9.8 Multivariate Fourier Series

In this section we extend the idea of Fourier analysis to "multivariate" functions: that is, functions of more than one independent variable.

9.8.1 Discovery Exercise: Multivariate Fourier Series

1. Have a computer plot the function $z(x,y) = \sin(3x)\cos(10y)$ on the domain $-2\pi/3 \le x \le 2\pi/3$, $-2\pi/3 \le y \le 2\pi/3$.
2. Have a computer make an animation of the function $y(x,t) = \sin(3x)\cos(10t)$ on the domain $-2\pi/3 \le x \le 2\pi/3$, $0 \le t \le 2\pi/3$. In other words make a video that starts as a plot of $y(x,0)$ and changes into a plot of $y(x,\Delta t)$, $y(x,2\Delta t)$, and so on up to $y(x,2\pi/3)$, where Δt is a number much smaller than $2\pi/3$.
3. Redo Part 1 changing $\sin(3x)$ into $\sin(5x)$. Then redo it changing $\cos(10y)$ into $\cos(20y)$ (using $\sin(3x)$ again for this one). How did each of those changes affect the plot?
4. Similarly, redo Part 2 first changing $\sin(3x)$ into $\sin(5x)$ and then changing $\cos(10t)$ into $\cos(20t)$. How did each of those changes affect the animation?

9.8.2 Explanation: Multivariate Fourier Series

A multivariate version of our guitar string problem is a rectangular drumhead: that is, a piece of rubber stretched across a rectangular wire frame with length L and width W. See Figure 9.20. The rubber is free to vibrate in the middle, but is tacked down along the border, so the height z must be zero along $x = 0$, $x = L$, $y = 0$, and $y = W$.

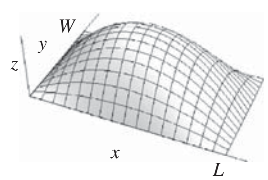


FIGURE 9.20 A rectangular drumhead stretched into the shape $z = k(Lx - x^2)(Wy - y^2)$.

Now imagine pulling the center of the drumhead straight up, just as we did with our guitar string in Section 9.4. The resulting shape might reasonably be modeled by the function below.

$$z = k(Lx - x^2)(Wy - y^2)$$

The basic idea here is the same as it was for the guitar string. If the drumhead happens to start in a perfect sine wave, it will vibrate at one particular frequency. When the starting condition is not a sine wave we represent it as a sum of sine waves, whose frequencies and amplitudes will result in particular sounds. But before we go further with this particular example, let's step back to discuss what multivariate Fourier series look like mathematically.

One of the Many Formulas for a Multivariate Fourier Series

We have seen four different types of Fourier series: sines-and-cosines, sines-only, cosines-only, and complex exponentials. This leads to sixteen different types of multivariate Fourier series, because you can choose independently for x and y! For instance, consider a function that is even in x with a period of 6π, and neither even nor odd in y with a period of 2π. Each term in the Fourier expansion of this function could look like $\cos(mx/3)\left[A_{nm}\cos(ny) + B_{nm}\sin(ny)\right]$. Then again, you might choose in such a case to use cosines for x and complex exponentials for y, creating $\cos(mx/3)\left[c_{nm}e^{iny} + c_{-nm}e^{-iny}\right]$.

In Appendix G we give the general sine-and-cosine form, and the complex exponential form, for a Fourier series in two variables. (You can easily generalize these formulas to more variables if you need to.) The sine-and-cosine form is messy, with four different terms and a variety of integrals, which is one reason most people stay with the complex exponential form. But the sines and cosines are useful, especially when working with a function defined on a finite domain. Below we give the specific formula that applies to a function that is odd in both x and y, so all the terms are sines. (This particular formula gives as good an example as any, and it happens to be the one we need for our drumhead. Do you see why?)

Multivariate Fourier Series, Sines Only (See Appendix G for More General Formulas)

The Fourier series for a function that is odd in both x and y, with a period of $2L$ in x and a period of $2W$ in y, is:

$$f(x,y)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}D_{mn}\sin\left(\frac{m\pi}{L}x\right)\sin\left(\frac{n\pi}{W}y\right) \tag{9.8.1}$$

The coefficients are given by the equation:

$$D_{mn}=\frac{1}{LW}\int_{-W}^{W}\int_{-L}^{L}f(x,y)\sin\left(\frac{m\pi}{L}x\right)\sin\left(\frac{n\pi}{W}y\right)\,dx\,dy \tag{9.8.2}$$

You will derive that formula in Problem 9.125, using as always the orthogonality of the sine functions.

Solving the Vibrating Drumhead

Because our drumhead is defined on a finite domain, we need to create a periodic extension of it. That means we extend out in the x-direction, and *also* extend out in the y-direction, to cover the whole plane. We have four options for how to do this; for instance, we could do an even extension in x and an odd extension in y. But remember that our drumhead is constrained to $z=0$ on all four edges, so its vibrational modes are all sines in both x and y. For this reason we will make odd extensions in both directions, as shown in Figure 9.21.

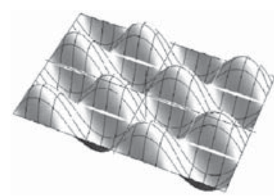


FIGURE 9.21 The rectangular drumhead's initial shape extended periodically as an odd function in both x and y.

We could express this odd extension as a piecewise function, and then calculate coefficients with four separate integrals in the four separate quadrants. However, the symmetry of the situation saves us a lot of the trouble. Our function is odd in both x and y (because we made it so!). That tells us that if we looked for cosine coefficients, they would all come out zero. It also tells us that we can calculate the sine coefficients by integrating in the first quadrant only, and then multiplying the result by four.

$$D_{mn}=\frac{4k}{WL}\int_{0}^{W}\int_{0}^{L}(Lx-x^2)(Wy-y^2)\sin\left(\frac{m\pi}{L}x\right)\sin\left(\frac{n\pi}{W}y\right)\,dx\,dy$$

The integral is separable, and you can then work out the individual integrals by parts, or hand them to a computer. After all due simplification, we arrive here.

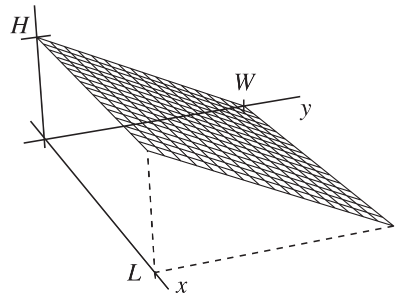
$$D_{mn}=\frac{64kW^2L^2}{\pi^6m^3n^3}\qquad \textit{odd } m \textit{ and } n \textit{ only}$$

So we conclude that our original function can be written in this form.

$$\frac{64kW^2L^2}{\pi^6}\left[\sin\left(\frac{\pi}{L}x\right)\sin\left(\frac{\pi}{W}y\right)+\frac{1}{27}\sin\left(\frac{\pi}{L}x\right)\sin\left(\frac{3\pi}{W}y\right)\right.$$
$$\left.+\frac{1}{27}\sin\left(\frac{3\pi}{L}x\right)\sin\left(\frac{\pi}{W}y\right)+\frac{1}{27^2}\sin\left(\frac{3\pi}{L}x\right)\sin\left(\frac{3\pi}{W}y\right)+\cdots\right]$$

EXAMPLE Multivariate Fourier Series

Problem:
The drawing shows a surface going directly up from 0 at $y = W$ to a height of H when $y = 0$. The function $f(x, y)$ is this figure repeated evenly in the y-direction and oddly in the x-direction. Find the Fourier series for $f(x, y)$.



Solution:
In the x-direction this function is odd with a period of $2L$. In the y-direction this function is even with a period of $2W$. We could use the full sines-and-cosines form in Appendix G, but the sines in y and cosines in x must drop out due to the symmetry, so our function will look like:

$$f(x,y) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} C_{mn} \sin\left(\frac{m\pi}{L}x\right)\cos\left(\frac{n\pi}{W}y\right) \qquad \text{where}$$

$$C_{m0} = \frac{1}{2WL}\int_{-W}^{W}\int_{-L}^{L} f(x,y)\sin\left(\frac{m\pi}{L}x\right)\,dx\,dy$$

$$C_{mn} = \frac{1}{WL}\int_{-W}^{W}\int_{-L}^{L} f(x,y)\sin\left(\frac{m\pi}{L}x\right)\cos\left(\frac{n\pi}{W}y\right)\,dx\,dy \qquad n > 0$$

As the appendix instructed us to, we used $Q = 2$ when $n = 0$ and $Q = 1$ when neither m nor n was zero. (In the drumhead example we were only dealing with sines so the $m = 0$, $n = 0$ terms dropped out. Here the $m = 0$ terms vanish, but the $n = 0$ terms do not.)

We could write the function $f(x, y)$ four times—one in each quadrant—and perform four separate integrals, but the symmetry of the situation allows us to integrate only in the first quadrant and quadruple the answer. In the first quadrant, our function looks like $f(x, y) = H(1 - y/W)$. The formula for the coefficients is therefore:

$$C_{m0} = \frac{2H}{WL}\int_{0}^{W}\int_{0}^{L}\left(1-\frac{y}{W}\right)\sin\left(\frac{m\pi}{L}x\right)\,dx\,dy$$

$$C_{mn} = \frac{4H}{WL}\int_{0}^{W}\int_{0}^{L}\left(1-\frac{y}{W}\right)\sin\left(\frac{m\pi}{L}x\right)\cos\left(\frac{n\pi}{W}y\right)\,dx\,dy \qquad n > 0$$

The x integrals are trivial. The y integrals can be done with integration by parts, or using the table in Appendix G. The result, after simplifying, is:

$$C_{m0} = \frac{2H}{\pi m} \quad \text{(odd } m \text{ only)} \qquad\qquad C_{mn} = \frac{16H}{\pi^3 mn^2} \quad (n > 0, \text{ odd } m \text{ and } n \text{ only})$$

So the original function can be expressed as:

$$f(x, y) = \frac{2H}{\pi} \sum_{m=1}^{\infty} \frac{1}{m} \sin\left(\frac{m\pi}{L}x\right) + \frac{16H}{\pi^3} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{mn^2} \sin\left(\frac{m\pi}{L}x\right) \cos\left(\frac{n\pi}{W}y\right)$$

odd m and n only

What do Those Things Look Like?

When you write a multivariate Fourier series, you write your function as a series of sine waves. If the independent variable is x then one term in your expansion might be $10\cos(2x)$, a sine curve with a wavelength of π meters. If the independent variable is t then a term might be $10\cos(2t)$, an oscillation with a period of π seconds. If you don't understand what those terms represent, finding a series of them won't do you much good.

So what do the terms in our multivariate series represent? If x and y are both spatial then a function like $\sin(3x)\cos(7y)$ can be graphed as a surface, as shown in Figure 9.22. If you start at any point and move in the x-direction, you trace out a sinusoidal oscillation with wavelength $2\pi/3$. If you move in the y-direction you see oscillation with a wavelength of $2\pi/7$. The function is odd with respect to x and even with respect to y. You can verify all that information from the function, but the graph gives you a way to see what it all means visually.

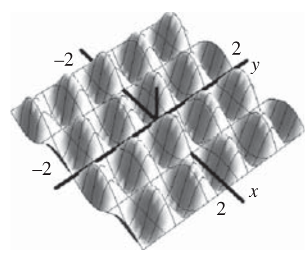


FIGURE 9.22 $f(x, y) = \sin(3x)\cos(7y)$.

We can visualize the same function in a very different way by thinking of one of the independent variables as time. Return to the image of a guitar string moving up and down in the plane of the page. The motion of such a string can be fully described by a function $y(x, t)$, which can in turn be decomposed into a multivariate Fourier series. One term in that series might be $\sin(3x)\cos(7t)$. What does that look like?

Well, at $t = 0$ it just looks like $y = \sin(3x)$. The function $\cos(7t)$ represents the changing amplitude of that function, and the wavelength does not change. So over time, the state evolves as shown in Figure 9.23.

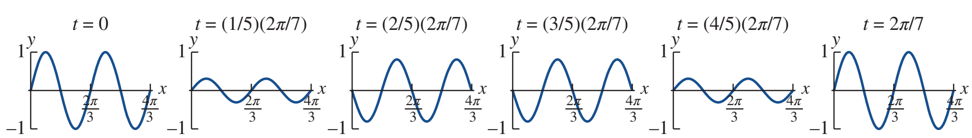


FIGURE 9.23 The function $f(x, t) = \sin(3x)\cos(7t)$ at several moments in time.

Motion like this where all the points on the string oscillate together is sometimes referred to as a "normal mode" of the system. The wavelength (2π divided by the x-frequency) is the distance from one crest to another; the period (2π divided by the t-frequency) is the time between oscillations. In an actual string the two frequencies are related by the "wave equation" which will be discussed more fully in Chapter 11.

Remember that we are not saying that a string always vibrates in a normal mode like $\sin(px)\cos(\omega t)$. We are saying, rather, that more complicated motion can be decomposed into a combination of such normal modes—that's what a Fourier series does—and that representation helps you model the behavior of the string and predict the resulting notes.

9.8.3 Problems: Multivariate Fourier Series

The formulas in Appendix G for multivariate Fourier series will be needed for many of these problems.

9.116 Walk-Through: Multivariate Fourier Series. Imagine an infinite chess board on which all the black squares are raised to the height $z = 1$ and all the white squares are lowered to the height $z = -1$. (Such boards are sometimes used by blind players so they can easily feel where each square begins and ends. Those boards, of course, are finite.) Such a function might be represented as:

$$f(x,y) = \begin{cases} 1 & x \in [0,1), \quad y \in [0,1) \\ -1 & x \in [-1,0), \quad y \in [0,1) \\ 1 & x \in [-1,0), \quad y \in [-1,0) \\ -1 & x \in [0,1), \quad y \in [-1,0) \end{cases}$$

… and so on forever in both directions.

(a) Is this function odd, even, or neither in x? What is its period in x?

(b) Is this function odd, even, or neither in y? What is its period in y?

(c) Your answers so far will tell you what types of functions will appear in the Fourier series: sines or cosines or both, and with what frequencies. Based on that information, write the form of the Fourier series you will need.

(d) One complete period of this function is a square extending from -1 to 1 in both x and y. Explain why, in this problem, you can integrate over a smaller part of that region and still find the complete answer.

(e) Find the coefficients of the Fourier series.

(f) Write the Fourier series for $f(x,y)$ including all terms with m and n less than 4.

In Problems 9.117–9.121 find the Fourier series for the specified function. If *both* x and y are symmetric (even or odd) then use sines and cosines, using only the appropriate one for each variable. Otherwise use the complex exponential form for both variables. The integrals in Appendix G will be helpful for some of these.

9.117

$$f(x,y) = \begin{cases} 1 & x \in [0,1), \quad y \in [0,1) \\ 2 & x \in [-1,0), \quad y \in [0,1) \\ 3 & x \in [-1,0), \quad y \in [-1,0) \\ 4 & x \in [0,1), \quad y \in [-1,0) \end{cases}$$

… repeated periodically in both directions.

9.118 The function $x^2 y$ on $-1 \le x < 1$, $-2 \le y < 2$ and repeated thereafter.

9.119 The function xy^2 on $0 \le x < 1$, $0 < y < 2$ and repeated thereafter.

9.120 The function $4x\sin(3y)$ on $-\pi \le x < \pi$ and repeated thereafter.

9.121 The function e^{2x-y} on $-2\pi \le x < 2\pi$, $-2\pi \le y < 2\pi$.

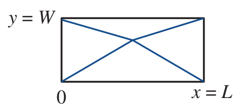
9.122 The Explanation (Section 9.8.2) showed the time evolution of a guitar string that follows the equation $z = \sin(3x)\cos(7t)$. In this problem you are going to do some similar sketches. In each case you will draw a sequence of pictures to show the time evolution of a different string. You can do these quickly by hand; the point is to see how they differ from each other, not to get every detail right.

(a) Draw the time evolution of $z = 2\sin(3x)\cos(5t)$. How does the behavior of the string differ from the version we drew?

(b) Draw the time evolution of $z = \sin(6x)\cos(5t)$. How does the behavior of the string differ from the version we drew?

(c) Draw the time evolution of $z = \sin(3x)\cos(10t)$. How does the behavior of the string differ from the version we drew?

9.123 The Explanation (Section 9.8.2) considered a rectangular drumhead pulled up into a paraboloid. It could instead have been pulled up into four planes.



(a) Assuming the drumhead extends along $0 \le x \le L$ and $0 \le y \le W$, and the middle point is raised to height H above the plane, write the equations for the four planes.

(b) To create a Fourier series for this shape, you need to create a periodic extension. Is it best to use an odd or even extension in x? In y? Explain.

(c) Write the Fourier series for your periodic function and write the integrals for finding the coefficients. You do *not* need to evaluate those integrals for this problem.

9.124 *[This problem depends on Problem 9.123.]*

(a) Evaluate the integrals you found in Problem 9.123 for the Fourier coefficients. (You could do this by hand but it would be very tedious. Whether you do it by hand or on a computer, you will need to consider the case $m = n$ separately from the case $m \neq n$.)

(b) Set $L = 4$, $W = 2$, and $H = 1$ and plot the Fourier series, including all terms up to $m = n = 20$. Verify that it matches the original shape of the drumhead.

(c) Compare this series to the one we found in the Explanation (for the same drumhead with a different starting condition). Do the same combinations of notes appear in both cases? Explain.

9.125 In this problem you will derive the formula for the coefficients of a multivariate Fourier series. We assume that the function $f(x, y)$ is odd in both x and y, and that it is periodic in x with period $2L$ and periodic in y with period $2W$. Under these circumstances the Fourier series looks like Equation 9.8.1.

(a) Multiply both sides of Equation 9.8.1 by $\sin\left(\frac{k\pi}{L}x\right)\sin\left(\frac{p\pi}{W}y\right)$. Then integrate both sides as x goes from $-L$ to L and y goes from $-W$ to W.

(b) The resulting integral is separable. Pull the constant out in front and separate it into one integral in terms of x and one in terms of y.

(c) The right side of your equation now has an infinite number of integrals. Use the orthogonality of the sine functions, and the fact that $\int_{-L}^{L} \sin^2\left(\frac{k\pi}{L}x\right)dx = L$, to evaluate all of them.

(d) Solve the resulting equation for the coefficient D_{kp}.

9.126 *[This problem depends on Problem 9.125.]* Derive the formula for the coefficients of a multivariate Fourier series for a function $f(x, y)$ that is periodic in x with period $2L$, periodic in y with period $2W$, odd in x, and *even* in y. You will not begin with Equation 9.8.1 exactly.

9.127 In the example on Page 11 we found a Fourier series for the function $f(x, y) = H(1 - y/W)$ on the domain $0 \le x \le L$, $0 \le y \le W$. Choose any positive values for H, L, and W, and plot the partial sum of that Fourier series that includes all terms up to $m, n = 11$. Verify that it reproduces the shape of the original function in the appropriate domain.