physical situations, however, multiple dependent variables depend on *each other*. Two reactants decrease in response to each other's concentrations. A planet and a moon change positions in response to each other's positions. In cases like these you need a differential equation for each quantity that depends on itself and on the other variables.

To see what it means to solve coupled differential equations, recall simultaneous algebraic equations.

$$4x^2 - 3y = -5$$
$$2x + y^2 = 53$$

You cannot say that $x = 2$ is a solution, or that it is *not* a solution, because either statement depends on what y is. You can say that $x = 2$, $y = 7$ is a solution because you can plug that in and it works. Similarly, if you have a pair of differential equations for two functions $x(t)$ and $y(t)$ a solution is a *pair* of functions that, taken together, make both equations work.

As an example, consider the famous love of Romeo and Juliet.[3] Romeo's love for Juliet, $R(t)$, grows when it is returned. The more Juliet loves Romeo, $J(t)$, the more his love for her grows. When she hates him, this makes his love diminish. Juliet, on the other hand, is coy. The more Romeo loves her the more bored she becomes with him, but when he despises her this inflames her love. Using positive numbers for love and negative for hatred, we can express all this in a pair of coupled differential equations.[4]

$$\frac{dR}{dt} = J \tag{1.7.1}$$
$$\frac{dJ}{dt} = -R \tag{1.7.2}$$

Try to think of a solution before reading further. We're looking for two functions with the property that the derivative of the first one equals the second one, and the derivative of the second one equals negative the first. One obvious answer is $R(t) = J(t) = 0$, which might correspond to "they haven't met," but see if you can find something more interesting than that.

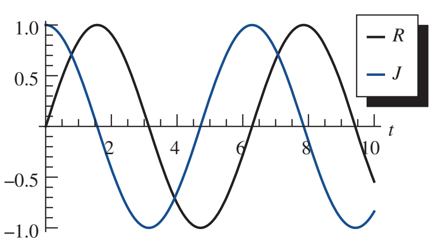


FIGURE 1.3 Romeo and Juliet's feelings for each other oscillate out of phase.

You may have landed on $R(t) = \sin t$, $J(t) = \cos t$. In this particular solution Romeo starts out indifferent to Juliet while Juliet starts out in love ($J > 0$). Juliet's love causes Romeo's love to grow, which in turn causes Juliet's to fade. They remain fond of each other until $t = \pi/2$, when Juliet begins to dislike Romeo. This causes his love to fade as well until starting at time $t = \pi$ they both dislike each other, which causes Juliet's disdain for Romeo to diminish, etc.

This solution only matches one initial condition, however, so it can't be the general solution. Experimenting with a few constants, we find that $R(t) = 3\sin t, J(t) = 3\cos t$ works but $R(t) = 2\sin t, J(t) = 3\cos t$ does not. You can multiply R and J by an arbitrary constant, but you must multiply them by the *same* constant. So now we have $R(t) = A\sin t$, $J(t) = A\cos t$. We give below the mathematical criterion for identifying

[3]This example is adapted with permission from *Nonlinear Dynamics and Chaos* by Steven Strogatz. It is not particularly adapted from Shakespeare, although we like to think he would be amused.
[4]We're writing this with incorrect units for simplicity. In Problem 1.158 you'll put in constants of proportionality to fix the units and re-solve the problem.

the general solution to a set of coupled differential equations, but even without the official rules you can see that we're not there yet. You could start with any combination of Romeo's love and Juliet's love, so you need two arbitrary constants to be able to match any possible pair of initial conditions.

We thus need to find another independent solution. Once again, this can be done by squinting at the equations for a while: $R(t) = \cos t, J(t) = -\sin t$. As before, you can multiply this solution by an arbitrary constant and it still works, so adding our two solutions gives us the general solution.

$$R(t) = A \sin t + B \cos t, \quad J(t) = A \cos t - B \sin t$$

This pattern occurs frequently in coupled equations: the same arbitrary constants appear in both solutions, but they appear in front of different functions.

The box below generalizes the definitions and rules for single differential equations to coupled differential equations. We present them all together without much explanation because they are similar to what we have seen before.

Linear Superposition and General Solutions for Coupled Equations

Linearity A set of coupled differential equations is said to be linear if every term in all of the equations is linear in one of the dependent variables. Equations 1.7.1–1.7.2 for Romeo and Juliet are linear because each term is linear in either R or J. Note that this means no term can include more than one dependent variable. If the equations for R and J included a term with R^2, RJ, or $R(dJ/dt)$ they would be non-linear.

Order of a set of equations A set of coupled differential equations is nth order in a given variable if the nth derivative is the highest derivative of that variable that appears anywhere in the set of equations. For example, Equations 1.7.1–1.7.2 are first order in both R and J.

General solution If the differential equations in a set are all linear, then the general solution to that set will have as many independent arbitrary constants as the sum of the orders of all the dependent variables.

Homogeneity A set of linear coupled differential equations is homogeneous if every term includes one of the dependent variables or one of their derivatives.

Linear superposition If a set of coupled differential equations is linear and homogeneous then any linear combination of solutions to the set of equations is also a solution. If a set of coupled differential equations is linear and inhomogeneous then you can define the complementary set of equations by replacing all inhomogeneous terms with zero. In this case the sum of any particular solution to the inhomogeneous equations with any linear combination of solutions to the complementary set of equations is a solution to the inhomogeneous equations. (That's a mouthful, but it's the same thing we said earlier about single differential equations.)

The rule about having as many arbitrary constants as the sum of the orders makes sense if you think about it. If your equations are second order in $y(x)$ you need to specify $y(0)$ and $y'(0)$ as initial conditions, and if they are first order in $z(x)$ you also need to specify $z(0)$, so a set of equations that's first order in one variable and second order in another needs three arbitrary constants.

EXAMPLE **A Molecule with Two States**

Problem:
Suppose a certain type of molecule can exist in two possible states. Let $a(t)$ represent the number of molecules in state A and $b(t)$ the number of molecules in state B. Let p be the rate at which state A converts to state B, meaning each second pa molecules change from A to B. Molecules in state B convert to state A at twice that rate, so every second $2pb$ molecules switch from B to A. Putting all that together:

$$\frac{da}{dt} = -pa + 2pb, \quad \frac{db}{dt} = pa - 2pb$$

Which of the following are valid solutions to this pair of equations?

1. $a(t) = 2e^{-2pt},\ b(t) = -e^{-2pt}$
2. $a(t) = 2 + e^{-3pt},\ b(t) = 1 - e^{-3pt}$
3. $a(t) = 6 - 2e^{-3pt},\ b(t) = 3 + 2e^{-3pt}$

What is the general solution?

Solution:
To check a solution you plug *both* functions in simultaneously.

1. First take derivatives of the two functions: $a'(t) = -4pe^{-2pt}$, $b'(t) = 2pe^{-2pt}$. Plugging these in to the first differential equations gives $-4pe^{-2pt} = -2pe^{-2pt} - 2pe^{-2pt}$, which works. Plugging these into the second differential equation gives $2pe^{-2pt} = 2pe^{-2pt} + 2pe^{-2pt}$. This equation is not satisfied, so this is not a solution.
2. The derivatives are $a'(t) = -3pe^{-3pt}$, $b'(t) = 3pe^{-3pt}$. Plugging these in gives $-3pe^{-3pt} = -2p - pe^{-3pt} + 2p - 2pe^{-3pt}$ and $3pe^{-3pt} = 2p + pe^{-3pt} - 2p + 2pe^{-3pt}$. Both equations are satisfied, so this is a solution.
3. We will leave the calculations to you, but you can similarly show that plugging in these solutions works, so this is also a solution.

Because these linear equations are first order in both a and b, the general solution must have two arbitrary constants. We can get it, as usual, by writing a linear combination of the two independent solutions we just found.

$$a(t) = C\left(2 + e^{-3pt}\right) + D\left(6 - 2e^{-3pt}\right)$$
$$b(t) = C\left(1 - e^{-3pt}\right) + D\left(3 + 2e^{-3pt}\right)$$

(This solution can be simplified by combining terms and renaming arbitrary constants; we'll leave that to you to think about.)

Solving Coupled Equations

You may be assuming by this point that we are building up to solving all sorts of coupled differential equations. As with uncoupled equations, there is no one method that applies to all of them, and computers can do that step quite well as a rule. Our main focus is on writing a set of equations for a given scenario, and on interpreting the equations and their solutions. Nonetheless, we will say a few words here about solving coupled differential equations.

The Romeo and Juliet example above demonstrated one important technique for solving coupled equations, which is just thinking about them. "The derivative of R is J, and the derivative of J is $-R$ … what would do that? … ah yes, a sine and cosine." The more you practice this skill the better you will be at it, and the better you will understand such equations.

(Try your hand at these: what if $f'(x) = g(x)$ and $g'(x) = f(x)$ with no negative signs? Or what if $f'(x) = -g(x)$ and $g'(x) = -f(x)$ with negative signs on both? See Problem 1.140.)

In Chapter 6 we will use matrices to represent and solve coupled equations. In Chapter 10 we will discuss Laplace transforms, a powerful technique for solving both single and coupled equations. Chapter 10 will also introduce "phase portraits" which do for coupled equations roughly what slope fields do for single equations: they give you a way of graphing and visualizing the entire space of solutions even if you cannot solve the problem. Here we will present one simple but fairly general method for approaching coupled equations, which is to "decouple" them.

To illustrate the method, let's return again to Romeo and Juliet, but imagine you failed to just think of the solutions. To decouple the equations take either one of them and take the derivative of both sides. Here we start with Equation 1.7.1.

$$R' = J \qquad \rightarrow \qquad R'' = J'$$

That equation gives us an expression for J'. When we substitute that in for J' in Equation 1.7.2, we get a single decoupled equation for R.

$$J' = -R \qquad \rightarrow \qquad R'' = -R$$

This is the simple harmonic oscillator equation, with solution $R = A \sin t + B \cos t$. Finally, plugging this into $R' = J$ gives $J = A \cos t - B \sin t$, the same solution we found before. In general, the way to decouple two differential equations is to differentiate one of them and then use the other to eliminate one of the two variables.

1.7.3 Problems: Coupled Equations

For Problems 1.135–1.139 check whether the given functions solve the set of coupled differential equations.

1.135 $x'(t) = y$, $y'(t) = x$; $x(t) = Ae^t$, $y(t) = Ae^t$

1.136 $x'(t) = -x + y$, $y'(t) = x - y$; $x(t) = A(1 + e^{-2t}) + B(1 - e^{-2t})$, $y(t) = A(1 - e^{-2t}) + B(1 + e^{-2t})$

1.137 $x'(t) = 2y$, $y'(t) = x + y$; $x(t) = Ae^{2t} + 2Be^{-t}$, $y(t) = Ae^{2t} - Be^{-t}$

1.138 $x'(t) = 2xy$, $y'(t) = x^2 + y^2$; $x(t) = 1/(t^2 - 1)$, $y(t) = -t/(t^2 - 1)$

1.139 $x''(t) = y$, $y''(t) = -x$; $x(t) = Ae^t$, $y(t) = Ae^t$

1.140 Find the general solutions (two independent arbitrary constants) to the following sets of coupled equations. You should be able to do these mostly by inspection, but you can decouple them if you get stuck.

(a) $f'(x) = g(x)$, $g'(x) = f(x)$

(b) $f'(x) = -g(x)$, $g'(x) = -f(x)$

(c) $f'(x) = g(x)$, $g'(x) = 4f(x)$

1.141 The superstates of Oceania and Eastasia have a complicated ever-changing relationship. For each scenario below write a set of coupled differential equations that might represent their populations $O(t)$ and $E(t)$. Then describe—using equations or words—how you would expect their populations to evolve over time.

(a) Oceania and Eastasia are mutually supportive allies. The greater their combined (total) population, the more their individual populations grow.

(b) Oceania and Eastasia are at war. The greater the population of Oceania, the more Eastasians are killed each year, and vice versa.

(c) Oceania and Eastasia are at war, but a different kind of war. If Oceania has more people than Eastasia, then the population of Oceania will grow while the population of Eastasia shrinks. The reverse if Eastasia outnumbers Oceania. The greater the difference in population, the greater the changes.

(d) Oceania and Eastasia have a treaty of peace and mutual support, but Oceania cheats. The greater the population of Eastasia, the more Oceania thrives;

the greater the population of Oceania, the more Eastasia suffers.

1.142 A vat contains a molecules of substance A and b molecules of substance B. Each second, kab reactions occur, each of which turns one molecule of A and two molecules of B into a molecule of C.

(a) Write differential equations for the number of molecules of a, b, and c.

(b) Describe in words how you would expect these numbers to evolve over time.

(c) Now assume that in addition to the chemical reaction described above, 10^{23} molecules of A are being added to the vat each second. Write the new differential equations and discuss how this will change the results over time.

1.143 Tanks A and B with volumes V_A and V_B are both filled with a mixture of water and brine. Pure brine is being poured into tank A at a rate of r gallons per minute. Meanwhile the water/brine mix from tank A is being poured into tank B at the same rate, and the mix from tank B is being poured into the nearby river at the same rate. Since the rates are all the same the volume of liquid in each tank stays the same, and you can assume that the tanks are well mixed, so the fraction of brine leaving each tank equals the total fraction of brine in the tank at that moment.

(a) Write a pair of coupled differential equations for the number of gallons of brine in each tank, G_A and G_B.

(b) Physically, what would you expect to happen to G_A and G_B in the long run, and why?

(c) Mathematically, how can you look at the differential equations you wrote and confirm that the physical behavior you described is what G_A and G_B will do at late times? *Hint*: Consider what must be true of G_A and G_B in order for $G_A' = G_B' = 0$, and what will happen to G_A and G_B when that condition isn't met.

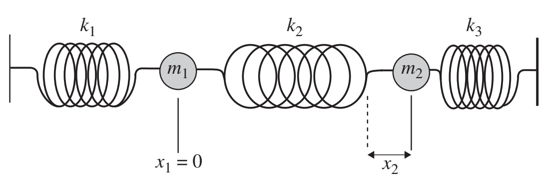
1.144 Tanks A and B, with volumes V_A and V_B, are both filled with a mixture of water and brine. Each minute r gallons of liquid flow from A to B and an equal amount flows from B to A. Since the rates are the same the volume of liquid in each tank stays the same, and you can assume that the tanks are well mixed, so the fraction of brine leaving each tank equals the total fraction of brine in the tank at that moment.

(a) Write a pair of coupled differential equations for the number of gallons of brine in each tank, G_A and G_B.

(b) Physically, what would you expect to happen to G_A and G_B in the long run, and why?

(c) Mathematically, how can you look at the differential equations you wrote and confirm that the physical behavior you described is what G_A and G_B will do at late times? *Hint*: Consider what must be true of G_A and G_B in order for $G_A' = G_B' = 0$, and what will happen to G_A and G_B when that condition isn't met.

1.145 The figure shows two balls connected to each other and to the walls by three springs. Ball 1 has mass m_1 and is displaced by an amount x_1 from its equilibrium position, and similarly for Ball 2.



(a) The position $x_1 = x_2 = 0$ represents the equilibrium position. Now imagine that Ball 1 is at this position precisely, but Ball 2 is slightly to the right of this position, as shown above. Which springs now exert force on Ball 1, and in which directions? Which springs now exert force on Ball 2, and in which directions?

(b) Now imagine that Ball 1 is displaced to the right by a distance x_1, and Ball 2 is displaced to the right by x_2.

i. The force of Spring 1 on Ball 1 depends only on the position x_1 (the other position is irrelevant). Write a formula for this force.

ii. How much is Spring 2 stretched? Your answer should be a function of x_1 and x_2. For example, if both balls are displaced the same amount to the right then Spring 2 isn't stretched at all. As a check on your answer, make sure it gives a positive answer when Spring 2 is longer than its equilibrium length, a negative answer when it is shorter, and 0 when Spring 2 is at its equilibrium length (neither stretched nor compressed.)

iii. Using your answer to Part (ii), write a formula for the force of Spring 2 on Ball 1.

iv. Putting the two forces together and using $F = ma$, write a differential equation for $x_1(t)$. Be sure you have the sign of each force correct.

(c) Repeat Part (b) for x_2.

You will solve this particular set of coupled differential equations twice: using matrices in Chapter 6 , and using Laplace transforms in Chapter 10.

1.146 *[This problem depends on Problem 1.145.]*

(a) Write the coupled differential equations for Problem 1.145 for the case where $k_1 = 1$, $k_2 = 4$, $k_3 = 28$, $m_1 = 1$, and $m_2 = 4$.

(b) The general solution to this problem would have four arbitrary constants. What are the four initial conditions you would need to find them?

(c) Verify that $x_1 = 4\cos(2t)$, $x_2 = \cos(2t)$ is one valid solution.

(d) Verify that $x_1 = \sin(3t)$, $x_2 = -\sin(3t)$ is one valid solution.

1.147 Define a coordinate system with the sun at the origin and the Earth's position given by (x, y). (The Earth orbits in a plane, so we can ignore the third direction.) We will for simplicity assume the sun doesn't move.

(a) The gravitational acceleration of a planet being pulled on by the sun has magnitude GM/r^2, where G is a constant, M is the sun's mass, and r is the distance between the two objects. Write down the magnitude of the gravitational acceleration of the Earth in terms of x and y.

(b) Using the fact that the gravitational acceleration points toward the origin (the sun), find the x- and y-components of the Earth's gravitational acceleration. *Hint*: You may find it helpful to draw a picture and label an angle θ, but your final answer should be in terms of x and y, not θ.

(c) Use the acceleration components you just wrote to write two coupled differential equations for $x(t)$ and $y(t)$.

(d) Verify that $x(t) = a\cos\left(\sqrt{GM/a^3}\;t\right)$, $y(t) = a\sin\left(\sqrt{GM/a^3}\;t\right)$ is a solution to the coupled equations you wrote for any value of a. What kind of motion does this solution represent?

1.148 **Walk-Through: Decoupling Equations.** In this problem you will solve the equations $f'(t) = f(t) + g(t)$, $g'(t) = 3f(t) - g(t)$. We will refer to these as the f' equation and the g' equation respectively.

(a) Differentiate both sides of the f' equation to get f'' in terms of f' and g'.

(b) Substitute g' from the g' equation into your answer to get f'' in terms of f, f', and g.

(c) Solve for g in the original f' equation and plug this into your answer to get a decoupled, second-order differential equation for f.

(d) Find the general solution for $f(t)$ by inspection.

(e) Plug this solution for f into the f' equation and solve (algebraically) for g.

(f) Plug your general solution for f and g into the original equations and verify that they work.

For Problems 1.149–1.152, find the general solutions to the differential equations by decoupling them. You should be able to solve the decoupled equations by inspection or guess-and-check. It may help to work through Problem 1.148 as an model.

1.149 $f' = 3g$, $g' = 6f$

1.150 $f' = f + g$, $g' = f$

1.151 $f' = af + bg$, $g' = cf + dg$

1.152 $f' = af^2/g$, $g' = 2af + bg$. *Hint*: You should end up with a simple linear equation for f''.

Newton's law of heating and cooling states that when you put two objects in contact the cold one will heat up at a rate proportional to the temperature difference between them. The hot object will cool down at a rate proportional to the same temperature difference; however, the constants of proportionality may be very different (the "heat capacity" of each object).[5]

In Problems 1.153–1.155 assume that any two objects in contact with each other obey Newton's law of cooling. Unless otherwise specified, assume that none of the objects gains or loses heat to any other part of the environment. The ODEs will include constants of proportionality, to which you should assign letters, but you should write your equations in

[5]In practice this is only an approximation, but it generally works well unless the temperature differences are large.

such a way that those letters have positive values. In general the constants of proportionality will be different for the two objects in contact.

1.153 Bars A and B are in contact with each other.

(a) Write a pair of coupled ODEs for T_A and T_B.

(b) Looking at your equations, describe what will happen to T_A and T_B over time. If they start with $T_A > T_B$ what will happen to the two temperatures initially? How will it change over time? What will happen at very late times?

(c) Based on your answers, draw a qualitative sketch showing T_A and T_B vs. time on the same plot. Your plot should show how the functions behave at early and late times. You do not need to include any numbers on your axes.

(d) Solve the differential equations you wrote by decoupling them. Verify that the solutions match the behavior you described qualitatively above.

1.154 Bars A, B, and C are stacked so that A and B are touching and B and C are touching, but not A and C.

(a) Write a set of coupled ODEs for T_A, T_B, and T_C.

(b) What would have to be true of T_A, T_B, and T_C in order for none of them to change? You can use physical intuition to help guide you, but you must explain your answer in terms of the ODEs you wrote.

(c) If the bars start out with $T_A > T_B = T_C$, describe what will happen to the temperatures initially. What will happen a short time later? What will happen a very long time later?

1.155 Bar A is touching bar B, which is in contact with a room at constant temperature T_R. (Bar A is insulated from the room, so it can only exchange heat with bar B. The room is large so it will affect bar B without being affected by it.)

(a) Write a pair of coupled ODEs for T_A and T_B.

(b) What do you expect to happen to T_A and T_B in the long term?

(c) Solve the ODEs you wrote by decoupling them. Take the limit of your solution as $t \to \infty$ and verify that they match your predictions.

1.156 The Discovery Exercise (Section 1.7.1) presented a simplified model of a predator-prey relationship. A more commonly used model, although still too simple for some situations, is the "Lotka–Volterra equations," sometimes called the "predator-prey equations."

$$dR/dt = \alpha R - \beta RF$$
$$dF/dt = \gamma RF - \delta F$$

R and F represents the populations of rabbits and foxes, and the Greek letters represent positive constants.

(a) What would happen to the rabbit population if there were no foxes? You may use common sense to check your answer, but you must explain how you can figure out your answer from the Lotka–Volterra equations.

(b) Similarly, explain using these equations what would happen to the fox population if there were no rabbits.

(c) What values would R and F have to have in order for their populations to be unchanging? Give two answers to this, one of which you can think of just by looking at the equations. The other one will require some calculation.

(d) The product RF appears in both equations, adding to the number of foxes and subtracting from the rabbits. Why?

(e) The simplest population growth for the foxes would be $dF/dt = \delta F$: "the more foxes you have, the more you get." But in the Lotka–Volterra equations, δF is subtracted from dF/dt: "the more foxes you have, the fewer you get." Why? (*Hint*: You cannot answer this question without thinking about the whole scenario.)

1.157 *[This problem depends on Problem 1.156.]* Decouple the Lotka–Volterra equations to get a differential equation for $R(t)$. Why is this technique not particularly useful for this case?

1.158 Consider the Romeo and Juliet problem with proper units. Here R and J are Romeo's and Juliet's love for each other and α and β are positive constants.

$$dR/dt = \alpha J$$
$$dJ/dt = -\beta R$$

(a) If R and J are each measured in love-units, what are the units of α and β?

(b) Solve for $R(t)$ and $J(t)$ by decoupling the equations. Your final answer should have two arbitrary constants in it.

(c) Using the units you found for α and β, what units must your arbitrary constants have in order for your answer to make sense? Make sure that all terms are added to things with equivalent units and that all arguments of trig functions are unitless.

(d) Suppose α is very large compared to β. What does that tell us about Romeo and Juliet? (Answer based on the original differential equations.) What effect will it have on their behavior? (Answer based on your solution.)