Discovery Exercise for Infinite Series

In this exercise you'll apply the geometric series trick to find the values of x that a particular Taylor series works for. (If you're not familiar with how to find the sum of a geometric series you should go through the geometric series exercise before this one.)

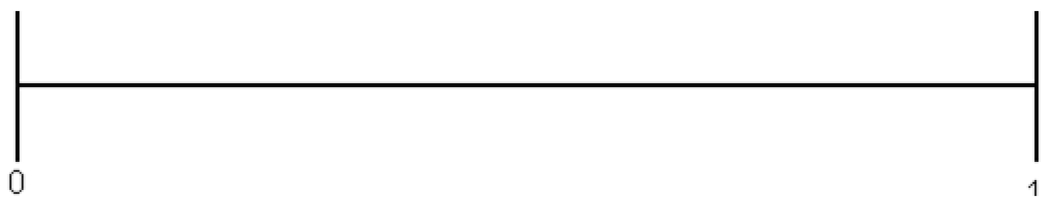
Our example is the following Maclaurin series.

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \ldots$$

This series works very well for $x = 1/3$, and does not work at all for $x = 2$. This brings up the practical question: how can we determine when a Taylor series works? But underlying that question is a more theoretical concern: what does it mean for an infinite series to add up to anything at all?

As an example consider the series $\sum_{n=1}^{\infty} (1/2)^n$, or $(1/2) + (1/4) + (1/8) + \ldots$.

We begin our examination of this series visually, on the number line below.



1. The series begins at $1/2$. Mark this point on the number line, and label it S_1 for the first partial sum.
2. After the second term, the series reaches $(1/2) + (1/4)$. This is the second partial sum: label it S_2 on the number line. (You are not labeling the number $1/4$; you are labeling $(1/2) + (1/4)$, the series after two terms.)
3. Now label S_3 which is $(1/2) + (1/4) + (1/8)$. (You can of course punch this into a calculator, but you will find the pattern more easily if you work with fractions instead of decimals.)
4. Find S_4, S_5, and S_6. Write down their values and label them on the number line.
5. As we add more and more terms, where are we heading?

This brings us to the mathematical definition of an infinite series: the sum of an infinite series is defined as the limit as $n \to \infty$ of the partial sums S_n. Let's see how that definition applies to a few different series.

6. Consider the series $\sum_{n=1}^{\infty} (1/3)^n$.
   (a) Write out the first five terms of the series.
   (b) Write out the first five partial sums of the series.

(c) Calculate the 20^{th} partial sum of the series. (*Hint*: don't do it by hand!)

(d) Where does it look like this is headed?

(e) Calculate the n^{th} partial sum of the series.

(f) Calculate $\lim_{n\to\infty} S_n$.

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7. For a geometric series whose first term is a_1 and whose common ratio is r, the sum of the first n terms is given by the formula $S_n = a_1(1 - r^n)/(1 - r)$.

   (a) If $r = 2$, what is the sum of the *infinite* geometric series? (That is, what is $\lim_{n\to\infty} S_n$?)

   (b) If $r = 1/2$, what is $\lim_{n\to\infty} S_n$?

   (c) If $r = 1$, what is $\lim_{n\to\infty} S_n$? (*Hint:* Rather than applying l'Hôpital's rule, write down a geometric series with $r = 1$ and it's easy to see where the partial sums are headed.)

   (d) Now, generalize: for what values of r ($r > 0$) does an infinite geometric series approach a finite sum?

   (e) Finally, do the same for $r < 0$.

8. The Maclaurin series for $1/(1-x)$ is, in fact, a geometric series. Based on your answer to Part 7, for which values of x does that series add up to a finite number? (This range of values is called the "interval of convergence" of the series.)