CHAPTER 4

Partial Derivatives (Online)

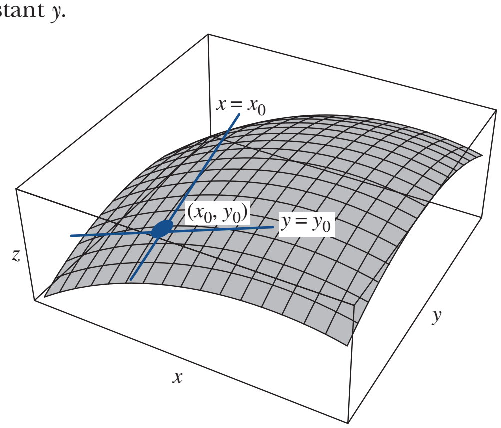
4.7 Tangent Plane Approximations and Power Series

It is often helpful to use a linear approximation to replace a complicated function $f(x)$ with a linear function that approximates f well when x is within a certain domain. If more accuracy is needed Taylor series can give higher order polynomial approximations. Such approximations were the main focus of Chapter 2.

In this section we apply a similar technique to multivariate functions, finding first a linear approximation (a plane), and then extending it to higher order terms.

4.7.1 Discovery Exercise: Tangent Plane Approximation

The drawing shows a function $z = f(x, y)$. Our goal is to find a plane that will approximate this function near the point (x_0, y_0, z_0): a tangent plane to the surface. The drawing does not show the tangent plane, but it does show two tangent lines at that point, one with a constant x and one with a constant y.



1. For a given function $f(x, y)$, how would we find the slope of the line labeled $y = y_0$? (Remember that this is the slope of the function in the x-direction, holding y constant.)
2. How would we find the slope of the line labeled $x = x_0$?
3. Recall that we are looking for a plane that we can use to approximate f. The equation for a plane can be written in the form $z = a(x - x_0) + b(y - y_0) + c$. Use this equation to answer the following questions:
   (a) At the point (x_0, y_0), what is the value of z?
   (b) What is the slope of z at that point as you move in the x-direction?
   (c) What is the slope of z at that point in the y-direction?

4. Find the values of a, b, and c for which the plane $z(x, y)$ has the same value, slope in the x-direction, and slope in the y-direction as $f(x, y)$ at the point (x_0, y_0).
   See Check Yourself #24 in Appendix L
5. Once we have made the proper choice, will our plane also match the slopes of the original function in all *other* directions at that point? How do you know?

4.7.2 Explanation: Tangent Plane Approximations and Power Series

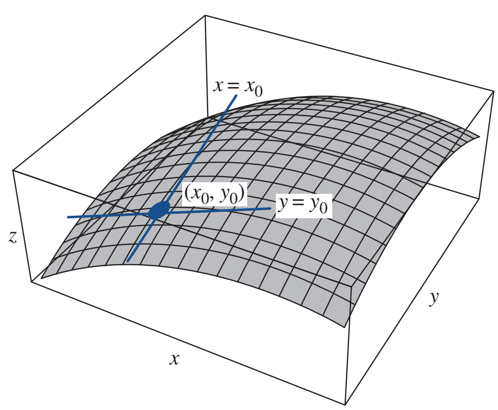
In Chapter 2 we found the tangent line to a curve at a given point. That's not a useless geometric exercise: the tangent line is useful because it serves as a *linear approximation* to the original function, and we can solve many important problems for linear functions that we cannot solve for more complicated functions. If a linear approximation is not sufficient, we can add more terms—a Taylor series—creating a higher order polynomial to approximate the function as accurately as necessary.

In this section we extend these ideas to multivariate functions. Our initial goal is to find the tangent plane to a surface. Once again, the real purpose of this exercise is to approximate a complicated function with something easier to work with. And once again, we will end with a formula that can be used to extend the approximation to higher order terms if necessary.

A Formula for the Tangent Plane

What is the definition of a tangent line to a curve? What makes it… tangent? Our answer is that the tangent line and the curve share a point, and they share the same derivative at that point. Based on that definition we can arrive quickly at a formula: the tangent line to $y = f(x)$ at the point (x_0, y_0) is $y = y_0 + f'(x_0)(x - x_0)$. The tangent line works as a good approximation to the original curve for values close to x_0 because both functions start at the same y-value and move up (or down) from there at the same rate.

A similar argument applies in higher dimensions. We begin with a definition: a tangent plane to the surface $z = f(x, y)$ at the point (x_0, y_0, z_0) must contain that point, and must match the original function at that point in both its partial derivatives. If the two functions share both their partial derivatives, then *all* their directional derivatives will be the same at that point. (Remember that $D_{\vec{u}}f = \vec{\nabla}f \cdot \vec{u}$ and $\vec{\nabla}f = \left(\partial f/\partial x\right)\hat{\imath} + \left(\partial f/\partial y\right)\hat{\jmath}$.) Such a plane works as a good approximation for the original surface for points close to (x_0, y_0) because both functions start at the same z-value and, no matter which direction you travel in, they move up (or down) from there at the same rate.



These considerations are enough to arrive at a formula.

The Tangent Plane to a Surface

Given a surface S defined by a function $z = f(x, y)$ that is differentiable at the point (x_0, y_0), the tangent plane to S at (x_0, y_0) is given by the following formula.

$$z = f(x_0, y_0) + \left(\frac{\partial f}{\partial x}(x_0, y_0)\right)(x - x_0) + \left(\frac{\partial f}{\partial y}(x_0, y_0)\right)(y - y_0) \tag{4.7.1}$$

We present this formula with no derivation, although you may have arrived at something similar on your own if you worked through the Discovery Exercise (Section 4.7.1). As always, however, you shouldn't take our word for it. Convince yourself of the following facts.

- Equation 4.7.1 does in fact define a plane. (There are of course rigorous ways to prove this but you can see it intuitively by considering some possible values for the constants in the formula, which is everything on the right-hand side except x and y, and seeing what the function looks like.)
- The plane and the original function $f(x, y)$ intersect—have the same z-value—at (x_0, y_0).
- At that point, the plane and the original function also have the same $\partial z/\partial x$ and the same $\partial z/\partial y$.

If those conditions are satisfied, then we have found the tangent plane we are looking for.

EXAMPLE **Tangent Plane**

Problem:
Find the tangent plane to the function $f(x, y) = 3y + \ln(2x + y)$ at the point $(0, 1)$, and use it to approximate $f(0.1, 0.96)$.

Solution:
$f(0, 1) = 3$.
$\partial f/\partial x = 2/(2x + y)$, so $\partial f/\partial x(0, 1) = 2$.
$\partial f/\partial y = 3 + 1/(2x + y)$, so $\partial f/\partial y(0, 1) = 4$.
The formula for the tangent plane is therefore $z = 3 + 2x + 4(y - 1)$.
This formula gives $f(0.1, 0.96) \approx 3.04$. (The actual value is roughly 3.03.)

If a function depends on more than two variables, add a term for each variable. For example, the linear approximation to a function $f(x, y, z)$ about the point (x_0, y_0, z_0) is given by

$$f(x_0, y_0, z_0) + \left(\frac{\partial f}{\partial x}(x_0, y_0, z_0)\right)(x - x_0) + \left(\frac{\partial f}{\partial y}(x_0, y_0, z_0)\right)(y - y_0) + \left(\frac{\partial f}{\partial z}(x_0, y_0, z_0)\right)(z - z_0)$$

Linearizing Higher Order Differential Equations

As with single-variable linear approximations, one of the most important applications of multivariate linear approximations is to turn non-linear (and unsolvable) differential equations into linear ones that can actually be solved. In many cases the "variables" in the linear approximation are the dependent variable in the problem and its derivative(s).

EXAMPLE **Linearizing a Differential Equation**

Problem:
Find and solve a linear approximation to the differential equation

$$\ddot{x} = 1 - e^{3x+4\dot{x}}$$

(Recall that $\dot{x}$ means the derivative of x with respect to time.)

Solution:
The problem presents us with a function $\ddot{x}(x, \dot{x})$. If x and $\dot{x}$ are small then we can replace this function with a linear approximation around $(0, 0)$. Note how the following numbers all come directly from the differential equation itself.

$$\ddot{x}(0,0) = 0, (\partial \ddot{x}/\partial x)(0,0) = -3, \quad \text{and} \quad (\partial \ddot{x}/\partial \dot{x})(0,0) = -4, \quad \text{so} \quad 1 - e^{3x+4\dot{x}} \approx 0 - 3x - 4\dot{x}.$$

The equation $\ddot{x} = -4\dot{x} - 3x$ can be solved by guessing an exponential solution, which leads to $x(t) = Ae^{-3t} + Be^{-t}$. Of course, it's important to remember that this solution is only useful for small values of both x and $\dot{x}$! Fortunately this solution shows that if x and $\dot{x}$ start out small they will remain so since they will decay exponentially.

Higher Order Terms

A Taylor series begins with a linear approximation but adds higher order terms to match the second, third, and higher order derivatives of the function, providing a more accurate estimation tool. You can expand a function $f(x)$ into a Taylor series around the value $x = x_0$ with the formula:[3]

$$f(x) = \sum_{n=0}^{\infty} \left(\frac{d^n f}{dx^n}(x_0) \right) \frac{1}{n!}(x - x_0)^n$$
$$= f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2 + \frac{f'''(x_0)}{6}(x - x_0)^3 + \ldots$$

If you want to find the third-order term in the expansion of $\sqrt{x}$ around $x = 25$, this formula tells you to evaluate the third derivative of $\sqrt{x}$ at $x = 25$ and divide it by 3!, and that gives you the coefficient of $(x - 25)^3$. In this fashion you can build a third-order polynomial that matches the original function's y-value and its first three derivatives at $x = x_0$. Note that the "0th derivative" of a function $f(x)$ is defined to be the function $f(x)$ itself (and recall that $0! = 1$), so the first term in this series is $f(x_0)$ as shown above.

The formula for a multivariate Taylor series looks similar.

The Taylor Series for a Multivariate Function

If a function $f(x, y)$ can be expanded into a polynomial around the point (x_0, y_0), then the formula is given by:

$$f(x, y) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left(\frac{\partial^{n+m} f}{\partial x^n \partial y^m}(x_0, y_0) \right) \frac{1}{n!m!}(x - x_0)^n (y - y_0)^m \qquad (4.7.2)$$

To compute a Taylor polynomial of order 5, you write out all the terms for which $n + m \leq 5$.

(The extension of this formula to functions of more than two variables is straightforward; see Problem 4.123.)

[3]Some people write the first term separately and start the series at $n = 1$, which avoids 0^0 appearing in the first term for $x = x_0$.

In Problem 4.122 you will show that this formula makes a logical extension of our tangent plane. It reduces to Equation 4.7.1 in the case of a first-order approximation. For a second-order approximation, it matches the function $f(x, y)$ at the point (x_0, y_0) with the same z-value, the same (two) first derivatives, and the same (three) second derivatives. A third-order approximation matches all those plus all four third derivatives, and so on. In Problem 4.114 you'll go through part of the argument for why these requirements lead to this particular formula.

EXAMPLE Multivariate Taylor Series

Problem:
Find the second-order approximation to the function $z = 3y + \ln(2x + y)$ at the point $(0, 1)$, and use it to approximate $z(0.1, 0.96)$.

Solution:
First calculate the relevant derivatives, remembering that the 0th derivative is just the function itself.

$$\frac{\partial^0 z}{\partial x^0 \partial y^0} = z(x, y) = 3y + \ln(2x + y) \qquad \text{so} \qquad \frac{\partial^0 z}{\partial x^0 \partial y^0}(0, 1) = 3$$

$$\frac{\partial^1 z}{\partial x^1 \partial y^0} = \frac{\partial z}{\partial x} = \frac{2}{2x + y} \qquad \text{so} \qquad \frac{\partial^1 z}{\partial x^1 \partial y^0}(0, 1) = 2$$

$$\frac{\partial^1 z}{\partial x^0 \partial y^1} = \frac{\partial z}{\partial y} = 3 + \frac{1}{2x + y} \qquad \text{so} \qquad \frac{\partial^1 z}{\partial x^0 \partial y^1}(0, 1) = 4$$

$$\frac{\partial^2 z}{\partial x^2 \partial y^0} = \frac{\partial^2 z}{\partial x^2} = -4/(2x + y)^2 \qquad \text{so} \qquad \frac{\partial^2 z}{\partial x^2 \partial y^0}(0, 1) = -4$$

$$\frac{\partial^2 z}{\partial x^0 \partial y^2} = \frac{\partial^2 z}{\partial y^2} = -1/(2x + y)^2 \qquad \text{so} \qquad \frac{\partial^2 z}{\partial x^0 \partial y^2}(0, 1) = -1$$

$$\frac{\partial^2 z}{\partial x^1 \partial y^1} = \frac{\partial^2 z}{\partial x \partial y} = -2/(2x + y)^2 \qquad \text{so} \qquad \frac{\partial^2 z}{\partial x^1 \partial y^1}(0, 1) = -2$$

Plug this into the formula for a second-order Taylor series.

$$z(x, y) = 3 + 2x + 4(y - 1) - 2x^2 - (1/2)(y - 1)^2 - 2x(y - 1)$$

This formula puts $z(0.1, 0.96)$ at 3.027. (The actual value is roughly 3.028.)

As with Taylor series for one variable, you can find Taylor series for multivariate functions by multiplying other Taylor series, differentiating or integrating other Taylor series, or plugging in combinations of variables into them. This is shown in the example below and explored further in the problems.

EXAMPLE

Building a Complicated Taylor Series from Simpler Ones

Problem:
Find the second-order Maclaurin series for $f(x,y) = e^x \sin(x+y)$.

Solution:
We can find the Maclaurin series for $\sin(x+y)$ by plugging $x+y$ into the series for sin:

$$\sin(x+y) = (x+y) - \frac{(x+y)^3}{6} + \ldots$$

Next we multiply this by the Maclaurin series for e^x, being careful to keep all terms up to the second order.

$$f(x,y) = \left[1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots\right]\left[(x+y) - \frac{(x+y)^3}{6} + \ldots\right] \approx x + y + x^2 + xy$$

You'll show in Problem 4.124 that you get the same answer using Equation 4.7.2.

4.7.3 Problems: Tangent Plane Approximations and Power Series

4.111 Let $f(x,y) = \sqrt{x+y^2}$, and let $g(x,y)$ be the tangent plane to $f(x,y)$ at the point $(40,3)$.

(a) Find the formula for $g(x,y)$.

(b) Show that $f(40,3) = g(40,3)$ and $\vec{\nabla} f(40,3) = \vec{\nabla} g(40,3)$.

(c) Calculate $f(41,2.9)$ and $g(41,2.9)$.

(d) Calculate $f(50,5)$ and $g(50,5)$.

(e) In which case, Part (c) or (d), did $g(x,y)$ serve as a better approximation of $f(x,y)$? Why?

4.112 *[This problem depends on Problem 4.111.]* Let $h(x,y)$ be the second-order Taylor approximation to the function $f(x,y)$ at the point $(40,3)$.

(a) Find the formula for $h(x,y)$.

(b) Show that at the point $(40,3)$, $\partial^2 f/\partial x^2 = \partial^2 h/\partial x^2$ and $\partial^2 f/\partial x\,\partial y = \partial^2 h/\partial x\,\partial y$ and $\partial^2 f/\partial y^2 = \partial^2 h/\partial y^2$.

(c) Calculate $h(41,2.9)$ and $h(43,2.5)$.

(d) At both points, did g or h work better as an approximation for f?

4.113 One term in the Taylor series for a function $f(x,y)$ around $(0,0)$ is

$$\left(\frac{\partial^5 f}{\partial x^2 \partial y^3}(0,0)\right)\frac{1}{2! \times 3!}x^2y^3$$

(a) Write down the term involving x^7 and y^4.

(b) Write down the term involving the same powers in a Taylor series around $(-3,\pi)$.

4.114 One term in the Taylor series for a function $f(x,y)$ around $(0,0)$ is $C_{23}x^2y^3$ where C_{23} is a constant.

(a) Find d^2/dx^2 of this term evaluated at $(0,0)$.

(b) Find d^3/dx^3 of this term evaluated at $(0,0)$.

(c) Find $d^2/(dx\,dy)$ of this term evaluated at $(0,0)$.

(d) Find $d^6/(dx^3\,dy^3)$ of this term evaluated at $(0,0)$.

(e) We just asked you four different questions—four different derivatives of this function, all evaluated at $(0,0)$. Write down and answer another such question. Your answer should not be zero. (*Hint*: there is only one correct question you can ask here!)

(f) The Taylor series and the function $f(x,y)$ should give the same answer for the derivative you wrote in Part (e). What value of C_{23} accomplishes this goal?

4.115 Find the tangent plane approximation to the function $f(x,y) = \sin(2x)\cos(3y)$

at the point $(\pi/6, \pi/6)$ and use it to approximate $f(1/2, 1/2)$.

4.116 Find the second-order approximation to the function $f(x, y) = \sin(2x)\cos(3y)$ at the point $(\pi/6, \pi/6)$ and use it to approximate $f(1/2, 1/2)$.

4.117 Find the tangent plane approximation to the function $z = x/y$ at the point $(6, 2, 3)$.

4.118 Find the second-order approximation to the function $z = x/y$ at the point $(6, 2, 3)$.

4.119 Find the fourth-order Taylor series approximation for $\sin(x + y^2)$ around $(0, 0)$. (*Hint*: There's a quick and easy way to do this. Just be sure that you toss out all terms above the fourth order.)

4.120 It is possible to do this entire problem without using Equation 4.7.2. (The second part can come quickly from the first, and the third from the second.)

- **(a)** Find the third-order Taylor series approximation for $\sin(x + y)$ around $(0, 0)$.
- **(b)** Find the third-order Taylor series approximation for $\sin(x + y)$ around $(0, \pi)$.
- **(c)** Find the second-order Taylor series approximation for $\cos(x + y)$ around $(0, \pi)$.

4.121
- **(a)** Find the third-order Taylor series approximation for e^{x+2y} around $(0, 0)$.
- **(b)** Take $\partial/\partial x$ of your answer to part (a). The result is the second-order Taylor series approximation for what function?
- **(c)** Take $\partial/\partial y$ of your answer to part (a). The result is the second-order Taylor series approximation for what function?

4.122 Write all the terms of Equation 4.7.2 for which $n + m \leq 1$—in other words a first-order series. Show that this results in Equation 4.7.1, the tangent plane approximation.

4.123 Equation 4.7.2 gives the formula for the Taylor series of a function of two variables $f(x, y)$.

- **(a)** By extending this formula, write the formula for a Taylor series of a three-variable function: $f(x, y, z) = \ldots$.
- **(b)** Use your formula to calculate the first-order- and second-order Maclaurin series for the function $f(x, y, z) = x^2 + ye^{5z}$.
- **(c)** Use your first-order- and second-order expansions to approximate $f(0.01, 0.02, -0.01)$. As a check on your formula, your answers should both be close to the correct value and your second-order one should be closer than the first-order one.

4.124 Find the second-order Maclaurin series for $f(x, y) = e^x \sin(x + y)$ by plugging it into Equation 4.7.2 and verify that you get the same answer we derived for it by easier methods in the Explanation (Section 4.7.2).

4.125 Suppose an object A is moving with a velocity v_{AB} relative to an object B, and B is moving with a velocity v_{BC} (in the same direction) relative to an object C. According to special relativity, the velocity of A with respect to C is:

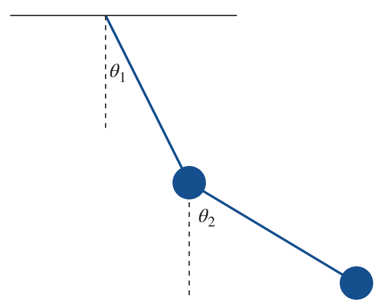
$$v_{AC} = \frac{v_{BC} + v_{AB}}{1 + v_{BC}v_{AB}/c^2}$$

where c, the speed of light, is a constant.

- **(a)** Find the linear approximation to v_{AC} when both velocities are much smaller than c. Explain why your answer makes sense physically.
- **(b)** Find the second-order approximation to v_{AC} when both velocities are close to the speed of light. Use your approximation to confirm that, as both velocities approach c, v_{AC} also approaches c (not $2c$ as classical mechanics would predict).

4.126 Find an approximate general solution to the differential equation $d^2x/dt^2 = (1 + x + \dot{x})/(1 + x - \dot{x})$ using a linear approximation valid when x and $\dot{x}$ are both close to 0.

4.127 Two coupled pendulums of length L are connected as shown in the figure below.



The equations describing this system are

$$2\ddot{\theta}_1 + \ddot{\theta}_2 \cos\left(\theta_1 - \theta_2\right) + \dot{\theta}_2^2 \sin\left(\theta_1 - \theta_2\right) + 2\frac{g}{L}\sin\theta_1 = 0 \tag{4.7.3}$$

$$\ddot{\theta}_2 + \ddot{\theta}_1 \cos\left(\theta_1 - \theta_2\right) - \dot{\theta}_1^2 \sin\left(\theta_1 - \theta_2\right) + \frac{g}{L}\sin\theta_2 = 0 \tag{4.7.4}$$

These equations have no solution in terms of simple functions. If you assume the amplitude of oscillations is small, however, then you can find approximate solutions.

(a) The first equation begins with a function of θ_1, θ_2, $\dot{\theta}_2$, $\ddot{\theta}_1$, and $\ddot{\theta}_2$. Write the linear approximation for that five-variable function.

(b) Do the same for the second equation (with a slightly different list of variables) and then write the two resulting simpler differential equations.

The differential equations you just wrote do have relatively simple solutions, which describe the motion of these pendulums for small oscillations. One such solution takes the following form.

$$\theta_1 = Ae^{it\sqrt{\left(2+\sqrt{2}\right)(g/L)}} \qquad (4.7.5)$$
$$\theta_2 = Be^{it\sqrt{\left(2+\sqrt{2}\right)(g/L)}}$$

(c) Plug this solution into your linear approximation to Equation 4.7.3 and solve for A in terms of B. Plug all the numbers into a calculator and express your answer in the form $A =$<*a number*>B.

(d) Repeat Part (c) for your approximation to Equation 4.7.4 and verify that you get the same relationship between A and B. This tells you that for any two numbers A and B with the relationship you found, Equation 4.7.5 is a solution to this pair of differential equations.

(e) If the motion of the coupled pendulum is described by this solution and the upper pendulum is oscillating with an amplitude of 5°, what will be the amplitude of oscillation of the lower pendulum?

4.128 Generate plots of the function $z = \sin(x^2y)$ in the range $-2 \le x \le 2$, $-2 \le y \le 2$ and of its power series at different orders. What order do you need to go to before the power series plot looks nearly identical to the plot of the actual function?