10.5 Exact Differential Equations

In this section we will encounter differential equations written in the unfamiliar looking form $P\,dx + Q\,dy = 0$. In the specific case where $\partial P/\partial y = \partial Q/\partial x$ this is called an "exact differential equation" and has a simple solution.

This section relies on your prior understanding of differentials, as discussed in Chapter 4.

10.5.1 Discovery Exercise: Exact Differential Equations

The buoyancy B of a hot air balloon is a function of the temperature T of the air inside the balloon and the volume V of the balloon.

1. If the air temperature changes while the volume stays constant, the resulting change in buoyancy is given by:
$$dB = \frac{\partial B}{\partial T} dT$$
Explain why. Your explanation should focus on the meaning of that partial derivative.
2. If the volume changes while the air temperature stays constant, what is the resulting change in buoyancy?
3. If the temperature and volume both change, what is the total resulting change in buoyancy?

10.5.2 Explanation: Exact Differential Equations

The following, believe it or not, is a differential equation.

$$3x^2y\,dx + x^3\,dy = 0 \qquad (10.5.1)$$

This section will answer two questions about that equation and others like it.

- What on Earth does that even mean?
- And oh yeah, how do I solve it?

One way to answer both questions is to rearrange the terms into a more familiar-looking form.

$$\frac{dy}{dx} = \frac{-3x^2y}{x^3} \qquad (10.5.2)$$

We all know what *that* means, and how to solve it. Mathematically it's perfectly valid to turn Equation 10.5.1 into Equation 10.5.2. But for reasons we will explain soon, we want instead to take Equation 10.5.1 on its own terms.

How to Read (and Solve) a Funny-Looking Equation Like That One

Equation 10.5.1 presents a relationship between x and y, but we're going to introduce a new variable f that depends on both x and y. What happens to f if you change x by a small amount dx and change y by a small amount dy? Such a "total differential" is found by using partial derivatives. (Partial derivatives, and equations such as the following, are explained in Chapter 4.)

$$df = \left(\frac{\partial f}{\partial x}\right) dx + \left(\frac{\partial f}{\partial y}\right) dy$$

That's starting to look a bit like Equation 10.5.1. And if we happen to choose $f(x, y) = x^3y$, then it looks a *lot* like Equation 10.5.1. With that insight we are ready to address the two questions that started us off.

- What does Equation 10.5.1 mean? It means that if we change x by a small amount dx, and change y by a small amount dy, then the function $f(x, y) = x^3 y$ will not change at all: its df will be zero.
- And how do we solve it? If $df = 0$ then f must be a constant, and it doesn't matter what constant. So we write $x^3 y = C$ and we have the general solution, arbitrary constant and all.

EXAMPLE **Verifying a Solution to an Exact Differential Equation**

Problem:
We said above that $x^3 y = C$ is the solution to Equation 10.5.1. Verify this solution.

Solution:
Let's pretend for the moment that x and y are both functions of a third variable—we will call it t and think of it as time, but it could be anything. Now take the derivative of both sides of the solution with respect to time. (Remember that C is *not* a function, but a constant.)

$$x^3 y = C$$
$$3x^2 \frac{dx}{dt} y + x^3 \frac{dy}{dt} = 0$$

Multiply both sides by dt and you have the differential equation we set out to solve.

You can think about this result geometrically. For any given value of C the equation $x^3 y = C$ traces out a curve. (These are called the "level curves" of the function $x^3 y$.) If you start at the point (x, y) and take a small step along the curve, your dy is related to your dx by the equation of the curve. The differential equation defines that family of curves by saying how dy and dx are related at each point (x, y).

Of course we can rewrite that solution as $y = C/x^3$. You can verify that this is a valid solution to Equation 10.5.2 (or you can solve Equation 10.5.2 by separating variables and verify that you end up with C/x^3). But the method we are presenting does not depend on being able to solve for y; it is a more general technique than that, and we present it now in general form.

Exact Differential Equations: First Definition, and Solution

The equation:

$$P(x, y)\ dx + Q(x, y)\ dy = 0 \qquad (10.5.3)$$

is an "exact differential equation" if there exists a function $f(x, y)$ such that

$$\partial f/\partial x = P \quad \textit{and} \quad \partial f/\partial y = Q \qquad (10.5.4)$$

In that case, the general solution is:

$$f(x, y) = C$$

You may still be wondering why we prefer this new, abstract formulation to the more familiar approach represented by Equation 10.5.2. One reason is that many differential equations can't easily be solved in the form $dy/dx =$<*something*>. If we write an equation in the form of Equation 10.5.3 it's easy to check if it is exact and solve it if it is, as we'll explain below. Another reason is that exact differential equations are not limited to two variables. You can solve $P(x, y, z)\ dx + Q(x, y, z)\ dy + R(x, y, z)\ dz = 0$ with the same approach, but you cannot write it in terms of a simple derivative of x, y, or z. Such equations come up in applications where differentials play an important role, and it is important to understand and be able to work with them. (We discuss the role of such equations in thermodynamics in Section 4.10: see felderbooks.com.)

How Do You Know If Your Equation is Exact?

Here's a different definition of "exact."

Exact Differential Equations: Second (Equivalent) Definition

Equation 10.5.3 represents an exact differential equation if and only if:

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \tag{10.5.5}$$

You can verify this quickly on the example we gave above. The function $P = 3x^2y$ so $\partial P/\partial y = 3x^2$. And $Q = x^3$ so $\partial Q/\partial x = 3x^2$. Because the two come out the same, we know the equation is exact.

Some textbooks use Equation 10.5.4 as a definition of exact (as we have), and others use Equation 10.5.5. We will show below that the two definitions are equivalent, but first let's talk about why it is useful to have both.

You start with a problem in the form of Equation 10.5.3: that is, you are given the functions $P(x, y)$, and $Q(x, y)$. Equation 10.5.4 tells us that they define an exact differential equation if "there exists a function $f(x, y)$ such that…" If you can find such a function, you have the whole problem solved. But how do you know if such a function exists or not?

By contrast, Equation 10.5.5 gives you an easy method based only on the given P and Q to determine if a given equation is exact. But even if it is, it doesn't give you any hint of the solution.

So a common approach is to begin with Equation 10.5.5. If $\partial P/\partial y$ and $\partial Q/\partial x$ are not equal, then the equation is not exact, and you must move on to other methods. If they are equal, then you integrate to find the function f that you *know* must exist, and that will give you the solution. We discuss later in the section how to generalize this rule to equations with more than two variables.

How Do You Solve an Exact Differential Equation?

The example below illustrates the method we just discussed. First we use Equation 10.5.5 to check that the equation is exact. Once we know that, our goal is to solve the equations $\partial f/\partial x = P$ and $\partial f/\partial y = Q$—a process that is easier to demonstrate (below) than to explain (here).

EXAMPLE Exact Differential Equation

Question: Solve this differential equation.

$$2x \ln y \, dx + \left(\frac{x^2}{y} + 6e^{2y} \right) dy = 0$$

Solution:

We begin by determining if this is an exact equation, based on Equation 10.5.5.

$$P(x,y) = 2x \ln y, \quad \text{so} \quad \frac{\partial P}{\partial y} = \frac{2x}{y}$$
$$Q(x,y) = \frac{x^2}{y} + 6e^{2y}, \quad \text{so} \quad \frac{\partial Q}{\partial x} = \frac{2x}{y}$$

Because the two answers are equal, this is an exact differential equation. We now set out to find the function f that Equation 10.5.4 promises.

We begin with $\partial f/\partial x = 2x \ln y$. This means that f could be $x^2 \ln y$, but it doesn't have to be exactly that. If we add any constant to that function, or any pure function of y, then $\partial f/\partial x$ will remain unchanged. So we write

$$f(x,y) = x^2 \ln y + g(y)$$

Based on that equation, $\partial f/\partial y = x^2/y + g'(y)$. But $\partial f/\partial y$ must be Q, which in this case is $x^2/y + 6e^{2y}$. So we see that $g'(y) = 6e^{2y}$, meaning $g(y) = 3e^{2y}$.

We have now found that $f(x,y) = x^2 \ln y + 3e^{2y}$ has the appropriate partial derivatives. The solution to our differential equation is therefore:

$$x^2 \ln y + 3e^{2y} = C$$

(When we integrated $g'(y)$ to get $g(y)$ we could have included $+C$, but it would have gotten absorbed in the arbitrary constant we set f equal to anyway.)

If the middle step in that example made you suspect that our original differential equation was carefully contrived, you're absolutely right. What if the coefficient of dy had not looked like x^2/y plus a pure function of y? That would mean the original equation was not exact. It would have failed the $\partial P/\partial y = \partial Q/\partial x$ test, and no $f(x,y)$ function could be found.

We've given two completely different definitions of "exact." Why are they the same?

Equation 10.5.4 tells us there exists a function $f(x,y)$ such that $\partial f/\partial x = P$. Take the derivative of both sides of that equation with respect to y and you get this.

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial P}{\partial y}$$

Similarly, you can start with $\partial f/\partial y = Q$ and take the derivative of both sides of that equation with respect to x.

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial Q}{\partial x}$$

Since partial derivatives commute, those two quantities must be equal, so Equation 10.5.4 leads us to Equation 10.5.5.

If you're familiar with vector calculus, we can reframe everything we have said more concisely. Equation 10.5.3 is exact if $P(x, y)\hat{i} + Q(x, y)\hat{j}$ represents a *conservative vector field.* (See Section 8.11.) Recall that a vector field $\vec{V}$ is conservative if there exists a function f such that $\vec{V} = \vec{\nabla} f$, which is what Equation 10.5.4 says. We also know that $\vec{V}$ is conservative if and only if $\vec{\nabla} \times \vec{V} = \vec{0}$, which is what Equation 10.5.5 says.

This interpretation also suggests the way to generalize Equation 10.5.5 to three-variable equations.

Exact Differential Equations: Three-Variable Definition

The equation $P(x, y, z)dx + Q(x, y, z)dy + R(x, y, z)dz = 0$ is exact if and only if $\vec{V} = P\hat{i} + Q\hat{j} + R\hat{k}$ represents a conservative vector field: in other words, if $\vec{\nabla} \times \vec{V} = \vec{0}$.

The curl operator is not defined in more than three dimensions, however, so for four or more variables just use the brute force method. See Problems 10.92–10.93.

Integrating Factors

Recall from Section 10.4 that we sometimes multiply both sides of a differential equation by the same thing—an "integrating factor." One use of this technique is to *make* an exact equation where there was none.

EXAMPLE Integrating Factor

Problem:

Solve the equation $3y\ dx + x(2y + 1)\ dy = 0$

Solution:

As given, the equation is not exact. ($\partial P/\partial y \neq \partial Q/\partial x$.) You can try multiplying both sides of the equation by x, and it's still not exact. Try a few other things. (Go ahead, we'll wait.)

Now we'll multiply both sides of the equation by $x^2 e^{2y}$.

$3x^2 ye^{2y}\ dx + x^3 e^{2y}(2y + 1)\ dy = 0$

Ta-da! $\partial P/\partial y$ is now the same as $\partial Q/\partial x$. With a bit more work (as in the previous example) you can solve it.

$$x^3 ye^{2y} = C$$

We know that is the solution because if $f(x, y) = x^3 ye^{2y}$ then $\partial f/\partial x = 3x^2 ye^{2y}$ and $\partial f/\partial y = x^3 e^{2y}(2y + 1)$. You should confirm, however, that this also solves the original differential equation.

We hope you can easily confirm for yourself that the original equation in that example was not exact, that multiplying by x would not have made it exact, and that multiplying by $x^2 e^{2y}$ did. But none of that suggests how you can come up with integrating factors on your own. There is no general answer to that question, but there is a formula you can use in some cases.

The Integrating Factor to Make an Exact Differential Equation

Given an equation that is in the form of Equation 10.5.3 but is not exact, you want to find an integrating factor I that will make it exact. If such a factor exists that is a function of x and not of y, then it is:

$$I(x) = e^{\int \left[\frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)\right] dx} \tag{10.5.6}$$

If a factor exists that is a function of y and not of x, then it is:

$$I(y) = e^{\int \left[\frac{1}{P}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\right] dy} \tag{10.5.7}$$

In many cases—such as the example we worked above—the integrating factor is a function of both x and y, in which case those formulas won't help. But a formula that works sometimes is better than no formula at all.

These formulas follow from Equation 10.5.5. In general, an integrating factor $I(x, y)$ makes a differential equation exact if:

$$\frac{\partial}{\partial y}(IP) = \frac{\partial}{\partial x}(IQ)$$

Applying the product rule turns this into:

$$I\frac{\partial P}{\partial y} + P\frac{\partial I}{\partial y} = I\frac{\partial Q}{\partial x} + Q\frac{\partial I}{\partial x}$$

If it happens that I is a function of x only, then $\partial I/\partial y = 0$. Dropping that term, dividing both sides by QI, and rearranging leads to:

$$\frac{1}{I}\left(\frac{dI}{dx}\right) = \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$$

The left side of this equation is the derivative with respect to x of $(\ln I)$, so integrating both sides and then exponentiating leads to Equation 10.5.6. A similar argument leads to Equation 10.5.7 for integrating factors that only depend on y. (See Problem 10.101.)

EXAMPLE Finding an Integrating Factor

Problem:

Solve the equation $2xy \sin y\, dx + x^2 y \cos y\, dy = 0$

Solution:

The equation as given is not exact. Applying Equation 10.5.6 we begin with:

$$\frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = \frac{1}{x^2 y \cos y}\left(2x \sin y + 2xy \cos y - 2xy \cos y\right)$$

It looks promising, with most of the terms in the parentheses canceling, but there's a problem. When you simplify this, you're going to end up with a function of both x and y. Integrating with respect to x, and exponentiating, is not going to make those

"y"s go away. Equation 10.5.6 only gives us a useful integrating factor if it gives us a function of x only, so this won't help. (You can finish integrating and exponentiating this, but you'll find that it doesn't make the differential equation exact.)

Let's see if Equation 10.5.7 works out any better.

$$\frac{1}{P}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) = \frac{1}{2xy\sin y}\left(2xy\cos y - 2x\sin y - 2xy\cos y\right)$$

That's more like it! That simplifies to $-1/y$; $\int(-1/y)dy = -\ln y$, so $e^{-\ln y} = 1/y$ is the integrating factor. This turns our original equation into:

$$2x\ \sin y\ dx + x^2\ \cos y\ dy = 0$$

which is exact, as promised. The solution is $x^2 \sin y = C$.

10.5.3 Problems: Exact Differential Equations

10.80 Walk-Through: Exact ODE. $2\cos(2x + y)\ dx + \left(\cos(2x+y) + 3\sin y\right)\ dy = 0$

(a) This problem is in the form of Equation 10.5.3. What are the functions $P(x,y)$ and $Q(x,y)$?

(b) Use Equation 10.5.5 to show that this is an exact differential equation.

(c) Because this is an exact differential equation, there must exist a function $f(x,y)$ such that $\partial f/\partial x = P$ and $\partial f/\partial y = Q$. To begin finding it, write down the general solution to the equation $\partial f/\partial x = P$ for the $P(x,y)$ that you wrote above. Note that your solution at this stage will involve an arbitrary function $g(y)$.

(d) Take the partial derivative with respect to y of the $f(x,y)$ function you wrote in Part (c). Set the result equal to your $Q(x,y)$ function and solve to find $g(y)$.

(e) Write the function $f(x,y)$ and confirm that it fulfills Equation 10.5.4.

(f) Write the solution to the differential equation.

(g) Assuming that x and y are both functions of t, verify that your answer solves the original differential equation.

10.81 *[This problem depends on Problem 10.80.]* Rewrite the differential equation in Problem 10.80 in the form $dy/dx =$ <*some function of x and y*>. Show that your final solution to Problem 10.80 is a valid solution to this differential equation.

In Problems 10.82–10.88 determine if the given differential equation is exact or not. If it is, solve it. You may find it helpful to first work through Problem 10.80 as a model.

10.82 $x\ dx + y\ dy = 0$

10.83 $y\ dx + x\ dy = 0$

10.84 $y\ dx - x\ dy = 0$

10.85 $(6x + 6xy + 10y)\ dx + (3x^2 + 10x + 14y)\ dy = 0$

10.86 $(10x + 3y + 8)\ dx + (4x + 4y)\ dy = 0$

10.87 $\dfrac{y}{(x+y)^2}dx - \dfrac{x}{(x+y)^2}dy = 0$

10.88 $\left(\dfrac{x}{\sqrt{x^2 - y}} + 2x\right)dx - \left(\dfrac{1}{2\sqrt{x^2-y}} + 2y\right)dy = 0$

10.89 In special relativity the length of an object is given by the formula $L = L_0\sqrt{1 - v^2/c^2}$ where L_0 is the "rest length" of the object, v is its speed, and c, the speed of light, is a constant.

(a) If the rest length of the object increases by a small dL_0, calculate the resulting change dL in the length.

(b) If the object increases its speed by a small dv, calculate the resulting change dL in the length.

(c) Write a differential equation that says "Both L_0 and v increased by small amounts in such a way that there was no net change in the length." Your differential equation should be in the form of Equation 10.5.3.

(d) Solve your differential equation.

10.90 A light source of strength S is shining on an object x meters away. You have measured that when you increase the strength of the source the illumination of the object increases according to $\partial I/\partial S = k/x^2$, where k is a positive constant. You've also measured that when you move the object farther from the source the illumination deceases: $\partial I/\partial x = -2kS/x^3$. Write and solve a differential equation of the form <*something*>dS+<*something*>$dx = 0$ that says "Both S and x increased by small amounts in such a way that there was no net change in the illumination."

10.91 Measurements of the electric field in a region give $\vec{E} = (-2xe^{-x^2-y^2}+1)\hat{\imath} - 2ye^{-x^2-y^2}\hat{\jmath}$. The electric potential V is related to the electric field by $\partial V/\partial x = -E_x$ and $\partial V/\partial y = -E_y$. Find $V(x, y)$, up to an arbitrary additive constant.

10.92 In this problem you'll solve the equation $y\,dx + (x + 2y\sin z)dy + y^2\cos(z)dz = 0$.

(a) To check that this equation is exact, define a vector $\vec{V} = y\hat{\imath} + (x + 2y\sin z)\hat{\jmath} + y^2\cos(z)\hat{k}$ and show that $\vec{\nabla}\times\vec{V} = \vec{0}$.

(b) Since the equation is exact there must be a function $f(x, y, z)$ such that $\vec{\nabla} f = \vec{V}$. To find that function, first solve $\partial f/\partial x = y$. Your answer should contain an arbitrary function $g(y, z)$.

(c) Using your answer to Part (b), calculate $\partial f/\partial y$ and set it equal to $x + 2y\sin z$. By solving the resulting equation you should be able to find f up to an arbitrary function $h(z)$.

(d) Finally, set $\partial f/\partial z = y^2\cos z$ and solve to find f up to an arbitrary constant.

(e) Write the solution to the differential equation in the form $f = C$, where C is the (only) arbitrary constant in the solution.

10.93 With four or more variables, you can't use the curl to test if a differential equation is exact, so you simply have to start trying to solve it and see if it works. Consider two differential equations that we will call D_1 and D_2.

$$D_1 : (yzt + e^z)dx + xzt\,dy + (xyt + xe^z)dz + (xyz + 2t)\,dt = 0$$
$$D_2 : (yzt + e^z)dx + xzt\,dy + (xyt + e^z)dz + (xyz + 2t)\,dt = 0$$

Show that one of them is not exact. Show that the other one is exact, and solve it.

10.94 In this problem you will solve the equation $y\,dx + 2\tan x\,dy = 0$ by using an integrating factor.

(a) Show that the equation as given is *not* exact.

(b) Multiply both sides of the equation by $y\cos x$.

(c) Show that the resulting equation is exact, and solve it.

10.95 In this problem you will solve the following equation by using an integrating factor.

$$\frac{3y^2+2y}{x}\,dx + (xy^2 - 3y)\,dy = 0$$

(a) Show that the equation as given is *not* exact.

(b) Multiply both sides of the equation by $1/(xy)$.

(c) Show that the resulting equation is exact, and solve it.

In Problems 10.96–10.99 use Equations 10.5.6 and 10.5.7 to find an appropriate integrating factor and solve the equation.

10.96 $dx + 2x\cos y\,dy = 0$

10.97 $e^{x+2y}dx + \left(2 + 2/y\right)e^{x+2y}dy = 0$

10.98 $2xy\,dx + 3x^2\,dy = 0$

10.99 $\dfrac{2\ln(x^2+y)}{x}dx + \dfrac{\ln(x^2+y)}{x^2}dy = 0$

10.100 $P(x, y)dx + Q(x, y)dy = 0$ is an exact differential equation with solution $f(x, y) = C$. Is $P(x, y)dx + (Q(x, y) + 7)dy = 0$ also an exact differential equation? If not, why not? If so, what is the solution?

10.101 Show that if an integrating factor exists that is a function of y only, it must be given by Equation 10.5.7.

10.102 The "thermodynamic identity" for a gas in a sealed container (constant number of molecules) relates the change in internal energy U of the gas to changes in its entropy S and volume V: $dU = T\,dS - P\,dV$ where T and P are the temperature and pressure of the gas. For a monatomic ideal gas these are given by

$$T = \frac{C}{V^{2/3}}e^{(2/3)S/(Nk_B)}, \quad P = \frac{CNk_B}{V^{5/3}}e^{(2/3)S/(Nk_B)}$$

The constants N and k_B are the number of molecules and Boltzmann's constant, and C is a another constant that depends on N. Starting from the thermodynamic identity, derive a relationship between S and V that must hold for an ideal gas undergoing a process at constant internal energy. (If you know enough thermodynamics there are easier ways to derive this. You could do that to check yourself, but for this problem you should derive it by setting $dU = 0$ and solving the resulting equation.)

10.103 The "thermodynamic identity" for a gas relates the change in internal energy U of the gas to changes in its entropy S, volume V, and number of molecules N: $dU = TdS - PdV + \mu dN$. T, P, and μ are the temperature, pressure, and "chemical potential" of the gas. For a monatomic ideal gas these are given by

$$T = \frac{h^2 N^{2/3}}{2\pi e^{5/3} m k_B V^{2/3}} e^{(2/3)S/(Nk_B)}$$
$$P = \frac{h^2 N^{5/3}}{2\pi e^{5/3} m V^{5/3}} e^{(2/3)S/(Nk_B)}$$
$$\mu = -\left(\frac{S}{Nk_B} - \frac{5}{2}\right) \frac{h^2 N^{2/3}}{2\pi e^{5/3} m V^{2/3}} e^{(2/3)S/(Nk_B)}$$

Here m is the mass of a molecule, h is Planck's constant, and k_B is Boltzmann's constant. Starting from the thermodynamic identity, derive a relationship between S, V, and N that must hold for an ideal gas undergoing a process at constant internal energy. (If you know enough thermodynamics there are easier ways to derive this. You could do that to check yourself, but for this problem you should derive it by setting $dU = 0$ and solving the resulting equation.) *Hint*: rather than taking the curl, it's easier in this case to just look for a function that has the right partial derivatives.

10.104 You are in charge of the production line at Spacely Sprockets. Mr. Spacely has told you to increase production, but he refuses to increase your budget. You decide to accomplish this by putting less metal in each sprocket. Let C be your total cost, S be the number of sprockets you produce, and M be the grams of metal in each sprocket. Taking into account the grams of metal per sprocket M and the discounts you get for bulk buying, $\partial C/\partial S = M(S+10)/\sqrt{(S+10)^2 - 100}$ and $\partial C/\partial M = \sqrt{(S+10)^2 - 100}$. Find an equation relating S and M that will keep your total costs fixed.

10.105 You're conducting experiments on a flat table. The experiment produces varying amounts of heat in different places, and a series of measurements tells you that $\partial T/\partial x = e^y$ and $\partial T/\partial y = xe^y$. Sketch the isotherms (curves of constant temperature) on the surface.

10.106 You're conducting experiments on a flat table. The experiment produces varying amounts of heat in different places, and a series of measurements tells you that $\partial T/\partial x = \sin(y^2 + x) + x\cos(y^2 + x)$ and $\partial T/\partial y = 2xy\cos(y^2 + x)$. Sketch the isotherms (curves of constant temperature) on the surface.

10.107 **Make Your Own.**

(a) Write an exact differential equation that isn't in this section (including the problems) and solve it using the techniques from this section.

(b) Write a differential equation that is not exact, but that can be made exact by multiplying both sides by $y/(\ln x)$. *Hint*: This is very easy once you have done Part (a).

(c) Find a function $Q(x, y)$ to make $\sin(x + 2y)\,dx + Q(x, y)\,dy = 0$ an exact differential equation.