2.9 Additional Problems

2.191 The first two terms in the Taylor series for a function $f(x)$ make up the linear approximation $f(x_0 + \Delta x) \approx f(x_0) + f'(x_0)\Delta x$.

(a) Under what circumstances would this expression be exactly correct for all values of Δx?

(b) When the linear approximation is not exact the next order approximation is $f(x_0 + \Delta x) \approx f(x_0) + f'(x_0)\Delta x + (1/2)f''(x_0)\Delta x^2$. Explain why this next term appears with a positive sign. In other words, why is the linear approximation generally too low when $f''(x_0) > 0$ and too high when $f''(x_0) < 0$?

For Problems 2.192–2.200 find the fourth-order Taylor series of the given function about the given point.

2.192 $1/(1+x)^3$ about $x = 0$

2.193 $x^2 + e^x$ about $x = 0$

2.194 $\tan\left(x^2\right)$ about $x = 0$

2.195 $1/(\sin x + \cos x)$ about $x = 0$

2.196 $e^{\ln x + 2}$ about $x = 0$. (Think about this one for a moment before starting.)

2.197 $\ln\left(x^2\right)$ about $x = 2$

2.198 e^{x+x^2} about $x = 1$

2.199 $1/(1 + \tan x)$ about $x = \pi/2$

2.200 $1/(1 - x^2)$ about $x = -2$

For Problems 2.201–2.204 find the 15th-order Maclaurin series of the given function. *Hint*: There's an easy way and a hard way to do each of these. We recommend you find the easy way.

2.201 $\sin\left(x^3\right)$

2.202 $\sin\left(x^2\right)/x^2$

2.203 $\ln(5e^x)$

2.204 $x^3 e^{-x^2}$

For Problems 2.205–2.208 use a computer to calculate partial sums of the Taylor series for the function about the midpoint of the domain. On one plot, show the function in black and its partial sums in different colors. You should show enough partial sums to clearly see how they are changing as you add more terms, and your final partial sum should match the function well throughout the domain.

2.205 $\sin x$, $-5 \le x \le 5$

2.206 e^{-x^2}, $-2 \le x \le 2$

2.207 e^{-x^2}, $0 \le x \le 2$

2.208 $\sin^3 x$, $-\pi/2 \le x \le \pi$

In Problems 2.209–2.213 you are given a non-linear differential equation that, like most non-linear differential equations, has no simple solution. For each one replace the right-hand side of the equation with a linear function that approximates it well under the specified assumptions, and solve the resulting approximate differential equation.

2.209 $d^2f/dx^2 = 1 - e^f$. Assume you know that $f(x)$ is going to stay close to 0.

2.210 $d^2x/dt^2 = -e^x$. Assume $x(t)$ stays close to $x = 1$.

2.211 $d^2x/dt^2 = -\ln x$. Assume $x(t)$ stays close to $x = 1$.

2.212 $dz/dt = 1 + \ln z$. Assume $z(t)$ is close to $z = 2$ for the period of time you are interested in. Explain why this approximation can *not* be used out to arbitrarily late times.

2.213 $d^2x/dt^2 = -k \sinh x$. Assume x is the displacement from equilibrium of a mass on a non-ideal spring and that it is oscillating with a small amplitude. (If you don't know what sinh is see Appendix J.)

For Problems 2.214–2.224 show whether the given series converges or diverges.

2.214 $\sum_{n=1}^{\infty} n^{-5}$

2.215 $\sum_{n=1}^{\infty} n^5$

2.216 $\sum_{n=1}^{\infty} 1/(1+n^5)$

2.217 $\sum_{n=1}^{\infty} n/(1+n^5)$

2.218 $\sum_{n=1}^{\infty} 1 + n^{-5}$

2.219 $\sum_{n=1}^{\infty} (\tanh n)/n$ (see Appendix J for tanh)

2.220 $\sum_{n=1}^{\infty} n/(1+n^3)$

2.221 $\sum_{n=1}^{\infty} (\sin n)/(n!)$

2.222 $\sum_{n=2}^{\infty} 2!(n-2)!/(n!)$

2.223 $\sum_{n=1}^{\infty} e^{1/n}/n^2$

2.224 $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n \cos(\pi n)}$

For Problems 2.225–2.232 determine the interval of convergence of the given power series. (In other words, for which x-values does this series converge?)

2.225 $\sum_{n=0}^{\infty} x^{2n}$

2.226 $\sum_{n=0}^{\infty} x^{2n}/(n!)$

2.227 $\sum_{n=0}^{\infty}(x+1)^n/(n!)$

2.228 $\sum_{n=0}^{\infty}(-1)^n(x+1)^n/(n!)$

2.229 $\sum_{n=0}^{\infty}\sin(\pi n/2)x^n/(n!)$

2.230 $\sum_{n=0}^{\infty}(x-1)^n/n^2$

2.231 $\sum_{n=0}^{\infty}(-1)^n(x-1)^{2n}/n$

2.232 $\sum_{n=0}^{\infty}e^{-n}(x-2)^n$

Problems 2.233–2.237 deal with finding bounds on the errors in series approximations. All of these problems use the formulas for those errors given in Appendix B.

2.233 Let $f(x)=e^x+e^{x^2}$. Use a first-order Maclaurin series to estimate $f(0.1)$. Use the Lagrange remainder to place an upper bound on the error of this approximation, and verify that the upper bound is correct.

2.234 Let $f(x)=e^x+e^{x^2}$. Use a 10th-order Maclaurin series to estimate $f(.8)$. Use the Lagrange remainder to place an upper bound on the error of this approximation, and verify that the upper bound is correct.

2.235
- **(a)** Use a third-order Maclaurin expansion of $\sin x$ to estimate $\sin(1.5)$.
- **(b)** Use a third-order Taylor series for $\sin x$ around $x=\pi/2$ to estimate $\sin(1.5)$.
- **(c)** Which answer would you expect to be more accurate? Why?
- **(d)** Use the Lagrange remainder to show that the error in your second approximation must be less than 1.1×10^{-6}.

2.236 Let $f(x)=e^{-x}$.
- **(a)** Find the third-order Maclaurin series for $f(x)$ and use it to estimate $e^{-0.2}$.
- **(b)** Use the rule for errors in alternating series to put an upper bound on the error in this estimate. Verify that your error is less than the upper bound.
- **(c)** Use your series for e^{-x} to estimate $e^{0.2}$. Explain why you can't use the same technique to put an upper bound on this value. Instead, find an upper bound using the Lagrange remainder. Express your answer as a ratio of the possible error in your estimate to the correct value for $e^{0.2}$.

2.237 Let $f(x)=1/(1+x)$. Use a second-order Maclaurin series to estimate $f(0.1)$ and find the bounds on that estimate using each of the three methods described in Appendix B. Verify that the actual error is lower than all three bounds. Which technique gives you the strictest bound? Which one is easiest to find?

2.238 Does the series $\sum_{n=1}^{\infty}(\pi/3)^n\sin(n\pi/2)/n!$ converge or diverge? If it does converge, what (exactly!) does it converge to?

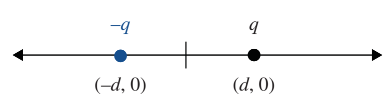
2.239 The differential equation $dx/dt=x+\sin x+\cos x$ is non-linear and has no simple solution.
- **(a)** Use a Maclaurin series to approximate the right side of this equation with a linear function of x, valid when $x\approx0$.
- **(b)** Solve this approximate differential equation with the initial condition $x(0)=0$.
- **(c)** Do you expect your approximation to remain valid at late times? Explain, using the solution you found.
- **(d)** Plot the approximate solution you found and the numerical solution to the original differential equation, from $t=0$ to $t=1$. Does the behavior match your prediction? Explain.

2.240 Use Maclaurin series to prove "Euler's Formula" which states that, for any real number x,

$$e^{ix}=\cos x+i\sin x$$

where i is an imaginary number, defined by the property $i^2=-1$.

2.241 The picture below shows an "electric dipole." Two equal and opposite charges sit on the x-axis.



The electric field on the x-axis due to the presence of these two charges is given by the equation

$$E=\frac{kq}{(x-d)^2}-\frac{kq}{(x+d)^2}\qquad(x>d)$$

This is very similar to the equation from the Motivating Exercise (Section 2.1), but in this case we are interested in points on the x-axis very far away from the crystal.
- **(a)** Factor out kq/x^2 from the equation for E. What you are left with should only depend on the fraction d/x.
- **(b)** If $x\gg d$ then d/x is small and you can expand the expression you just found

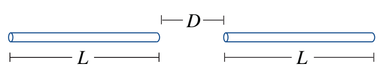
in a Maclaurin series for d/x. Find the first two non-vanishing terms of this expansion. (*Hint: You can make this much easier by using the binomial series.*)

(c) Using your Maclaurin series, argue that the electric field from a dipole drops off proportionally to $1/x^3$ at large distances.

2.242 Like π, e is an irrational number that has been calculated to large numbers of digits. (As of late 2010 the first trillion digits were known.) For this problem we'll let you get away with doing the first 10,000. Use the Maclaurin series for e^x with $x = 1$ to find successively better approximations of e. Keep adding terms up to and including the first term that is smaller than $10^{-10{,}000}$. Give as your final answer the 9997th through 10,000th digits. (Count the initial 2 as the first digit.)

2.243 Two rods, each of length L and charge per unit length λ, lie along the same line with a distance D between them. The electric force between these rods is

$$F = \frac{\lambda^2}{4\pi\epsilon_0} \ln\left(\frac{(L+D)^2}{D(2L+D)}\right)$$



(a) Rewrite this expression so L only appears in the combination L/D.

(b) Use a Maclaurin series in L/D to find an expression for the force valid when you pull the rods very far apart compared to their lengths. Keep only the first non-zero term.

(c) Rewrite $\lambda = Q/L$ where Q is the charge on each rod and simplify your answer. Explain why your answer makes physical sense in the limit $D \gg L$.

2.244 The "MIDI tuning standard" assigns a "midi note number" to each pitch: the note A440 is assigned $p = 69$, the A♯ a half-step above it is $p = 70$, and so on. This note number p is related to the frequency f of the sound (in Hz) by $p = 69 + (12/\ln 2)\ln(f/440)$. If a wind instrument plays A440, however, the actual frequency produced is $440\sqrt{T/T_0}$, where T is the temperature at which the instrument is being played and T_0 is the temperature at which the instrument was tuned.

(a) Write a first-order Taylor series for $p(T)$ about $T = T_0$.

(b) The formulas above assume temperature is measured in Kelvin. Suppose you tuned your flute at a comfortable room temperature of 290 K. Roughly how much would the pitch rise per degree of increase in T? Roughly how much would the temperature have to rise in order to increase your pitch from A to A♯? Use your Taylor series to answer both questions.

2.245 The relationship between the volume, pressure, and temperature of a gas can be written in the form $PV = nRT\left[1 + B(T)(n/V) + C(T)(n/V)^2 + \dots\right]$ where the coefficients B and C are called the second and third "virial coefficients." (You may be familiar with the "ideal gas law" that results from $B(T) = C(T) = 0$.) Calculate the second and third virial coefficients for the Van der Waals equation of state $(P + an^2/V^2)(V - nb) = nRT$. *Hint*: Start by writing the equation in the form $PV = nRTf(n/V, T)$ and then expand f in a Maclaurin series in n/V.

2.246 **Exploration: Simple Harmonic Oscillations.** Oscillations occur in a wide variety of situations, ranging from atoms vibrating around their positions in a crystal to wrecking balls swinging back and forth to giant waves moving up and down on the surface of stars. In all of these situations there is an object (the atom, the wrecking ball, the fluid on the surface of the star, ...) that experiences a force as a function of its position. In general these forces can be very complicated and can vary widely from one situation to another. In many situations, however, they can be well approximated by a simple equation.

(a) Using Newton's second law, write a differential equation for the position $x(t)$ of an object experiencing a force $F(x)$.

(b) Any oscillator moves back and forth across some equilibrium point. For simplicity you can always define that equilibrium point to be at $x = 0$, in which case it is natural to expand $F(x)$ in a Maclaurin series. Write a second-order Maclaurin series for $F(x)$ and plug this into the equation you wrote down in Part (a). The result should be a differential equation with d^2x/dt^2 on the left and three terms on the right.

(c) The force on an oscillator is always zero at the equilibrium point. Use that fact to eliminate one of the terms from your differential equation.

(d) Explain why for small amplitude oscillations one of the remaining terms will be much larger than the other one. Use that fact to eliminate one more term from your equation.

(e) In order for an object to oscillate it must have a "restoring force," which means that if $x > 0$ the force should be negative and if $x < 0$ the force should be positive. Use that fact to determine the sign of the constant in the one remaining term on the right-hand side of your differential equation.

(f) A simple harmonic oscillator is defined by the differential equation

$$\frac{d^2x}{dt^2} = -\omega^2 x$$

where ω is any (real) constant. Putting together everything you've done so far in this problem, explain why almost any oscillator can be well approximated by a simple harmonic oscillator for small amplitude oscillations. Write an equation for ω that depends on $F(x)$ and m.

(g) How could you have an oscillator that could *not* be well approximated by a simple harmonic oscillator for small amplitudes? In other words, write a force $F(x)$ that would describe an oscillator that would be an exception to the argument you just presented. Note that the answer is *not* to add damping because then the force F wouldn't just be a function of x.

2.247 Exploration: The Method of Power Series. We have seen that Taylor series can be used to help solve difficult differential equations by simplifying complicated functions. Taylor series can also be used in a more direct way to solve differential equations. In the "Method of Power Series" you assume a solution in the form of a power series and then solve for the coefficients. (This technique is explored further in Chapter 12.) In this problem you will solve $d^2y/dx^2 = -y$ with the conditions $y(0) = 0$ and $y'(0) = 1$ by plugging in the "guess" $y = c_0 + c_1x + c_2x^2 + c_3x^3 + \ldots$.

(a) Plug the initial condition $y(0) = 0$ into the "guess" and solve for c_0.

(b) Take the derivative of both sides of the guess and then use the initial condition $y'(0) = 1$ to solve for c_1.

(c) Now plug the guess into the differential equation you are solving. The resulting equation will set two different power series equal to each other.

(d) Your answer to Part (c) set two power series equal to each other. For two power series to be equal their constant terms must be equal. Write the resulting equation and solve it for c_2.

(e) Your answer to Part (c) set two power series equal to each other. For two power series to be equal their coefficients of x must be equal. Write the resulting equation and solve it for c_3.

(f) Following a similar process, solve for all coefficients up to c_7. Then write the solution to this differential equation as a Maclaurin series up to the seventh power.

(g) What function has that particular Maclaurin series? Does that function solve the given differential equation and initial conditions?

You probably knew the answer to that problem before you started. Next you will use the same technique to solve Airy's equation, which is used in optics to model the intensity of a rainbow and in quantum mechanics to represent a particle confined within a triangular potential well.

(h) Solve $\dfrac{d^2y}{dx^2} = xy$ with conditions $y(0) = 1$ and $y'(0) = 0$. Your solution will be in the form of a Maclaurin series up to the sixth power.