CHAPTER 11

Partial Differential Equations (Online)

11.12 Additional Problems

11.232 The temperature along a rod is described by the function $T(x,t) = ae^{-(bx^2+ct^2)}$.

- **(a)** Sketch temperature as a function of position at several times. Your vertical and horizontal axes should include values that depend on (some of) the constants a, b, and c.
- **(b)** Sketch temperature as a function of time at several positions. Your vertical and horizontal axes should include values that depend on (some of) the constants a, b, and c.
- **(c)** Make a three-dimensional plot of temperature as a function of position and time. *For this part and the next you may assume any positive values for the constants in the temperature function.*
- **(d)** Make an animation of temperature along the rod evolving in time.
- **(e)** Describe the behavior of the temperature of the rod.
- **(f)** Does this temperature function satisfy the heat equation 11.2.3?

11.233 The solution to the PDE $4(\partial z/\partial t) - 9(\partial^2 z/\partial x^2) - 5z = 0$ with boundary conditions $z(0,t) = z(6,t) = 0$ is $z(x,t) = \sum_{n=1}^{\infty} D_n \sin(\pi nx/6) e^{(-\pi^2 n^2+20)t/16}$. In this problem you will explore this solution for different initial conditions. If you approach this problem correctly it requires almost no calculations.

- **(a)** From this general solution we can see that one particular solution is $z(x,t) = 2\sin(\pi x/6)e^{(-\pi^2+20)t/16}$. What is the initial condition that corresponds to this solution?
- **(b)** Describe how the solution from Part (a) evolves in time.
- **(c)** If the initial condition is $z(x,0) = \sin(5\pi x/6)$, what is the full solution $z(x,t)$?
- **(d)** Describe how the solution from Part (c) evolves in time.
- **(e)** If the initial condition is $z(x,0) = \sin(\pi x/6) + \sin(5\pi x/6)$, what is the full solution $z(x,t)$?
- **(f)** Two of the solutions above look nearly identical at late times; the third looks completely different. (If you can see how these solutions behave from the equations, you are welcome to do so. If not, it may help to get a computer to make these plots for you.) Which two plots look similar at late times? Explain why these two become nearly identical and the third looks completely different.
- **(g)** If you had a non-sinusoidal function such a triangle or a square wave as your initial condition, you could write it as a sum of sinusoidal normal modes and solve it that way. Based on your previous answers, which of those normal modes would grow and which would decay? Explain why almost any initial condition would end up with the same shape at late times. What would that shape be?

In Problems 11.234–11.236 assume that the string being described obeys the wave equation (11.2.2) and the boundary conditions $y = 0$ at both ends.

11.234 A string of length π begins with zero velocity in the shape $y(x,0) = 5\sin(2x) - \sin(4x)$.

- **(a)** Guess at the function $y(x,t)$ that will describe the string's motion. The constant v from the wave equation will need to be part of your answer.

(b) Demonstrate that your solution satisfies the wave equation and the initial and boundary conditions. (If it doesn't go back and make a better guess!)

11.235 A string of length π is given an initial blow so that it starts out with $y(x,0)=0$ and

$$\frac{\partial y}{\partial t}(x,0)=\begin{cases} 0 & x<\pi/3 \\ s & \pi/3\le x\le 2\pi/3 \\ 0 & 2\pi/3<x \end{cases}$$

(a) Rewrite the initial velocity as a Fourier sine series.

(b) Write the solution $y(x,t)$. Your answer will be expressed as a series.

(c) Let $v=s=1$ and have a computer numerically solve the wave equation with these initial conditions and plot the result at several different times. Then make a plot of this numerical solution and of the 10^{th} partial sum of the series solution at $t=2$ on the same plot. Do they match?

11.236 A string of length L begins with $y(x,0)=0$ and initial velocity $\frac{\partial y}{\partial t}(x,0)=\begin{cases} x & 0\le x\le L/2 \\ L-x & L/2\le x\le L \end{cases}$. Find the solution $y(x,t)$.

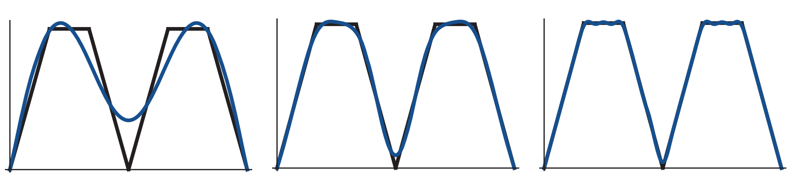
For Problems 11.237–11.239

(a) Solve the given problem using separation of variables. The result will be an infinite series.

(b) Plot the first three non-zero *terms* (not partial sums) of the series at $t=0$ and at least three other times. For each one describe the shape of the function and how it evolves in time.

(c) Plot successive partial sums at $t=0$ until the plot looks like the initial condition for the problem. Examples are shown below of what constitutes a good match.

(d) Having determined how many terms you have to include to get a good match at $t=0$, plot that partial sum at three or more other times and describe the evolution of the function. How is it similar to or different from the evolution of the individual terms in the series?



This is a bad match   Still not good enough   This is a good match

11.237

$$\frac{\partial^2 y}{\partial t^2}-\frac{\partial^2 y}{\partial x^2}+y=0,\ y(0,t)=y(4,t)=0,$$

$$y(x,0)=\begin{cases} 1 & 1<x<2 \\ -1 & 2\le x<3 \\ 0 & \textit{elsewhere} \end{cases},\quad \frac{\partial y}{\partial t}(x,0)=0$$

11.238 $$\frac{\partial y}{\partial t}-t^2\frac{\partial^2 y}{\partial x^2}=0,\ y(0,t)=y(3,t)=0,$$

$$y(x,0)=\begin{cases} x & 0\le x<1 \\ 1 & 1\le x\le 2 \\ 3-x & 2x\le 3 \end{cases}$$

11.239 $\partial^2 y/\partial t^2-\partial y/\partial t-\partial^2 y/\partial x^2+y/4=0$, $y(0,t)=y(1,t)=0$, $y(x,0)=x(1-x)$, $\partial y/\partial t(x,0)=x(x-1)$

Appendix I gives a series of questions designed to guide you to the right solution method for a PDE. For Problems 11.240–11.243 answer the questions in that appendix until you get to a point where it tells you what solution method to try, and then solve the PDE using that method. As always, your answer may end up in the form of a series or integral.

As an example, if you were solving the example on Page 648 you would say

1 The equation is linear, so we can move to step 2 and consider separation of variables.

2(a) The equation is not homogeneous, which brings us to…

2(e) We can't find a particular solution because the domain is infinite and anything simple (e.g. a line) would diverge as x goes to infinity. So we can't use separation of variables.

3 The domain is infinite so we can't use eigenfunction expansion.

4 The equation involves a first derivative of x, so we can't use the Fourier transform method.

5 Time has a semi infinite domain ($0\le t<\infty$) and t appears only in the derivatives, so we can try a Laplace transform.

Having come to this conclusion, you would then finish the problem by solving the PDE using the method of Laplace transforms.

11.240 $\partial y/\partial t - \alpha^2(\partial^2 y/\partial x^2) + \beta^2 y = 0$, $y(0,t) = y(L,t) = 0$, $y(x,0) = \kappa\left(x^3 - Lx^2\right)$

11.241 $\dfrac{\partial y}{\partial t} - \dfrac{\partial^2 y}{\partial x^2} = 1$, $y(0,t) = y(3,t) = 0$, $y(x,0) = \begin{cases} 1 & 1 \le x \le 2 \\ 0 & \textit{elsewhere} \end{cases}$

11.242 $\partial^2 u/\partial t^2 - c^2(\partial^2 u/\partial x^2) = \cos(\omega t)\sin^2(\pi x/L)$, $u(0,t) = u(L,t) = u(x,0) = \dot{u}(x,0) = 0$

11.243 $\partial^2 y/\partial t^2 - c^2(\partial^2 y/\partial x^2) = \alpha^2\cos(\omega t)e^{-\beta^2(x/x_0)^2}$, $y(x,0) = \dot{y}(x,0) = 0$, $\lim_{x\to-\infty} y(x,t) = 0$, $\lim_{x\to\infty} y(x,t) = 0$

In Problems 11.244–11.257 solve the given PDE with the given boundary and initial conditions. The domain of all the spatial variables is implied by the boundary conditions. You should assume t goes from 0 to ∞.

If your answer is a series see if it can be summed explicitly. If your answer is a transform see if you can evaluate the inverse transform. Most of the time you will not be able to, in which case you should simply leave your answer in series or integral form.

11.244 $\partial y/\partial t - 9(\partial^2 y/\partial x^2) = 0$, $y(0,t) = y(3,t) = 0$, $y(x,0) = 2\sin(2\pi x)$

11.245 $\partial y/\partial t - 9(\partial^2 y/\partial x^2) = 0$, $y(0,t) = y(3,t) = 0$, $y(x,0) = x^2(3-x)$

11.246 $\partial y/\partial t - 9\left(\partial^2 y/\partial x^2\right) = 9$, $y(0,t) = y(3,t) = 0$, $y(x,0) = 2\sin(2\pi x)$. *Hint*: After you find the coefficients, take special note of the case $n = 6$.

11.247 $\partial u/\partial t - \partial^2 u/\partial x^2 + \partial u/\partial x = e^{-t}$, $u(x,0) = u(0,t) = u(1,t) = 0$ *Hint*: depending on how you solve this, you may find the algebra simplifies if you use hyperbolic trig functions.

11.248 $\partial u/\partial t - \alpha^2(\partial^2 u/\partial x^2) - \beta^2(\partial^2 u/\partial y^2) = 0$, $u(0,y,t)=u(L,y,t) = u(x,0,t) = u(x,L,t) = 0$, $u(x,y,0) = \sin(\pi x/L)\sin(4\pi x/L)$

11.249 $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 - \partial^2 u/\partial z^2 = 0$, $u(0,y,z) = u(L,y,z) = u(x,0,z) = u(x,L,z) = u(x,y,0) = 0$, $u(x,y,L) = V$

11.250 $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 - \partial^2 u/\partial z^2 = 0$, $u(0,y,z) = u(L,y,z) = u(x,0,z) = u(x,L,z) = 0$, $u(x,y,0) = V_1$, $u(x,y,L) = V_2$ *Warning: the answer is long and ugly looking.*

11.251 $\partial y/\partial t - \partial^2 y/\partial x^2 = t\sin(3\pi x)$, $y(0,t) = y(1,t) = y(x,0) = 0$

11.252 $\partial y/\partial t = \partial^2 y/\partial x^2 + (1/x)(\partial y/\partial x) - y/x^2$, $y(0,t) = y(1,t) = 0$, $y(x,0) = x(1-x)$. You can leave an unevaluated integral in your answer.

11.253 $\dfrac{\partial y}{\partial t} - \alpha^2\dfrac{\partial^2 y}{\partial x^2} + \beta^2 y = \begin{cases} x & 0 < x < 1 \\ 1 & 1 \le x \le 2 \\ 3-x & 2 < x < 3 \end{cases}$, $y(0,t) = y(3,t) = 0$, $y(x,0) = x(3-x)$

11.254 $\dfrac{\partial y}{\partial t} - \dfrac{\partial^2 y}{\partial x^2} = 0$, $y(0,t) = y(3,t) = 0$, $y(x,0) = \begin{cases} 1 & 1 \le x \le 2 \\ 0 & \textit{elsewhere} \end{cases}$

11.255 $\partial y/\partial t - \partial^2 y/\partial x^2 = 0$, $y(0,t) = 0$, $y(1,t) = 1$, $y(x,0) = x^2$

11.256 $\partial^2 y/\partial t^2 - \alpha^2 t(\partial^2 y/\partial x^2) = 0$, $y(0,t) = y(\pi,t) = 0$, $y(x,0) = 5\sin(3x)$, $\partial y/\partial t(x,0) = 0$

11.257 $\partial^2 y/\partial t^2 - v^2(\partial^2 y/\partial x^2) = e^{-(x+t)^2}$, $y(x,0) = \partial y/\partial t(x,0) = 0$, $\lim_{x\to\infty} y(x,t) = \lim_{x\to-\infty} y(x,t) = 0$

11.258 The electric potential in a region without charges obeys Laplace's equation (11.2.5). Solve for the potential on the domain $0 \le x < \infty$, $0 \le y \le L$, $0 \le z \le L$ with boundary conditions $V(0,y,z) = V_0$, $V(x,0,z) = V(x,y,0) = V(x,L,z) = V(x,y,L) = 0$, $\lim_{x\to\infty} V = 0$.

11.259 You are conducting an experiment where you have a thin disk of radius R (perhaps a large Petri dish) with the outer edge held at zero temperature. The chemical reactions in the dish provide a steady, position-dependent source of heat. The steady-state temperature in the disk is described by Poisson's equation in polar coordinates.

$$\rho^2\frac{\partial^2 V}{\partial \rho^2} + \rho\frac{\partial V}{\partial \rho} + \frac{\partial^2 V}{\partial \phi^2} = \frac{\rho}{R}\sin\phi$$

(a) Begin by applying the variable substitution $\rho = Re^{-r}$ to rewrite Poisson's equation.

(b) What is the domain of the new variable r?

(c) Based on the domain you just described, the method of transforms is appropriate here. You are going to use a Fourier sine transform. Explain why this makes more sense for this problem than a Laplace transform or a Fourier cosine transform.

(d) Transform the equation. The formula is in Appendix G. You can evaluate the integral using a formula from that appendix (or just give it to a computer). Then solve the resulting ODE. Your general solution will have two arbitrary functions of p.

(e) The boundary conditions for ϕ are implicit, namely that $\hat{V}_s(\phi)$ and $\hat{V}_s'(\phi)$ must both have period 2π. You should be able to look at your solution and immediately see what values the arbitrary functions must take to lead to periodic behavior.

(f) Take the inverse transform. (You can get the formula from Appendix G and use a computer to take the integral.) Then substitute back to find the solution to the original problem in terms of ρ.

11.260 Exploration: A Time Dependent Boundary
A string of length 1 obeys the wave equation 11.2.2 with $v = 2$. The string is initially at $y = 1 - x + \sin(\pi x)$ with $\partial y/\partial t(x, 0) = x - 1$. The right side of the string is fixed $(y(1, t) = 0)$, but the left side is gradually lowered: $y(0, t) = e^{-t}$.

(a) First find a particular solution $y_P(x, t)$ that satisfies the boundary conditions, but does not necessarily solve the wave equation or match the initial conditions. To make things as simple as possible your solution should be a linear function of x at each time t.

The complete solution will be $y(x, t) = y_C(x, t) + y_P(x, t)$ where y_P is the solution you found in Part (a) and y_C is a complementary solution, still to be found.

(b) What boundary conditions and initial conditions should y_C satisfy so that y satisfies the boundary and initial conditions given in the problem?

(c) Using the fact that $y = y_C + y_P$ and $y(x, t)$ solves the wave equation, figure out what PDE y_C must solve. The result should be an inhomogeneous differential equation.

(d) Using the method of eigenfunction expansion, solve the PDE for y_C with the boundary and initial conditions you found.

(e) Put your results together to write the total solution $y(x, t)$.

(f) Based on your results, how will the string behave at very late times $(t >> 1)$? Does the particular solution you found represent a steady-state solution? Explain.