CHAPTER 12

Special Functions and ODE Series Solutions (Online)

12.9 Proof of the Orthgonality of Sturm-Liouville Eigenfunctions

In Section 12.8 we claimed that the eigenfunctions of a Sturm-Liouville problem, Equations 12.8.2–12.8.3, are orthogonal and complete. The proof of completeness is beyond the scope of this chapter[11], but we prove orthogonality below.

12.9.1 Explanation: Proof of the Orthgonality of Sturm-Liouville Eigenfunctions

When we claim that the sine functions are orthogonal, we mean the following.

$$\int_{-\pi}^{\pi} \sin(nx)\sin(mx)dx = 0 \text{ if } n \text{ and } m \text{ are distinct integers}$$

(The word "distinct" here means $m \neq n$, so we are dealing with two different functions.) That isn't too hard to prove. And you can imagine that with a bit more work we might be able to prove the same thing about any two distinct Legendre polynomials $P_n(x)$ and $P_m(x)$, or about two distinct Bessel functions $J_p(nx)$ and $J_p(mx)$. Each new function, each new normal mode, would have a different proof.

One of the remarkable accomplishments of Sturm-Liouville theory is to prove orthogonality relationships for all of these functions and many more with one relatively short bit of algebra. The proof is not based on the functions themselves, but on the differential equations that they solve.

Let y_m and y_n be two eigenfunctions of the Sturm-Liouville problem, Equations 12.8.2–12.8.3, with distinct eigenvalues λ_m and λ_n. To say they are solutions of the ODE means

$$\frac{d}{dx}\left(p(x)\frac{dy_m}{dx}\right) + q(x)y_m(x) + \lambda_m w(x)y_m(x) = 0$$
$$\frac{d}{dx}\left(p(x)\frac{dy_n}{dx}\right) + q(x)y_n(x) + \lambda_n w(x)y_n(x) = 0$$

We multiply each of these equations by the other eigenfunction and then subtract them.

$$\left[y_n\frac{d}{dx}\left(p(x)\frac{dy_m}{dx}\right) - y_m\frac{d}{dx}\left(p(x)\frac{dy_n}{dx}\right)\right] + \left(\lambda_m - \lambda_n\right) w(x)y_m(x)y_n(x) = 0 \qquad (12.9.1)$$

[11] See e.g. Birkhoff, Garrett and Rota, Gian-Carlo, "On the Completeness of Sturm-Liouville Expansions," The American Mathematical Monthly, Vol. 67, No. 9 (Nov., 1960), pp. 835–841.

Next we integrate from $x = a$ to $x = b$. You'll show in Problem 12.135 that this gives

$$p(x)\left[\left(y_n(x)y_m'(x) - y_m(x)y_n'(x)\right]_a^b + \left(\lambda_m - \lambda_n\right)\int_a^b w(x)y_m(x)y_n(x)dx = 0 \qquad (12.9.2)$$

We're trying to prove that the integral on the right equals zero, so we need to show that the term in square brackets on the left is zero. Since that term is evaluated at $x = a$ and $x = b$ we need to show that $p(a)[y_n(a)y_m'(a) - y_m(a)y_n'(a)]$ is zero, and likewise at $x = b$. The tool we can use here is the boundary condition.[12]

$$\begin{aligned} c_1 y_m(a) + c_2 y_m'(a) &= 0 \\ c_1 y_n(a) + c_2 y_n'(a) &= 0 \end{aligned} \qquad (12.9.3)$$

We multiply the first boundary condition by y_n and the second one by y_m and subtract, and we conclude that the term in square brackets is zero at $x = a$. A similar argument holds at $x = b$. (We have to divide by c_2 in the process, so this argument isn't valid if $c_2 = 0$. You'll show in Problem 12.136 that the conclusion still holds in that case.)

In sum, the fact that y_m and y_n both satisfy the original ODE led us to Equation 12.9.2. The fact that they both satisfy the boundary conditions at a and b led us to conclude that the first term in that equation is zero, so we were left with the orthogonality condition we were trying to prove.

12.9.2 Problems: Proof of the Orthgonality of Sturm-Liouville Eigenfunctions

12.133 In this problem you will prove that any distinct solutions y_m and y_n of the simple harmonic oscillator equation $y''(x) + \lambda y(x) = 0$ with boundary conditions $x(0) = x(L) = 0$ are orthogonal on the interval $0 \le x \le L$. You can do this by showing that the solutions are sines and then showing that sines are orthogonal, and you can also just say "this is an example of a Sturm-Liouville problem," but you're not going to do either of those. Instead you're going to follow the steps of the general Sturm-Liouville proof in the Explanation (Section 12.9.1).

- **(a)** Write an equation that asserts "If you write the SHO differential equation with eigenvalue λ_m then the solution is eigenfunction y_m."
- **(b)** Write an equation that asserts "If you write this differential equation with eigenvalue λ_n then the solution is eigenfunction y_n."
- **(c)** Multiple your equation from Part (a) by $y_n(x)$ and your equation from Part (b) by $y_m(x)$. Then subtract the two equations.
- **(d)** Integrate both sides of the resulting equation from 0 to L. Then use integration by parts to reduce second derivatives to first derivatives.
- **(e)** Use the boundary conditions to show that part of the resulting expression must be zero, and complete the proof of the orthogonality of the solutions to this equation. *Hint*: the first boundary condition for the generic Sturm-Liouville problem involves two constants called c_1 and c_2. It is possible for neither constant to be zero, and it is possible for one constant to be zero, but it is not possible for both constants to be zero because then you would be missing a boundary condition. This fact will be useful towards the end of the proof.

12.134 Why is there no $q(x)$ term in Equation 12.9.1?

12.135 Fill in the steps to get from Equation 12.9.1 to Equation 12.9.2. Start by integrating both sides of the equation from a to b

[12] You may recall from Section 12.8 that these boundary conditions don't apply at $x = a$ if $p(a) = 0$, but in that case the term on the left of Equation 12.9.2 is trivially zero at $x = a$.

and then use integration by parts to take the d/dx off of the py' terms.

12.136 Write out the steps in going from Equations 12.9.3 to showing that $p(a)[y_n(a)y_m'(a) - y_m(a)y_n'(a)] = 0$. You will have to handle $c_2 = 0$ as a special case.

12.137 We proved that $\int_a^b w(x)y_m(x)y_n(x)dx = 0$ for any two eigenfunctions y_m and y_n of the Sturm-Liouville problem where $m \neq n$. What step in our proof is invalid if $m = n$?

12.138 The functions $y_k = e^{kx}$ are solutions of the equation $y''(x) + \lambda y(x) = 0$ for $\lambda = -k^2$, but they are not orthogonal to each other. Explain why this does not violate what we've said about Sturm-Liouville theory.