CHAPTER 3

Complex Numbers (Online)

3.6 Special Application: Electric Circuits

Figure 3.5 shows a circuit diagram with standard electrical symbols for a resistor, capacitor, and inductor. A more complicated circuit might have hundreds of these elements, each with a resistance R, a capacitance C, or an inductance L.

However simple or complicated the circuit, you provide a "stimulus" (a voltage $V(t)$ that might come for instance from a battery or outlet) and the circuit "response" is the resulting current $I(t)$. When you analyze the circuit, you determine what response it will have to a given stimulus.

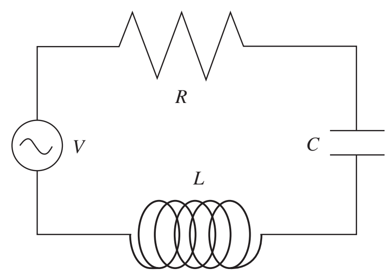


FIGURE 3.5 A simple RLC circuit.

In this section we will be looking at the response to a sinusoidal stimulus $V(t) = V_0 \sin(\omega t)$. This is a particularly important case because it models the voltage that comes out of a typical household outlet. Furthermore, more complicated functions can be built up as sums of sine waves (Problems 3.127–3.128), so a solution for a sinusoidal voltage turns out to be generally useful.

Before we go through the math, we're going to jump to the end and present most of the answer.

> **Response of an RLC circuit to a Sinusoidal Voltage**
>
> Given a voltage $V(t) = V_0 \sin(\omega t)$ the resulting current will be $I(t) = I_0 \sin(\omega t - \phi)$. That is, the current will oscillate with the same frequency ω as the voltage. However, the current will lag behind the voltage by a phase ϕ.

The current can also be expressed as $I(t) = A\sin(\omega t) + B\cos(\omega t)$ which may look like a more familiar solution to a differential equation. But the form $I(t) = I_0 \sin(\omega t - \phi)$ is mathematically equivalent (Problem 3.126) and lends itself to more direct physical interpretation. It is important to note that the "phase lag"[4] ϕ is measured in radians, not in seconds. A phase lag of zero means that the two oscillations are perfectly in sync; a phase lag of $\pi/2$ means that the current reaches its peak just as the voltage reaches zero.

In Figure 3.6, the period is 4 s (so $\omega = 2\pi/4$). The current lags the voltage by $1/3$ of a second, but it is more useful to view the lag as $\pi/6$ radians or as $1/12$ of a cycle.

But how do we compute the amplitude I_0 and phase lag ϕ of the response for any given circuit layout? This is where complex numbers come in.

[4] also called the "phase shift" or "phase difference".

Complex Impedance

The RLC circuit in Figure 3.5 comprises each of the basic circuit elements. The voltage drop[5] across a capacitor is proportional to the built-up charge ($V = Q/C$); the voltage drop across a resistor is proportional to the current ($V = IR$ or $V = Q'(t)R$); the voltage drop across an inductor is proportional to dI/dt ($V = LI'(t)$ or $V = LQ''(t)$). If we set the total voltage drop equal to the stimulus voltage $V(t)$ we get a differential equation for the charge Q. But it is preferable to work with the current I (which is easier to directly measure) so we take the derivative of both sides.

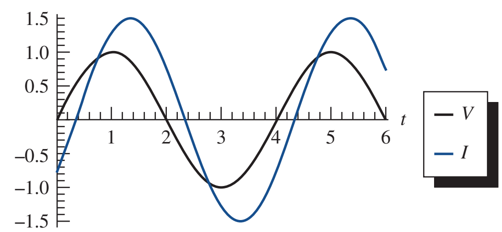


FIGURE 3.6 In this figure current lags voltage by $\pi/6$ radians. (A positive phase lag means current is to the right of voltage, so the voltage peaks first.)

$$LI''(t) + RI'(t) + \frac{1}{C}I = V'(t) \tag{3.6.1}$$

If we let $V(t) = V_0 \sin(\omega t)$ and solve this differential equation we can find everything we need to know about this particular circuit. But more complicated circuits have more complicated differential equations. So here is the easier approach. Because $V(t)$ is a sine, we can view it as the imaginary part of a complex exponential function $\mathbf{V}$. So Equation 3.6.1 becomes the imaginary part of the complex equation

$$L\mathbf{I}''(t) + R\mathbf{I}'(t) + \frac{1}{C}\mathbf{I} = \mathbf{V}'(t)$$

where $\mathbf{V} = V_0 e^{i\omega t}$. Since $V = Im(\mathbf{V})$ we know that the physical current I will be given by $I = Im(\mathbf{I})$.[6]

This equation is much easier to solve. We begin by guessing a solution of the form $\mathbf{I} = \mathbf{I_0} e^{i\omega t}$. (As we said above we are assuming that the current oscillates with the same frequency as the voltage. Mathematically or physically it's hard to imagine any other behavior, but as always the guess will prove itself by working.) When we plug in this guess and do a bit of algebra we end up here.

$$V_0 = \left(R + i\omega L - \frac{i}{\omega C}\right)\mathbf{I_0} \tag{3.6.2}$$

The quantity in parentheses is called the "impedance" Z, a complex number that represents the circuit layout of resistors, capacitors, and inductors. So the behavior of the circuit can be captured in a very simple-looking equation reminiscent of Ohm's law.

$$V_0 = \mathbf{I_0} Z \tag{3.6.3}$$

Because V_0 is real and Z is complex, we know that $\mathbf{I_0}$ must also be complex.

Equations 3.6.1 and 3.6.3 represent two very different approaches to analyzing a circuit.

Without complex numbers every circuit element looks mathematically different, as we saw before. A different arrangement of elements leads to a different differential equation, generally second order and not easy to solve.

[5] We use V for voltage; electrical engineers often use E.
[6] We could have considered a cosine for the potential and taken real parts instead of imaginary parts, but it's more common to use the imaginary part.

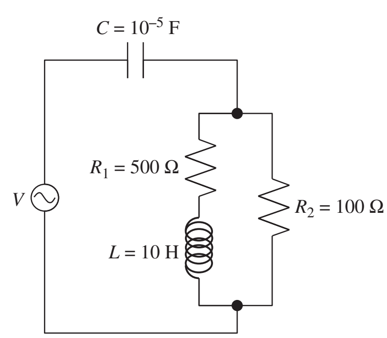
With complex numbers we approach RLC circuits very similarly to how we approach circuits that have only resistors: find the total impedance of all the circuit elements and write $V_0 = \mathbf{I_0} Z$. The resulting (complex) $\mathbf{I_0}$ tells us the (real) amplitude and phase lag of the current. Here are the steps we've left out of that brief description.

- How do you find the impedance? You can read the impedances of the individual elements directly from Equation 3.6.2 as R, $i\omega L$, and $-i/(\omega C)$. We combine elements just as we would combine resistors. Two elements "in series" (the current goes through one and then the other) add their impedances, so $Z_{\text{series}} = Z_1 + Z_2$. Two elements "in parallel" (the current goes through each element separately) add as $Z_{\text{parallel}} = \dfrac{1}{(1/Z_1) + (1/Z_2)}$.
- How do we find the amplitude of the current? Remember that $\mathbf{I}(t) = \mathbf{I_0} e^{i\omega t}$ and as we saw in Section 3.5 the amplitude of that function is the modulus of $\mathbf{I_0}$.
- Finally, what about the phase lag? Recall that the voltage is the imaginary part of $V_0 e^{i\omega t}$ and the current is the imaginary part of $\mathbf{I_0} e^{i\omega t}$. The exponentials are identical in the two expressions, so the phase difference between $\mathbf{V}$ and $\mathbf{I}$ is the phase difference between V_0 and $\mathbf{I_0}$. But $\mathbf{I_0} = V_0/Z$. Remembering how numbers divide on the complex plane, we conclude the phase lag is just the phase of Z.

The example below shows how we can predict the behavior of a circuit once all these pieces are in place.

EXAMPLE A Complicated Circuit

Question: If the circuit shown below is driven by a voltage $V = 3\sin(50t)$ find the amplitude of the resulting steady-state current and find the phase lag between the voltage and the current.



Solution:
We can find the equivalent impedance of this circuit by considering it to be made of two elements in series, the second of which is made up of two elements in parallel, one of which is made of two elements in series. That sounds messy, but using the rules for series and parallel circuit elements the total impedance is simply given by

$$Z = -\frac{i}{\omega C} + \frac{1}{\frac{1}{R_1 + i\omega L} + \frac{1}{R_2}}$$

Plugging in numbers gives $Z = 400 - 1800i$. Because $\mathbf{I} = \mathbf{V}/Z$ the amplitude of the current is given by $|\mathbf{V}|/|Z| = 3/1844 = 1.6 \times 10^{-3}$. The phase lag is the phase of Z, which in this case is $\tan^{-1}(-1800/400) = -1.4$ rad, or about $-80°$. That means the current leads the voltage by almost a quarter cycle, so when the voltage is at its peak the current has decreased nearly to zero.

And so what? If this circuit is part of an old-fashioned radio tuner, that might be perfectly fine. But if the source of this circuit is the power company, that $-80°$ phase lag is a disaster; it means you are getting almost none of the power you're paying for. See Problem 3.125.

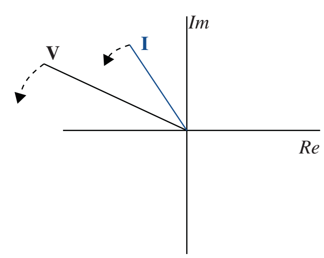
Note that the impedance of a resistor is just its resistance, but the impedance of an inductor or capacitor depends on the frequency of the voltage source. Physically this corresponds to the fact that a capacitor tends to impede low frequencies, an inductor impedes high frequencies, and a resistor is an equal opportunity impediment. Mathematically it reminds us that we are computing the response of a given circuit to a specific sinusoidal stimulus. A more complicated stimulus can be represented as a combination of different-frequency sine waves (a "Fourier series"); see Problems 3.127 and 3.128.

Stepping Back

We have arrived at a very general description of the "response" (the current) of a circuit to a sinusoidal "stimulus" (voltage source). For a circuit with voltage source $V = V_0 \sin(\omega t)$, the current will be $I = I_0 \sin(\omega t - \phi)$. To find the amplitude I_0 and the phase lag ϕ of the circuit response:

- Assign each circuit element an impedance of R, $i\omega L$, or $-i/(\omega C)$. Impedance represents how much a given element will obstruct the response of a circuit, but as you can see this depends on the frequency. Inductors tend to block high frequencies, capacitors block low frequencies, and resistors are equal opportunity obstructions.
- Add impedances in parallel or series using the same rules used for resistors to find the total impedance Z of the circuit.
- The amplitude of the oscillating current is given by $I_0 = V_0/|Z|$.
- The phase lag ϕ between voltage and current is the phase of Z.

The picture below shows the stimulus and response of a circuit with a phase lag of $\pi/6$ or thereabouts. Don't read anything into the relative lengths of the two lines, because $\mathbf{V}$ and $\mathbf{I}$ are measured in different units. What is important is that the two quantities will rotate around the complex plane with the same frequency ω and therefore keep the same phase lag between them. The real voltage and current are the imaginary parts of these complex quantities, so they will also oscillate with a constant phase lag between them.



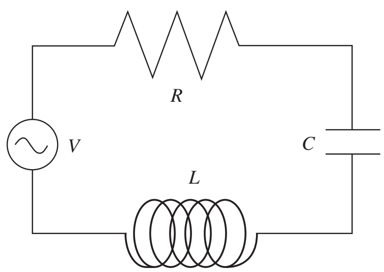
When we draw $\mathbf{V}$ and $\mathbf{I}$ as points in the complex plane this way, those points are referred to as "phasors" because they are like vectors that show the relative phases of the two oscillations.

Frequently $e^{i\omega t}$ is left out of the phasor, so the phasors for **I** and **V** show their *initial* positions on the complex plane. It's understood that the actual, time-dependent quantities rotate in circles in the complex plane.

Note that our entire analysis is based on a *particular* solution to the differential equation. There is also a *complementary* solution that contains arbitrary constants and depends on initial conditions. In most cases, however, the complementary solution is "transient"—an exponential that quickly decays—leaving the particular solution as the "steady-state" or long-term solution.

3.6.1 Problems: Electric Circuits

3.119 Walk-Through: Analyzing a Circuit. The electric circuit shown below has voltage $V = 3\sin(500t)$, resistance 2000, capacitance 10^{-6}, and inductance 15 (all measured in SI units).



(a) Find the equivalent impedance of the resistor, capacitor, and inductor.

(b) Find the complex current **I**.

(c) Find the modulus of **I** to get the amplitude of the oscillating current.

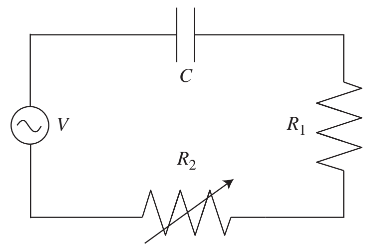
(d) Find the phase lag between the oscillating voltage and current by taking the phase of the impedance.

(e) Sketch the (real) current as a function of time. Include numbers on your plot that reflect the correct amplitude and frequency of the oscillation, and be sure to start at $t = 0$ with the correct phase. (Recall that V is a sine function, so the phase lag between V and I is minus the phase of I.)

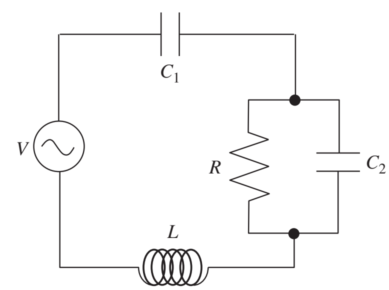
(f) How would your answers above change if the voltage was $V = 3\cos(500t)$?

In Problems 3.120–3.122 find the impedance of the indicated circuit and use that to find the amplitude and phase lag of the resulting current.

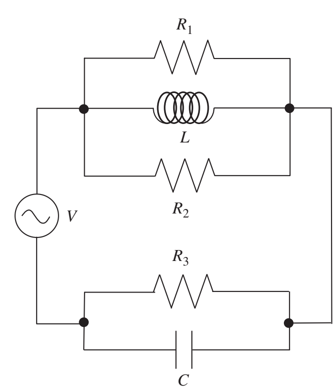
3.120 $V = 2\sin t$, $R_1 = 50$, $R_2 = 200$, $C = 10^{-6}$



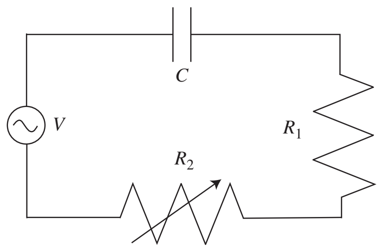
3.121 $V = 500\cos(400t)$, $R = 5000$, $C_1 = 3 \times 10^{-7}$, $C_2 = 10^{-6}$, $L = 25$



3.122 $V = \sin(t/10)$, $R_1 = 50$, $R_2 = 100$, $R_3 = 200$, $C = 10^{-7}$, $L = 5$.



3.123 In the circuit shown below $R_1 = 200\Omega$ and $C = 4 \times 10^{-8}F$. The resistor R_2 is a "rheostat," a resistor that can be tuned to different resistance values. The voltage oscillates at frequency $\omega = 120\pi$ sec^{-1}.

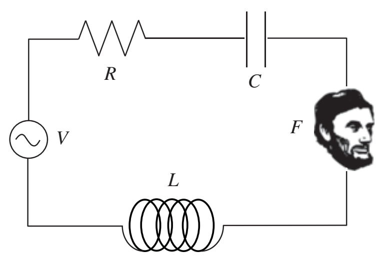


(a) How much does the current lag the voltage if $R_2 = 0$?

(b) What happens to the phase lag as $R_2 \to \infty$?

(c) What value of R_2 would you choose to make the current lead the voltage by $\pi/4$ (or, equivalently, lag the voltage by $7\pi/4$)?

3.124 Suppose someone were to invent a device called a "Feldor" whose voltage drop was proportional to $I''(t)$. The circuit shown below would obey the differential equation $FI'''(t) + LI''(t) + RI'(t) + (1/C)I = V'(t)$, where F is the "Feldance" of the Feldor. Find the impedance of a Feldor.



3.125 **The Power Factor.** The quantity VI (voltage times current) measures the power being delivered by a power source. When VI is negative power is actually being delivered *to* the source. If positive power alternates perfectly with negative power, the total "real power" delivered from the source to the load is zero.

(a) Consider first a circuit with only a power source and resistors: no capacitors or inductors.
  - **i.** What is the phase lag of such a circuit?
  - **ii.** For what fraction of the time is the power positive?

(b) Now consider a circuit with a phase lag of 180°. For what fraction of the time is the power positive?

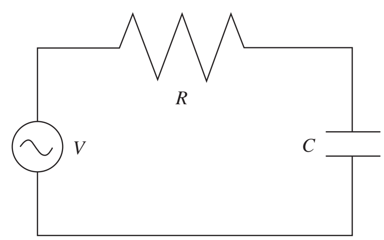
(c) Now consider a circuit with a phase lag of 90°. For what fraction of the time is the power positive?

(d) The "power factor" of a circuit is a unitless quantity defined as $\cos\phi$ where ϕ is the phase lag. Based on your answers above, describe the significance of the power factor. Why does the power company want this number to be as close to 1 as possible?

(e) In a circuit with a low power factor the energy is stored in the circuit and then fed back into the source. Where in the circuit (or "in what form") is the energy stored? (There are two important answers to this question. Discuss them both.)

3.126 Show that the differential equation solution $I(t) = A\sin(\omega t) + B\cos(\omega t)$ (where A and B are arbitrary constants) is equivalent to the solution $I(t) = I_0 \sin(\omega t - \phi)$ by finding I_0 and ϕ in terms of A and B. You can do this by looking up some trig identities, or you can use complex exponentials.

3.127 Consider the circuit shown below.



If the voltage source produces a voltage like $V = V_0 \sin(\omega t)$ then you know by now how to solve for the resulting current. In this problem, however, you will consider a voltage source $V = V_1 \sin(\omega_1 t) + V_2 \sin(\omega_2 t)$, and you'll solve for the current using the principle of linear superposition.

(a) Find the complex impedance of this circuit. (The answer will depend on ω, the frequency of the voltage source. Leave ω as a variable in this part; you'll fill in specific frequencies in the next parts.)

(b) Using the impedance you found in Part (a), find the complex current **I** that would result from a voltage source $V = V_1 \sin(\omega_1 t)$.

(c) Using the impedance you found in Part (a), find the complex current **I** that would result from the voltage source $V = V_2 \sin(\omega_2 t)$.

(d) Write the differential equation satisfied by **I** and show that the sum of the two currents you found is a solution to this differential equation.

(e) Write your solution $\mathbf{I}(t)$ for $R = 10^5 \Omega$, $C = 10^{-6}$ F, $V_1 = 3$ V, $V_2 = 5$ V, $\omega_1 = 2\ \mathrm{s}^{-1}$, and $\omega_2 = 3\ \mathrm{s}^{-1}$. Plot $V(t)$ and the imaginary part of $\mathbf{I}(t)$ on the same plot. Include a range of times sufficient to see the behavior of the functions.

3.128 **Exploration: Other Driving Functions** *[This problem depends on Problem 3.127.]*. The circuit from Problem 3.127, with the impedance you calculated in Part 3.127(a), is driven by a different voltage source. This source produces a "square wave": $V = 1$ for $0 \le t < 1$, then $V = -1$ for $1 \le t < 2$, and this pattern repeats indefinitely with period 2. Such a function can be represented as a sum of an *infinite number* of sine waves, a "Fourier sine series." Finding the coefficients is a topic for another chapter, so here we just give them to you:

$$\begin{aligned} V(t) &= \sum_{n=1}^{\infty} \frac{4}{n\pi} \sin(n\pi t) \text{ (odd } n \text{ only)} \\ &= \frac{4}{\pi}\sin(\pi t) + \frac{4}{3\pi}\sin(3\pi t) + \frac{4}{5\pi}\sin(5\pi t) + \dots \end{aligned}$$

(As in Problem 3.127, just leave the resistance and capacitance as R and C until we tell you to put in numbers. Leaving them as letters for now will keep your equations more readable.)

(a) Plot the driving function $V(t)$ from $t = 0$ to $t = 6$. On the same plot show the first term of the series expansion: $(4/\pi)\sin(\pi t)$.

(b) Make two more plots, each showing the function $V(t)$ and a partial sum of its series expansion. On the first one put the sum through $n = 5$. On the second one put the sum through $n = 101$. Describe what happens to the partial sum as you include more terms.

(c) Write the complex solution $\mathbf{I}(t)$ that you get if you replace the voltage $V(t)$ with the first term in its series expansion (the $n = 1$ part of the series).

(d) Write the complex solution $\mathbf{I}(t)$ that you get if you replace the voltage $V(t)$ with the sum of the first two non-zero terms in its series expansion.

(e) Write the complex solution $\mathbf{I}(t)$ that you get if you replace the voltage $V(t)$ with the $\sin(n\pi t)$ term *(not the partial sum up to that term)* of its series expansion.

(f) Write an infinite series for the solution $\mathbf{I}(t)$ that you get using the complete infinite series expansion for $V(t)$.

(g) Now let $R = 10^5 \Omega$ and $C = 10^{-6}$ F. Plot the partial sums of $Im(\mathbf{I}(t))$ for $n = 1$, $n = 11$, $n = 21$, $n = 31$, and $n = 101$, five plots in all. (Remember to plot partial sums, not individual terms.) Each of your plots should go from $t = 0$ to $t = 4$.

(h) Describe the behavior of the $n = 101$ plot. What does the current do at $t = 0$? What does it do shortly after that? What does it do at $t = 1$, and shortly after that? Describe what the capacitor is doing physically to produce the behavior you see. (If you skipped the computer part you can still do this part by *predicting* the behavior of $I(t)$ based on what you would expect the capacitor to do.)