10.4 Linear First-Order Differential Equations

The simplest type of differential equations are linear and first order, and there is a general formula for solving all such equations (if you can integrate them). Such equations are important in their own right, but they are also important because we often approach more difficult equations by turning them into linear first-order equations—either exactly or approximately.

10.4.1 Discovery Exercise: Linear First-Order Differential Equations

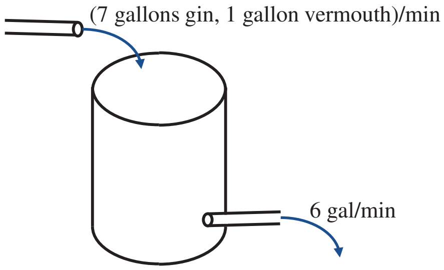
A linear first-order differential equation can always be written in the form:

$$\frac{dy}{dx} + a_0(x)y = f(x) \tag{10.4.1}$$

1. It would seem that a more general form would be $a_1(x)y' + a_0(x)y = f(x)$. Why can we claim that Equation 10.4.1 is fully general?
2. Write the complementary homogeneous equation for Equation 10.4.1.
3. Solve your homogeneous equation by separation of variables. Your answer will have an integral with respect to x in it.
4. Confirm that your answer works.

In this section we will find a general formula for solving the *inhomogeneous* Equation 10.4.1. But keep an eye on your homogeneous solution from this exercise; you will see it showing up as part of the general solution, as you might expect.

10.4.2 Explanation: Linear First-Order Differential Equations



The tank in the picture begins with 5 gallons of a mixture of gin and vermouth. Every minute, 7 gallons of gin and 1 gallon of vermouth are added to the tank. The tank is also being drained at 6 gallons/minute. Assuming the mixture in the tank is stirred constantly, find the total amount of gin in the tank as a function of time.

This problem calls on us to create a differential equation for $G(t)$, the total amount of gin in the tank, measured in gallons. There are two effects causing G to change over time.

- The inflow is very simple: every minute, 7 gallons of gin are added to the tank. If this were the only effect, we would write $dG/dt = 7$.
- Assuming the tank is kept evenly mixed, the outflow each minute is the total amount of gin in the tank (G) multiplied by the fraction of the mixture that is drained out. (For instance, if 1/10 of the mixture flows out, then $G/10$ gallons of gin are drained.) The total amount of the mixture is increasing by 2 gallons every minute, so the total volume after t minutes is $5 + 2t$. That means the fraction of the mixture that is drained every minute is $6/(5 + 2t)$.

We arrive at the following differential equation:

$$\frac{dG}{dt} = 7 - \left(\frac{6}{5+2t}\right) G \tag{10.4.2}$$

Equation 10.4.2 isn't separable, and doesn't suggest any obvious guess—so there go the only two methods we've seen so far. In this section we're going to introduce a new method that is specifically designed for first-order linear equations such as this one: that is, equations that can be expressed as follows.

$$\frac{dy}{dx} + a_0(x)y = f(x) \qquad \textit{the generic first-order linear ODE} \tag{10.4.3}$$

This method can be approached as either a *technique* or a *formula*, and we will solve Equation 10.4.2 both ways to show how the two are comparable.

The Technique

As an analogy to this technique we will quickly review the algebra trick called "completing the square," which is based on the following formula.

$$(x + a)^2 = x^2 + 2ax + a^2 \tag{10.4.4}$$

Every quadratic function involves $x^2 + 2ax$ for some number a. If you add a^2 then you can use Equation 10.4.4 to turn it into a perfect square, and then just take the square root.

$$\begin{aligned} x^2 + 10x &= 11 && \textit{the question} \\ x^2 + 10x + 25 &= 36 && \textit{add 25 to make the left side match Equation 10.4.4} \\ (x + 5)^2 &= 36 && \textit{because it does match!} \\ x + 5 &= \pm 6 && \textit{so } x \textit{ is } -11 \textit{ or } 1 \end{aligned}$$

Now: before you read further, grab a piece of scratch paper and take the derivative, with respect to x, of $ye^{\int a_0(x)dx}$. (You will need a chain rule inside a product rule.) The result provides a template for ODEs, just as Equation 10.4.4 is a template for quadratic functions.

$$\frac{d}{dx}\left(ye^{\int a_0(x)dx}\right) = y'e^{\int a_0(x)dx} + ya_0(x)e^{\int a_0(x)dx} = e^{\int a_0(x)dx}\left[y' + a_0(x)y\right] \tag{10.4.5}$$

Every first-order linear differential equation involves $y' + a_0(x)y$ for some function $a_0(x)$. If you multiply by $e^{\int a_0(x)dx}$ then you can use Equation 10.4.5 to turn it into a perfect derivative, and then just integrate.

As an example let's tackle Equation 10.4.2, starting by putting it in standard form.

$$\frac{dG}{dt} + \left(\frac{6}{5 + 2t}\right)G = 7 \tag{10.4.6}$$

Our a_0 function in this case is $6/(5 + 2t)$, so (after a few calculations) $e^{\int a_0(t)dt} = (5 + 2t)^3$. This function is called the "integrating factor."

Integrating Factor

An integrating factor for a differential equation is a function that you multiply by the equation to allow you to integrate both sides. For a first-order linear ordinary differential equation the integrating factor is

$$I(x) = e^{\int a_0(x)\, dx} \tag{10.4.7}$$

Equation 10.4.7 only works as an integrating factor if your differential equation is in the form of Equation 10.4.3. For instance, if your equation starts with $2y'(x)$ then you have to divide both sides by 2 before finding the integrating factor.

Multiplying both sides of Equation 10.4.6 by its $I(t)$ we get:

$$(5 + 2t)^3\left(\frac{dG}{dt}\right) + 6(5 + 2t)^2G = 7(5 + 2t)^3$$

That may not look like an improvement but Equation 10.4.5 promises us that the left side is the derivative of $G(5+2t)^3$, a fact we urge you to confirm for yourself. (Again, chain rule inside a product rule.) So we can now integrate both sides and solve.

$$G(5+2t)^3 = \frac{7}{8}(5+2t)^4 + C$$
$$G = \frac{7}{8}(5+2t) + C(5+2t)^{-3}$$

We leave it to you to confirm that this is a valid solution to Equation 10.4.2. Note that this technique has something in common with separation of variables: you put in the $+C$ when you integrate and carry it through the math from there, and you end up with the general solution. We never bothered finding separate "particular" and "complementary" solutions. But if you solve the complementary homogeneous equation on your own (by separating variables for instance) you will end up with $C(5+2t)^{-3}$, which comes as no great surprise. We see that $(7/8)(5+2t)$ is acting as the particular solution in this case.

The Formula

You can solve any specific quadratic equation by completing the square. If you apply the same technique to the generic quadratic equation $ax^2+bx+c=0$ you end up with the quadratic formula. Similarly, you can use the technique discussed above to solve specific differential equations. If you apply the same technique to the generic first-order linear differential equation, you end up with the following formula.

The General Solution to All Linear First-Order Differential Equations

The solution to the equation:

$$\frac{dy}{dx} + a_0(x)y = f(x) \tag{10.4.8}$$

is:

$$y = \frac{1}{I(x)}\int I(x)f(x)\,dx \quad \textit{where} \quad I(x) = e^{\int a_0(x)\,dx} \tag{10.4.9}$$

As we discuss below, you do *not* add $+C$ when you integrate a_0 to find I, but you *do* add $+C$ when you integrate If to find y.

This solution involves two integrals, either of which may be impossible. Alas, many differential equations are just impossible to solve, no matter how many techniques you learn. But when you can evaluate those integrals, by hand or by computer, solving a first-order linear ODE is now just a matter of plugging into a formula.

Let's see how that plays out in the example above, which started here.

$$\frac{dG}{dt} + \left(\frac{6}{5+2t}\right)G = 7$$

Note that Equation 10.4.9 prominently features $I(x)$, the integrating factor we saw in the technique above. So we start here with the same calculations we did there:

$$a_0 = 6/(5+2t) \quad \text{so} \quad \int a_0(t)\,dt = 3\ln(5+2t) \quad \text{so} \quad I(t) = (5+2t)^3$$

Now we plug into Equation 10.4.9:

$$G(t) = (5+2t)^{-3}\int 7(5+2t)^3 dt = (5+2t)^{-3}\left(\frac{7}{8}(5+2t)^4 + C\right)$$

And we have arrived at the same answer we got the first time.

The definition of I involves an indefinite integral, so why didn't we need an arbitrary constant in I? What would have happened if we had written $\int a_0(t)dt$ as $3\ln(5+2t)+17$ instead of $3\ln(5+2t)$? You might think we would get a different answer that would also work, but in fact we would have ended up with exactly the same answer. You may want to confirm this right now for this particular example; you will prove it in general in Problem 10.75.

EXAMPLE **First-Order Linear Differential Equation**

Question: Solve the equation $(dy/dx)/x^2 + y/x^3 = 1$.

Solution:

This is a first-order linear equation, but to use the technique of this section we have to put it in the form of Equation 10.4.3, in which nothing is multiplied or divided by dy/dx.

$$\frac{dy}{dx} + \frac{y}{x} = x^2$$

Now we can see that $a_0(x) = 1/x$, so $\int a_0(x)\,dx = \ln x$ and $I(x) = x$. We therefore multiply both sides of the equation by x:

$x(dy/dx) + y = x^3$

The left side of the equation is now the derivative, with respect to x, of xy. (Confirm that!) So we integrate both sides with respect to x and get:

$xy = (1/4)x^4 + C$

Dividing both sides by x solves the problem. Alternatively, we could approach the problem—and reach the same answer—with Equation 10.4.9. Again we need to use the form of the equation with no factor in front of dy/dx, so $f(x) = x^2$:

$$y = \frac{1}{x}\int x^3 dx = \frac{1}{x}\left(\frac{1}{4}x^4 + C\right) = \frac{1}{4}x^3 + \frac{C}{x}$$

We leave it to you to confirm that this is a valid solution.

10.4.3 Problems: Linear First-Order Differential Equations

10.63 **Walk-Through: Linear First-Order ODE.** In this problem you will solve the equation $(\csc x)dy/dx - y = 1$.

(a) Rewrite this equation in the form of Equation 10.4.3.

(b) Identify the function $a_0(x)$.

(c) Find the integrating factor $I(x)$ as defined in Equation 10.4.7.

(d) Multiply both sides of the equation by the integrating factor you found in Part (c).

(e) Show that the left side of the resulting equation is the derivative, with respect to x, of $I(x)y(x)$.

(f) Integrate both sides of your answer to Part (d) with respect to x. (Note that Part (e) does half the work for you.) Don't forget to put a $+C$ on the right side!

(g) Solve the resulting equation. You should end up with a formula for $y(x)$ that includes an arbitrary constant.

(h) Show that your final answer is a valid solution to the original differential equation.

10.64 *[This problem depends on Problem 10.63.]* Redo Problem 10.63 using separation of variables. Make sure you get the same answer!

In Problems 10.65–10.72 solve the given first-order linear differential equation. You may find it helpful to first work through Problem 10.63 as a model.

10.65 $dy/dx + 3y = e^{3x}$

10.66 $dy/dx + y = \sin x$ *Hint*: you may find the integrals on Page 485 useful.

10.67 $dy/dx + y/(2x) = e^x/\sqrt{x}$

10.68 $dy/dx = (\sin x)(\cos x - y/\cos x)$

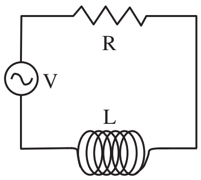
10.69 $\dfrac{dy}{dx} + \left(\dfrac{e^x}{e^x+1}\right)y = \dfrac{1}{e^x}$

10.70 $x^2(dy/dx) = 3xy - x - 1$

10.71 $(y-2)dx + dy/(2x^2+3) = 0$

10.72 $(x^2-4)(dy/dx) = 3x(y+2)$

10.73 The picture shows a circuit with three elements: a voltage source, a resistor, and an inductor.



The general equation governing this circuit is $L(dI/dt) + RI = V(t)$ where L is the inductance of the inductor, I is the current, R is the resistance of the resistor, and $V(t)$ is the voltage output of the source. Assume $R = 10\ \Omega$, $L = 2$ H, and $V(t) = V_0 \sin(\omega t)$.

(a) Write the differential equation for $I(t)$ for this circuit.

(b) Solve it. *Hint*: you may find the integrals on Page 485 useful.

10.74 Consider a special case of Equation 10.4.3 for which $a_0(x) = f(x)$.

(a) Evaluate Equation 10.4.9 using the substitution $u = \int f(x)\,dx$. You should get an answer for $y(x)$ that depends on $f(x)$.

(b) Show that the answer you found in Part (a) works.

(c) Use your result to write down the solution to $dy/dx + x^3y = x^3$ with little or no work.

10.75 When you calculate $I(x)$ in Equation 10.4.9, there is always an implicit $+C$ in the function. Show that including that constant does not change the final answer you get for the solution y. (Once you have proven this, you can ignore that $+C$ when applying this formula.)

10.76 A 5000 gallon tank is initially filled with pure water. A pipe at the top is pouring 100 gallons per minute of Gluppity-Glupp into the tank, while a drain at the bottom is dumping 200 gallons per minute of mixed water and Glupp into the pond where the Humming-Fish hum. Let $G(t)$ be the number of gallons of Gluppity-Glupp in the tank. You may assume throughout the problem that the tank is well mixed, meaning the ratio of Glupp to water going out the drain equals the ratio of Glupp to water in the entire tank.

(a) What is the total volume in the tank $V(t)$?

(b) What fraction of the tank mixture is Gluppity-Glupp? Your answer should depend on G and t.

(c) How many gallons of Glupp leave the tank each minute through the drain?

(d) Write a differential equation for $G(t)$ that takes into account increases in G through the input pipe and losses through the drain.

(e) Solve the equation to find $G(t)$ using the initial condition given in the problem.

(f) Find the total amount of Gluppity-Glupp dumped into the pond from the start of the process to the time when the tank is empty.

10.77 You invest A dollars per year in a bank account which gains interest at a rate R. That means every year the bank gives you an amount of money equal to R times the amount you have in the bank.

(a) Write a differential equation for the amount of money $M(t)$ you have in the bank. (Assume the investment and interest both occur continuously, not just once per year.)

(b) Assume the amount you invest and the interest rate are both dropping exponentially: $A = 1000e^{-t}$ and $R(t) = .05e^{-t}$ where t is measured in years. Solve for $M(t)$.

(c) If you start with nothing in the bank, how much money do you have after 10 years?

10.78 For as long as anyone can remember the birth and death rates in the city of Foom have been exactly equal, keeping the population constant. At a certain time, however, the birth rate started dropping. To try and compensate the leaders of Foom began allowing immigrants to enter the city each year.

(a) Assuming the birth rate dropped linearly, the death rate remained constant, and the immigration rate grew linearly, write and solve a differential equation for the population of Foom starting at the moment when the birth rate started dropping and immigrants began arriving. (Note that birth rate has units of year^{-1}—it is the number of people born each year divided by the total population—and similarly for death rate. Immigration, on the other hand, has units of people/year.)

(b) Describe the behavior indicated by your solution. Will the population grow without bound? shrink toward zero? Approach a constant value? Does the answer depend on the values of the constants?

10.79 **Make Your Own.** Write a first-order linear differential equation that isn't in this section (including the problems) and solve it using the techniques from this section. (The tricky part is making sure you can evaluate both integrals!)