CHAPTER 6

Linear Algebra I (Online)

6.10 Additional Problems

6.220 The "main diagonal" of a matrix goes from the upper left to the lower right. The "trace" of a square matrix is defined as the sum of the elements on the main diagonal. Write an expression in summation notation for the trace of an $N \times N$ matrix **M**. Your answer should be in terms of the elements M_{ij} of the matrix.

6.221 The "Kronecker delta" δ_{ij} is defined as 1 when $i = j$ and 0 when $i \neq j$. Using the Kronecker delta as the definition of the elements of a 3×3 matrix, write that matrix in standard form. You should recognize the matrix you write down.

6.222 Suppose a typical hive of honeybees contains 100,000 workers and 2000 drones, and a typical colony of carpenter ants contains 1500 workers and D_A drones.

(a) Write a matrix to convert from "number of bee hives and number of ant colonies" to "number of workers and number of drones."

(b) Write a matrix to convert from "number of workers and number of drones" to "number of bee hives and number of ant colonies."

(c) For what value of D_A is it impossible to answer Part (b)?

(d) Explain why your answer to Part (c) makes sense without making any reference to matrices or linear algebra.

6.223 Vector $\vec{A} = -\hat{i} - \hat{j}$ and vector $\vec{B} = 2\hat{i}$.

(a) Do vectors $\vec{A}$ and $\vec{B}$ form a basis for all vectors on the xy-plane? How do you know?

(b) Draw the vectors $3\vec{A} - \vec{B}$ and $-\vec{A} + 3\vec{B}$.

(c) The vector $\vec{X} = a\vec{A} + b\vec{B}$ can also be represented as $x\hat{i} + y\hat{j}$. Write a matrix to convert from a and b to x and y.

(d) Write a matrix to convert from x and y to a and b.

(e) Use your matrix from Part (d) to convert the matrix $3\hat{i} + 4\hat{j}$ into the $\vec{A}\vec{B}$ representation.

(f) Show graphically that your answer to Part (e) does in fact add up to $3\hat{i} + 4\hat{j}$.

6.224 In this problem you will prove that the determinant of the identity matrix **I** is 1 in any number of dimensions.

(a) Prove that $|\mathbf{I}| = 1$ for a 2×2 matrix.

(b) Prove that if $|\mathbf{I}| = 1$ for an $n \times n$ matrix, then $|\mathbf{I}| = 1$ for an $(n+1) \times (n+1)$ matrix. *Hint*: write out what **I** looks like!

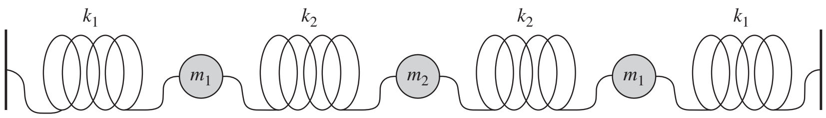
(c) Explain this result in terms of the effect this matrix has in transforming shapes.

6.225 You cannot use a matrix to convert Cartesian coordinates to spherical coordinates because the transformation is not linear. You can, however, use a matrix to convert a vector defined at a given point from the $\hat{i}\hat{j}\hat{k}$ basis to the $\hat{r}\hat{\theta}\hat{\phi}$ basis.

(a) At the point $(1, 0, 1)$ (in Cartesian coordinates) $\hat{r} = (1/\sqrt{2})(\hat{i} + \hat{k})$, $\hat{\theta} = (1/\sqrt{2})(\hat{i} - \hat{k})$, and $\hat{\phi} = \hat{j}$. Write a matrix for converting a vector defined at that point from Cartesian to spherical coordinates.

(b) Write a matrix for converting a vector defined at the point $(1, 1, 0)$ from Cartesian to spherical coordinates.

(c) Write a matrix for converting a vector defined at the point (x, y, z) from Cartesian to cylindrical coordinates. (We switched to cylindrical for the general case because it's easier than spherical.)



6.226 The image above shows three coupled oscillators. Take $m_1 = 1$ kg, $m_2 = 2$ kg, $k_1 = 3$ N/m, and $k_2 = 2$ N/m.

(a) Write differential equations for the three positions, find the normal modes of the system, and write the general solution. (This will require finding the eigenvalues and eigenvectors of a 3×3 matrix. That's normally cumbersome but in this case the characteristic equation includes one expression that you can easily factor out of every term.)

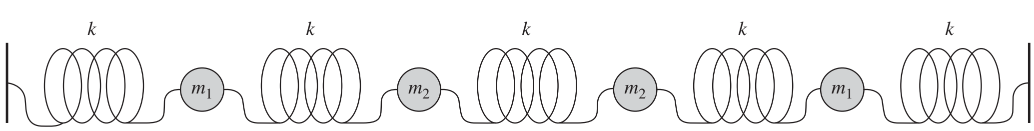
(b) One of the normal modes represents a motion in which you pull all three balls to the right (or left), pulling the middle one twice as far as the outer two, and then let go. They will then oscillate that way forever, always moving to the right and left together, with the middle one oscillating with twice as large an amplitude as the other two. Using that description as a guide, write similar descriptions for what the other two normal modes physically represent.

6.227 The image below shows four coupled oscillators. Take $k = 6$ N/m, $m_1 = 3$ kg, and $m_2 = 2$ kg.

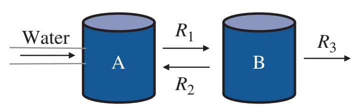
You should be able to write down their equations of motion and solve them to find the normal modes (with a computer to help you find eigenvectors and eigenvalues), but in this problem we want to focus on the physical interpretation instead, so we're just going to give the eigenvectors of the transformation matrix.

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \\ -3 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ -1 \\ 1 \\ 2 \end{pmatrix}$$

The first of these eigenvectors describes a normal mode in which you pull all four balls to the right (or left), pulling the middle two slightly farther than the outer two, and then let go. They will then oscillate that way forever, always moving to the right and left together, but with the middle ones oscillating with a larger amplitude than the outer two. Using that description as a guide, write similar descriptions for what the other three eigenvectors physically represent.



6.228 Two tanks contain a mixture of water and frobscottle. Let A and B be the amounts of frobscottle in each tank, and assume the tanks are well mixed so the concentration is uniform throughout each tank (but may be different in the two tanks). Throughout the problem we will measure all amounts in millions of gallons. In those units each tank has a volume of 1. Pipes carry fluid back and forth between each tank, and away from the second tank into the river. Clean water is pumped into the first tank to keep the volume constant.



Note that A and B represent amounts of frobscottle in each tank, while the R variables represents rates at which mixed amounts of water and frobscottle are pumped out of the tanks.

(a) Assuming the volume in each tank remains constant, express R_3 in terms of R_1 and R_2.

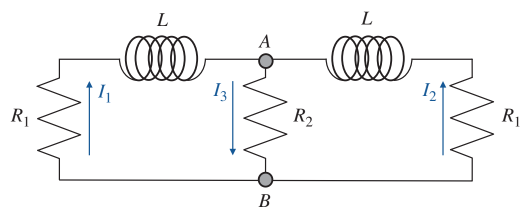
(b) Write a pair of coupled differential equations for A and B.

(c) Find the normal modes of the system.

(d) Let $R_1 = 9$, $R_2 = 4$, $A(0) = 3$, and $B(0) = 3$. Solve for $A(t)$ and $B(t)$.

(e) Both of the normal modes you found should be valid mathematical solutions to the equations, and you showed in the previous part that you can make physical solutions out of combinations of them, but it is not physically possible for the system to be entirely in one of the normal modes. Which one, and why not?

6.229 The picture below shows a circuit with inductors and resistors.



(a) The total current entering any given point must equal the total current leaving that point. Use that fact to write I_3 in terms of I_1 and I_2.

(b) The voltage drop across a resistor is IR, where I is the current through the resistor. The voltage drop across an inductor is $(dI/dt)L$. Write equations that express the fact that the voltage drop from A to B is the same whether you go along the left, middle, or right path. Rearrange your answers to get a pair of coupled differential equations for dI_1/dt and dI_2/dt in terms of I_1 and I_2.

(c) Find the normal modes of the system.

(d) In one of the normal modes the current on both sides of the circuit is equal, so at some point in time a current I is flowing in one direction (up or down) on the left, an equal current I is flowing in that same direction on the right, and a current $2I$ is flowing the opposite way in the middle. All of these currents die off exponentially, with the outer two always equal and the middle one always twice as large and opposite in direction. Using this description as a guide, describe the physical state represented by the other normal mode.

(e) Let $L = 10^{-7}$ H, $R_1 = 20\ \Omega$, and $R_2 = 10\ \Omega$, and assume that initially $I_1 = 3$ A and $I_2 = 2$ A. Find $I_1(t)$ and $I_2(t)$.

6.230 **Exploration: A Quantum Mechanical Well.** Many quantum mechanics problems start by solving Schrödinger's equation:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x)\psi = E\psi$$

The potential function $V(x)$ is specified in the problem (just as classical dynamics problems begin by specifying a force). In this problem you will solve Schrödinger's equation for the potential function:

$$V = \begin{cases} V_0 & x < 0 \\ 0 & 0 \le x \le a \\ V_0 & x > a \end{cases}$$

where m, $\hbar$, V_0, a and E are positive constants, and (this is very important) $E < V_0$. The value of ψ is generally complex, but in this problem make all your solutions real. The boundary condition is that $\lim_{x\to\pm\infty} \psi(x) = 0$.

(a) Begin by writing and solving Schrödinger's equation in the rightmost region. Your general solution will have two arbitrary constants, but the boundary condition at $x \to \infty$ will eliminate one of them.

(b) Repeat Part (a) for the leftmost region. Once again, your final solution will have only one arbitrary constant (but not the *same* arbitrary constant as in the first solution).

(c) Write and solve Schrödinger's equation in the middle region. This time you will be left with two arbitrary constants.

You now have four arbitrary constants: one on the left, one on the right, and two in the middle. But now we introduce a postulate of quantum mechanics: both $\psi(x)$ and $d\psi/dx$ must be continuous. So $\psi(a)$ calculated from Part (a) must agree with $\psi(a)$ calculated from Part (c) and so on.

(d) The requirement of continuity imposes four different restrictions, two on each boundary. Write the equations that represent those restrictions. *Hint*: you can save some writing if you define two new constants: $\alpha = \sqrt{2m(V_0 - E)}/\hbar$ and $\beta = \sqrt{2mE}/\hbar$.

(e) You should now have four homogeneous linear equations for four arbitrary constants. One solution is the trivial one where they are all zero, but that can't represent a physical state, so the only possible physical states are ones for which there are other solutions. Write an equation that must be satisfied in order for those nontrivial solutions to exist. This will require taking a 4×4 determinant, but it will have enough zeros in it that expansion by minors will not be too time-consuming to do by hand. Simplify the equation as much as possible (leaving it in terms of α and β). This equation describes the physically possible energies E for a particle in a finite potential well, although it cannot be analytically solved for E.

(f) At what step would your answers first have begun to look different if E were greater than V_0?

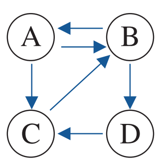
6.231 *[This problem depends on Problem 6.230.]* The mass of an electron is 9.1×10^{-31} kg and the constant $\hbar$ is 6.63×10^{-34} J·s. Consider an electron in a potential well with $V_0 = 10^{-18}$ J and $a = 10^{-8}$ m.

(a) Write the function $f(E)$ that must equal 0 at a physically allowed value of the energy.

(b) Plot that function from $E = 0$ to $E = V_0$. How many allowed values of energy are there for the electron in this well?

(c) Numerically find the value of the lowest allowed energy. (It must be positive.) Using these numbers your answer will come out in Joules. Convert them to the more standard particle physics energy unit of "electron volts": 1 eV $= 1.6 \times 10^{-19}$ J.

6.232 Exploration: Google PageRank
One of the techniques Google uses to select search results is "PageRank," invented by Google founders Sergey Brin and Larry Page (for whom it is named).[9] The basic idea is that each web page is given a rank based on what other pages link to it. The higher a page's rank is, the more rank it confers on other pages that it links to. To illustrate how the system works, consider the following web of four pages.



Initially each page is given a ranking of 1 over the number of pages, so in this case they each start with 1/4. In each iteration, each page distributes its current rank equally among all the other pages that it links to. For example, page A links to pages B and C, so it gives them each half of its current rank, or 1/8. Page C only links to page B, so it gives its entire current rank to B. After the first iteration, page B has a new rank of 3/8, the sum of the ranks it inherited from A and C. (It doesn't matter if page A links to page B once or twenty times; the algorithm only counts *whether* one page links to another. It also ignores any links from a page to itself.)

(a) Write the column vector $\mathbf{r}_1$ representing the rankings of the four pages after one iteration. Write the vector $\mathbf{r}_2$ giving their rankings after two iterations.

(b) Write the matrix $\mathbf{L}$ that you multiply by $\mathbf{r}_i$ to get $\mathbf{r}_{i+1}$.

(c) In the limit of infinitely many iterations, you approach a vector $\mathbf{r}$ that is no longer changing. From this you can conclude that $\mathbf{r}$ is an eigenvector of $\mathbf{L}$. What is its eigenvalue?

(d) Find the solution $\mathbf{r}$ for this particular web.

PageRank has a simple interpretation. If a user starts on a random page and randomly follows links, the rank of a given page after i iterations is the probability that he will be on that page after following i links. In practice, however, users sometimes jump to a new random page rather than following links. If d is the probability of a user following a link, and $1 - d$ is the probability of the user jumping to a new random page, then the probability of landing on the nth page after i iterations is given by:

$$r_i(n) = \frac{1-d}{N} + d \sum_{m=1}^{N} r_{i-1}(m) L_{mn}$$

Here $r_i(n)$ is the rank of page n after i iterations, and N is the total number of pages. That may sound complicated, but it's just what you did above. If a page links to seven other pages, then each iteration it gives 1/7 of its rank to each of those pages. The difference is that now it gives $d/7$ of its rank to each of those pages, and each page also receives a rank $(1 - d)/N$ for the chance that the user jumped to that page randomly instead of following a link. This formula can be written in matrix form, using $\mathbf{1}$ for a column matrix where all the entries are 1.

$$\mathbf{r}_i = \left(\frac{1-d}{N}\right)\mathbf{1} + d\,\mathbf{L}\,\mathbf{r}_{i-1} \qquad (6.10.1)$$

The probability d is called the "damping factor."

(e) Show that in the case $d = 1$ Equation 6.10.1 reduces to the simpler formula you were using above. What assumption does $d = 1$ represent?

(f) Once again the steady-state solution is the one where $\mathbf{r}$ doesn't change from one iteration to the next. Write a matrix

[9] Sergey Brin and Lawrence Page. 1998. The anatomy of a large-scale hypertextual Web search engine. Comput. Netw. ISDN Syst. 30, 1–7 (April 1998), 107–117.

equation for **r** and solve it. (In this part you're considering a general web, not the particular four-page example given above.) *Big hint*: Solving for **r** will require bringing both the **r** terms to one side of the equation. To combine those two terms you'll need to insert an identity matrix **I** in front of one of them. Finally, to get **r** by itself you'll multiply both sides by the inverse of the matrix that multiplies it.

(g) What would you expect the rankings to approach in the limit $d \to 0$ and why? What would you expect them to approach in the limit $d \to 1$ and why?

6.233 *[This problem depends on Problem 6.232.]* Unless otherwise specified, everything in this problem refers to the four-page web given in Problem 6.232.

(a) Find the solution **r** for the four-page web given above, using a damping factor $d = 0.85$ (which is the value recommended by Brin and Page). How do the numbers look different from the undamped solution you found above, and why do these differences make sense in light of your answers to Part (g) of Problem 6.232?

(b) Using the simple algorithm with no damping factor, calculate the first 100 iterations of $\mathbf{r}_i$. Make a plot showing each of the four ranks as a function of i. For reference, put horizontal lines on your plot representing the steady-state values **r**.

(c) Repeat Part (b) using a damping factor $d = 0.5$. (The horizontal lines in this plot should reflect the damped solution.)

(d) Calculate the steady-state solution **r** for 100 values of d from 0 to 1. (You may find it easier to include values close to 1, but not $d = 1$ itself.) Make a plot showing the steady-state solution for each of the four ranks as a function of d. Explain what your plot looks like and why it makes sense.

6.234 Exploration: Cramer's Rule
Cramer's rule is a method for solving n linear equations with n unknowns.

$$\mathbf{Mx} = \mathbf{b} \qquad (6.10.2)$$

Here **x** is a column vector made up of the unknowns x_1, x_2, etc., and **M** and **b** are a square matrix and a column matrix respectively. Let $\mathbf{N_i}$ be a matrix formed by replacing the i^{th} column of **M** with **b**. Cramer's rule says $x_i = |\mathbf{N}_i|/|\mathbf{M}|$.

(a) Use Cramer's rule to solve the equations $x_1 + 2x_2 = 3$, $2x_1 - x_2 = -1$. Verify that your answers work.

In the rest of the problem you'll show why Cramer's rule works. For definiteness, we'll let **M** be 3×3 and we'll derive Cramer's rule for finding x_2.

(b) If $\mathbf{M}^{-1}$ is the inverse of **M**, what is $\mathbf{M}^{-1}$ times the first column of **M**? Your answer should be a column vector.

(c) Multiply both sides of Equation 6.10.2 *on the left* with $\mathbf{M}^{-1}$ to find $\mathbf{M}^{-1}\mathbf{b}$. Once again your answer should be a column vector.

(d) Using your previous answers, what is $\mathbf{M}^{-1}\mathbf{N}_2$? Your answer, of course, should be a square matrix.

(e) Take the determinant of the square matrix you just wrote. Write your answer as an equation: $|\mathbf{M}^{-1}\mathbf{N}_2|$ equals such-and-such.

(f) Rewrite the equation you just wrote to derive Cramer's rule for this specific case. Explicitly state what properties of determinants you are using.

6.235 In Section 6.6 we asserted that only square matrices can be inverted.

(a) The definition of an inverse matrix requires that $\mathbf{AA}^{-1} = \mathbf{I}$ *and* $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$. Explain why it's only possible for both of these to be valid if **A** is a square matrix.

To see why this result makes sense we'll consider two non-square matrix transformations.

(b) As our first example, suppose you have 3 molecules of water (H_2O), 2 molecules of hydrogen peroxide (H_2O_2), and 4 molecules of hydrogen (H_2). How many atoms of hydrogen and oxygen do you have?

(c) Continuing with that example, now suppose you have 8 atoms of hydrogen and 4 atoms of oxygen. Write two possibilities for how many of each of those molecules you might have.

(d) Write a matrix for converting from molecules of water, hydrogen peroxide, and hydrogen to atoms of hydrogen and oxygen. Explain using your answer to Part (c) why this matrix can't be inverted.

(e) As our second example let's say you have 2 molecules of glucose ($C_6H_{12}O_6$) and 3 molecules of ethanol (C_2H_6O).[10]

[10] We really don't want to know what you have them for.

How many atoms of carbon, hydrogen, and oxygen do you have?

(f) Now suppose you have 12 atoms each of carbon, hydrogen, and oxygen. Prove that no combination of these two molecules can account for those numbers of atoms. This is true even if you allow fractional and negative numbers of molecules. *Hint*: set this up as a set of equations for the number of atoms as a function of G and E, the numbers of molecules.

(g) Write a matrix for converting from molecules of glucose and ethanol to atoms of carbon, hydrogen, and oxygen. Explain using your answer to Part (f) why this matrix can't be inverted.

(h) Explain in general why 2×3 and 3×2 matrices can't be inverted. Your answer should not be in terms of equations, but of the kinds of transformations these perform. Note that the answer is different for 2×3 and 3×2.