4.11 Additional Problems

For Problems 4.202–4.205 find $\partial f/\partial x$, $\partial^2 f/\partial x \partial y$, $\vec{\nabla} f$, and $D_{\vec{u}} f$.

4.202 $f(x, y) = x - y + e^{xy}$ and $\vec{u} = \hat{i} - \hat{j}$

4.203 $f(x, y) = x^2 y^3$ and $\vec{u} = \hat{j}$

4.204 $f(x, y) = ax + by^2 + c\sin(ey)$ and $\vec{u} = \hat{i} + 3\hat{j}$

4.205 $f(x, y) = x/(x + y)$ and $\vec{u}$ has magnitude 2 and direction 20° clockwise from the positive y-axis.

For Problems 4.206–4.208 say which of the given functions is a solution to the given partial differential equation. There may be more than one correct answer. Assume any letters other than f, x, and t represent constants.

4.206 $\partial f/\partial t = -\partial f/\partial x$

- **(a)** $f(x, t) = x^2 t^{-2}$
- **(b)** $f(x, t) = e^{x-t}$
- **(c)** $f(x, t) = \cos^2(t - x)$
- **(d)** $f(x, t) = (x + t)^2$

4.207 $\partial^2 f/\partial t^2 = c^2 \partial^2 f/\partial x^2 - (c/\sqrt{2})\left(\partial^2 f/\partial x \partial t\right)$

- **(a)** $f(x, t) = c$
- **(b)** $f(x, t) = e^{\sqrt{2}ax + act}$
- **(c)** $f(x, t) = \sin(x + \sqrt{2}ct)$
- **(d)** $f(x, t) = xt$

4.208 $\partial^2 f/\partial t^2 = (x^2/t^2)\partial^2 f/\partial x^2$

- **(a)** $f(x, t) = c$
- **(b)** $f(x, t) = x^3 t^3$
- **(c)** $f(x, t) = x^2 t^3$
- **(d)** $f(x, t) = \cos(ax) + \sin(bt)$

4.209 **Chemical Kinetics** The "Arrhenius equation" tells us that the rate k of a chemical reaction is given by $k = Ae^{-E_a/RT}$ where T is the temperature and E_a is the activation energy (the energy that must be overcome for the reaction to occur). A and R are positive constants.

- **(a)** Compute $\partial k/\partial T$. Based on the sign of your answer, you should be able to conclude that "If you increase the temperature, the reaction rate increases."
- **(b)** Compute $\partial k/\partial E_a$. Then write a sentence, similar to the quoted sentence in Part (a), explaining what the sign of your answer means.
- **(c)** Compute $\partial^2 k/\partial T^2$. How high does E_a have to be in order to make $\partial^2 k/\partial T^2$ positive? What does it tell you about the system when it is positive?
- **(d)** Compute $\partial^2 k/\partial T \partial E_a$. How *low* does E_a have to be in order to make $\partial^2 k/\partial T \partial E_a$ positive? What does it tell you about the system when it is positive? *Hint*: You can answer the verbal part two different ways, depending on whether you think of your second derivative as $\partial^2 k/\partial T \partial E_a$ or $\partial^2 k/\partial E_a \partial T$.

4.210 **The Wave Equation**

- **(a)** Verify that $f(x, t) = \sin(x + ct) + \ln(x - ct)$ is a solution to the wave equation $\left(\partial^2 f/\partial t^2\right) = c^2\left(\partial^2 f/\partial x^2\right)$.
- **(b)** Next show more generally that $f(x, t) = a(x + ct) + b(x - ct)$ is a solution, where a and b can be any functions whatsoever.

4.211 The current I in an RLC circuit depends on the applied voltage V, the resistance R, the inductance L, and the capacitance C. For an alternating current with a variable inductor, both V and L depend on time t. The resistance depends on temperature T which, in turn, depends on time. The capacitance does not depend on time or temperature.

- **(a)** Draw the dependency tree for I.
- **(b)** Write a formula for dI/dt.

4.212 Your company's profit P depends on the number of items you sell (N), the price you charge per item (I), and your costs (C). The number of items you sell depends on the price you charge, and your costs depend on the number of items you sell. (For simplicity we'll assume you make exactly as many as you sell.)

- **(a)** Draw the dependency diagram for your profit and write an expression for dP/dI.
- **(b)** One simple $N(I)$ function is $N = ae^{-bI}$ where a and b are constants. Show that this function makes sense at the low and high extremes. Give the units on the two constants.
- **(c)** Write plausible functions for the other dependencies in the problem and explain why they make sense. (These may be simpler than the possibility we proposed for $N(I)$.) Your answers may contain unknown constants. Assuming P, I, and C have units of dollars and N is unitless, specify the units of any constants you introduce.

(d) Using the functions you just wrote evaluate your expression for dP/dI. Your answer should contain all of the variables given in the problem as well as any constants you introduced in your function. Make sure each term in your answer has correct units.

4.213 The traffic density c on interstate highway I-95 depends on position x and time t, and your position on the highway depends on time.

(a) Write an expression for dc/dt, the traffic density you see as you drive.

(b) Your expression for dc/dt should contain $\partial c/\partial t$ in it. Explain what each of these two derivatives means.

(c) Describe a circumstance where you would expect $dc/dt > 0$ and $\partial c/\partial t < 0$. Don't use any mathematical terms like "with respect to" or "rate of change" in your answer. Just describe what's happening on I-95 and what your car is doing.

4.214 Consider the curve defined by the equation $\sin y = \cos x$.

(a) Use implicit differentiation to find dy/dx as a function of x and y.

(b) Use the identity $\sin^2 y + \cos^2 y = 1$ and the original equation $\sin y = \cos x$ to express dy/dx as a function of x only.

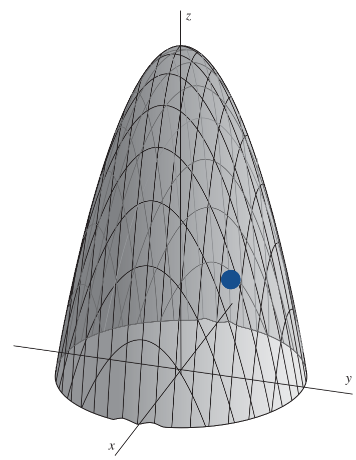
(c) Use the original equation to find $y(0)$. (There are infinitely many possibilities here—feel free to just give the simplest one.)

(d) Use your answers to Parts (b) and (c) to write the equation for this curve in a simpler form. Then confirm that your simplified equation satisfies the original relationship $\sin y = \cos x$.

4.215 The drawing shows the paraboloid $z = 50 - x^2 - y^2$ and shows the point $(4, 4, 18)$ on that paraboloid. "Which way does the gradient point at this location?" is an ill-defined question, because it depends on what function we use to represent the surface. We are going to ask this question for two different functions and find two different answers. Note that in *neither* case will the gradient point "up the hill" as many students expect.

(a) First consider the function $z(x, y) = 50 - x^2 - y^2$. Calculate the gradient of this function at the point $(4, 4, 18)$.

(b) Explain visually why the gradient pointed the way it did.



(c) If we picked a different point on this paraboloid, explain how we could determine visually (with no calculations) which way the gradient would point.

(d) Now consider the function $f(x, y, z) = x^2 + y^2 + z$, with the paraboloid representing one level surface of f. Calculate the gradient of this function at the point $(4, 4, 18)$.

(e) Explain visually why the gradient pointed the way it did.

(f) If we picked a different point on this paraboloid, explain how we could determine visually (with no calculations) which way the gradient would point.

4.216 Consider the curve $y + xe^y = 1$.

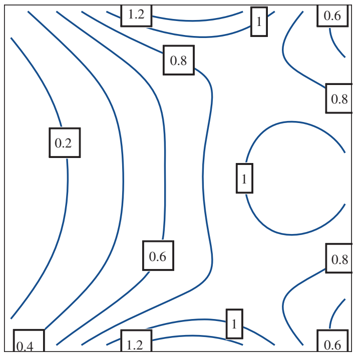
(a) Find the slope of this curve at the point $(1, 0)$.

(b) Find the gradient of the function $f(x, y) = y + xe^y$ at the point $(1, 0)$.

(c) Find the angle between the gradient vector $\vec{\nabla} f(1, 0)$ and the tangent line to the curve at that point. Explain why your answer makes sense.

(d) Find the concavity of the curve at the point $(1, 0)$.

4.217 The plot below shows the contour lines of a function $z(x, y)$. Copy this plot and add to it vectors at a variety of points (at least five, in different parts of the plot) showing the gradient at those points. Your vectors won't be precise, but they should all point in the correct direction and they should be larger in places with big gradients than they are in places with small gradients. (Pay careful attention to the numbers on the contours so you can tell where the function is increasing or decreasing.)



(If you want to print the picture from a computer instead of copying it, make a contour plot of $z(x, y) = e^{-(x-1)^2-y^2} + 0.3e^{-x^2+y^2}$ in the range $-1.2 \le x \le 1.2$, $-1.2 \le y \le 1.2$.)

4.218 Meteorologists study the "pressure gradient," the gradient of the air pressure in the atmosphere, to predict winds.

(a) Does the "pressure gradient force" point in the direction of the pressure gradient, or opposite that direction? Why?

(b) Suppose the air pressure in a local area is modeled by the equation $p(x, y, z) = x/(5\ln z)$. Write a unit vector in the direction of the pressure gradient force at the point $(2, 2, e)$.

(c) Compute the directional derivative of $p(x, y, z)$ in the positive y-direction. Explain why your result makes sense based on the pressure function.

4.219 Consider the function $f(x, y) = x/(2x + y + 1)$.

(a) Create the tangent plane to this function at the origin, and use it to approximate $f(0.01, 0.01)$.

(b) Create the second-order Taylor series to this function at the origin, and use it to approximate $f(0.01, 0.01)$.

(c) Calculate the actual value of $f(0.01, 0.01)$. Which approximation was closer?

(d) Create the tangent plane to this function at the point $(1, 1, 1/4)$, and use it to approximate $f(1.01, 0.9)$.

(e) Create the second-order Taylor series to this function at the point $(1, 1, 1/4)$, and use it to approximate $f(1.01, 0.9)$.

(f) Calculate the actual value of $f(1.01, 0.9)$. Which approximation was closer?

Problems 4.220–4.222 deal with "linear programming," meaning optimization where the objective function and all of the constraints are linear.

4.220 You are a manager at the Nezzer Chocolate Factory, responsible for the production of hand-crafted artistic chocolate bunnies and eggs. To make a Nezzer Egg requires 5 hours of labor and \$3 dollars of combined labor and material cost, and you sell them for \$5 dollars each (a \$2 profit). A Nezzer Bunny requires 3 hours and \$5 dollars and sells for \$6 (a \$1 profit). If you make g eggs and b bunnies your profit is $P = 2g + b$. Your total budget per day is \$200, and you have 250 hours of labor available to you each day.

(a) Draw the region in the g-b-plane that is allowed by these constraints. (*Hint*: Remember to consider the least possible number of eggs and bunnies you can make.)

(b) How many bunnies and eggs should you make to maximize your profit?

(c) Now suppose the market price of bunnies goes up to \$11 (so the profit is now \$6), and all the other numbers remain the same. How many eggs and bunnies should you produce?

(d) You should have found that with the price of bunnies at \$11 you could maximize your profits by making *only* bunnies. What is the minimum market price for bunnies that makes it optimal to produce all bunnies?

4.221 Your chemical plant has three reactors that use Unobtanium to make Gloppity-Glop. Reactor 1 uses 50 kg of Unobtanium and takes 2 kilowatt-hours to make a liter of Gloppity-Glop. Reactor 2 uses 20 kg of Unobtanium and takes 5 kilowatt-hours to make one liter. Reactor 3 only uses 10 kg of Unobtanium but it takes 10 kilowatt-hours per liter. (The reactors can make fractional numbers of liters of Gloppity-Glop.) You have 1000 kg of Unobtanium and enough fuel to produce 200 kilowatt-hours of energy.

(a) You want to make as much Gloppity-Glop as you can with your resources. Write the function you are trying to maximize. It should depend on the amounts R_1, R_2, and R_3 you produce from the reactors.

(b) Write the constraints on these variables. They should all be in the form of inequalities. In addition to the constraints described in the problem, there is also the requirement that none of these variables can be less than zero. Explain why.

(c) If you were to make a plot with R_1, R_2, and R_3 as the axes the constraints would define an allowed region on that plot. Show that the maximum value of Gloppity-Glop you can produce cannot occur in the interior of that region, which means it must be on the boundary. (Because you have three variables it's probably more trouble than it's worth to try to draw this region.)

(d) To test the boundaries consider each one in turn. First, assume that you use all 1000 kg of Unobtanium. This should turn one of your constraints into an equality. Use that constraint to eliminate R_3. Your remaining constraint is still an inequality and the constraint $R_3 \geq 0$ now becomes another inequality involving R_1 and R_2. Write these constraints and draw the allowed region in the $R_1 - R_2$-plane. Find the maximum amount of Gloppity-Glop you can make using all 1000 kg of Unobtanium.

(e) Repeat the process to find the maximum amount of Gloppity-Glop you can make using all 200 kilowatt-hours.

(f) How many liters of Gloppity-Glop should you make in each reactor, and how much total Gloppity-Glop can you make?

4.222 *[This problem depends on Problem 4.221.]* All possible solutions to Problem 4.221 could be plotted as points in a 3D space of R_1, R_2, and R_3. Each of the five constraints is a plane in that space, and between them they form the boundaries of the allowed region. (They form boundaries because they are all *inequalities*. If one of the constraints had been an *equation* such as $5R_1 - 2R_2 + 3R_3 = 7$, that would have forced the solution to lie on that surface instead of lying in a region bounded by it.) Because the objective function and the constraints are all linear functions, the optimal solution must lie on a vertex point where different boundary planes intersect.

(a) One vertex point can be described like this. "At the vertex of $R_2 = 0$, $R_3 = 0$, and $50R_1 + 20R_2 + 10R_3 = 1000$, Reactor 1 is used until it consumes all the Unobtanium. It makes $R_1 = 20$ L of Gloppity-Glop using 40 kilowatt-hours." List all the other possible vertices and give a similar description for each. Note in particular which vertices represent impossible solutions—that is, solutions that violate the constraints of the problem. (The example we gave is possible.)

(b) If you went through the same process you used in Problem 4.221 with an energy limit of 20 kilowatt-hours instead of 200 kilowatt-hours, you would find that the optimal solution used all of the available energy, but did *not* use all 1000 kg of available Unobtanium. Based on your answer to Part (a), how many reactors are used in that solution? (You can go through the whole process to find the solution, but you don't need to do that to answer this question.)

(c) Suppose another reaction used two chemicals, Unobtanium and Wonderflonium, to make Gloppity-Glopp. Once again you have three reactors, each of which uses a specified amount of Unobtanium, a specified amount of Wonderflonium, and a specified amount of energy to produce a liter of Gloppity-Glopp. For this reaction there are six constraints: none of the reactor outputs can be negative and you can't exceed the available amounts of Unobtanium, Wonderflonium, or energy. If the optimal solution involved using all of the energy and all of the Wonderflonium, but not using all of the Unobtanium, how many reactors would have to be idle in that solution?

4.223 The National Weather Service uses the following formula to calculate "wind chill," meaning the effective temperature you feel on a windy day: $W = 35.74 + 0.6215T - 35.75v^{0.16} + 0.4275Tv^{0.16}$. Here T is the temperature in Fahrenheit and v is the wind speed in mph.[9] This formula is only valid for $-50 < T < 50$ and $v > 3$.

(a) Find $\partial W/\partial v$.

(b) Explain how the formula you just found shows you that this formula for W cannot possibly be accurate for temperatures above 85° F.

4.224 In special relativity the Lorentz transformations give the position and time of an event in one reference frame in terms of its position and time in another: $x' = \gamma(x - vt)$,

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[9]See for example http://www.nws.noaa.gov/os/windchill/index.shtml. The authors would like to thank Courtney Lannert for suggesting this problem.

$t' = \gamma(t - vx/c^2)$. Here v is the velocity of the primed frame relative to the unprimed frame, $\gamma = 1/\sqrt{1 - v^2/c^2}$, and c is the speed of light. Suppose an object is moving at speed $u = dx/dt$ in the unprimed frame and you want to know the velocity an observer in the primed frame will measure.

(a) Using the Lorentz transformations, write expressions for dx' and dt' in terms of dx and dt. (Both v and c are constant.)

(b) Use those expressions to find dx'/dt'. Write your final answer only in terms of u, v, and c, and simplify as much as possible.

(c) The formula you just derived is sometimes called the "velocity addition" rule in special relativity. Evaluate it in the limiting cases $uv \ll c^2$ and $u = c$ and explain why your answer makes sense in both cases.

4.225 Exploration: The Earth's Equatorial Bulge If the Earth were a perfect sphere, it would create a gravitational potential of $-GM/r$ where M is the mass of the Earth and r is distance from the Earth's center ($r > R$). The Earth actually bulges around the equator, however, so a more accurate expression for its potential is

$$U = -\frac{GM}{r} + \frac{GMR^2 J_2}{2r^3}\left(3\sin^2 l - 1\right)$$

Here R is the Earth's radius at the equator, J_2 is a measure of "ellipticity" (how far its shape is from a perfect sphere), and l is latitude: 0 at the equator and $\pi/2$ at the poles.

(a) Find a second-order Taylor series for this potential about a point on the equator ($r = R$, $l = 0$).

(b) The gravitational force is given by $\vec{F} = -\vec{\nabla} U$, but the formula we gave for gradient in this chapter only applies to perpendicular distance variables, not angles (like latitude). To convert to a more amenable coordinate system, rewrite your Taylor series for U using the substitution $y = Rl$, where y is distance north from the equator.

(c) For $y << R$, we can treat $\hat{r}$ as a unit vector perpendicular to $\hat{y}$ and therefore write $\vec{\nabla} U = (\partial U/\partial y)\hat{y} + (\partial U/\partial r)\hat{r}$. Based on this approximation, find the gravitational force $\vec{F}(y, r)$. Simplify your answer as much as possible.

(d) For the Earth the ellipticity J_2 is roughly 10^{-3}. What direction does the Earth's gravitational field point at a latitude 5 degrees north of the equator? Express your answer as a difference between the actual direction you get from your formula for $\vec{F}$ and the direction it would be if the Earth were spherical (straight down). (*Hint*: when it's time to calculate the angle of difference, you will once again make the approximation that $\hat{r}$ is perpendicular to $\hat{y}$.)