This section is more mathematical than the other online sections for this book. For that reason, this is the only online section that ends with Questions and Problems, because you can't learn this kind of information by just reading about it.

For the same reason, it's important to come into this section with all the necessary background. The following sections are vital prerequisites; make sure you are familiar with them before reading this section.

- Section 5.5 discusses the complex exponential function.
- Section 5.6 discusses the time evolution of a quantum mechanical wavefunction.
- Section 6.1 discusses the math of traveling waves.
- Section 6.2 discusses the energy eigenfunctions of a free particle.

The content below provides valuable complements to this section.

- Section 6.3 presents the momentum eigenstates of a quantum mechanical particle, e^{ikx}. If you have not gone through that section, you will have to take our word for that fact.
- Section 6.4 discusses the idea of a wave packet, and the ideas of phase velocity and group velocity. There is considerable overlap between that section and this; going through both sections will deepen your understanding.
- The following two animations give you hands-on experience with the sums of traveling waves.

  Building a Wave Packet: `science.smith.edu/physics_demos/wave_packet/waveAddition.html`

  Group Velocity vs. Phase Velocity: `science.smith.edu/physics_demos/groupVSPhase/groupVSPhase.html`

Explanation: Wave Packets

Figure 1 shows a localized wavefunction. At the moment shown ($t = 0$), the particle represented by this wavefunction has a very high probability of being found close to $x = 0$, and a vanishing probability of being found more than 0.2 units or so away from that position. From here on, we're going to assume this is a "free particle," meaning there are no forces acting on it.

If you want to predict the motion of the particle—in other words, if you want to know how its wavefunction is going to evolve in time—you need to express that wavefunction in terms of energy eigenstates. Here's a quick review of what we know about those eigenstates (Problem 6).

- For a free particle, those eigenstates are of the form $\psi(x) = e^{ikx}$. Each such eigenstate has its own k-value: positive or negative (or zero), rational or irrational, but always real.

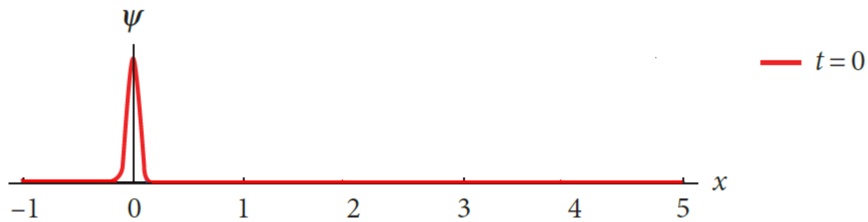

Figure 1: Wavefunction for a particle with a small uncertainty in position

- The eigenvalue of any one of those functions—that is, the energy represented by that wavefunction—is $E = \hbar^2k^2/(2m)$.
- The time dependence of an energy eigenstate is given by multiplying it by $e^{-iEt/\hbar}$, which in this case gives a traveling wave: $\Psi(x,t) = e^{i(kx-\omega t)}$, where $\omega = \hbar k^2/(2m)$. The speed of that traveling wave is $v = \omega/k = \hbar k/(2m)$. (Note that this velocity means that eigenstates with $k > 0$ move to the right, and those with $k < 0$ move to the left.)

Remember that each individual eigenstate extends periodically across the entire x-axis. They form a localized packet because they cancel each other out everywhere *except* at the packet around $x = 0$.

As all the eigenvalues travel along the x-axis, each with its own velocity, the locations of not-total-destructive-interference also travel. That moves the wave packet, and therefore the particle itself, as illustrated in Figure 2. This behavior—the particle moving at constant velocity—is essentially the same as that of a classical free particle. (We'll see later that under some circumstances a quantum free particle can act very differently from a classical one.)

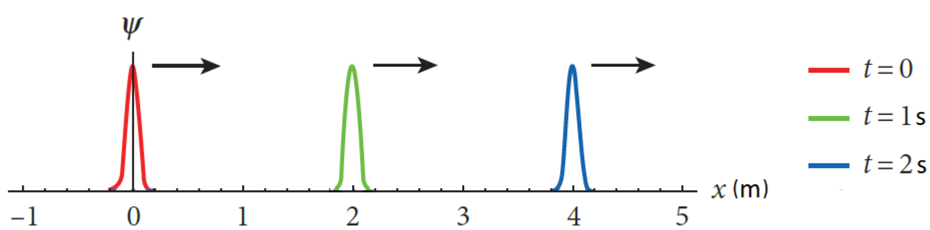

Figure 2: The same wavefunction shown at three different times. In this example the wave packet is moving to the right at 2 m/s.

We're not providing a full treatment of the math behind wave packets, a topic known as "Fourier transforms." But we are going to offer reasonable answers to two questions:

1. How can a *localized* wavefunction be written as a combination of complex exponentials, each of which extends over the entire x-axis?
2. How does the combined motion of all of those complex exponential functions (energy eigenstates) give rise to the time evolution of the wavefunction?

Building a Localized Wave Packet from Cosines

A complex exponential function does not represent exponential growth (like e^{kx}) or decay (e^{-kx}). Rather, it represents oscillation, as you can see from Euler's formula:

$$e^{ikx} = \cos(kx) + i\sin(kx).$$

So we can build our intuition for the sums of complex exponential functions by starting with sums of cosines. Such real-valued functions are (of course) much easier to visualize, but still, it's not obvious what the sum of many cosines—much less *infinitely* many cosines—should look like. So we're going to step through a sequence of exercises, from the simplest example to more elaborate cases. As always, answer each question carefully on your own before reading our answers!

Active Reading Exercise: The Sum of Two Cosines

1. What is the period of the function $\cos(x)$? What is the period of $\cos(2x)$?

Now consider the following combination of those two functions:

$$f_1(x) = \frac{1}{2}\cos(x) + \frac{1}{2}\cos(2x).$$

Answer both of the following questions *without* using a computer or calculator to graph $f_1(x)$ for you—but then, of course, feel free to use such a graph to check your answers.

2. At $x = 0$ our function reaches its maximum value, $f_1(0) = 1$, because of perfect constructive interference of the two cosines. What are all the other x-values at which this function attains that maximum?
3. What are all the x-values at which this function reaches perfect *destructive* interference, i.e. $\cos(x) = -\cos(2x)$ (which implies $f_1 = 0$)?

Because $\cos x$ has a period of 2π, and $\cos(2x)$ has a period of π, both functions reach their maxima at $x = 2\pi$, and again at $x = 4\pi$, 6π, and so on. We can say in general that $f_1(x) = 1$ when $x = 2n\pi$, where n is any integer (positive, negative, or zero).

Directly in between those points—at $x = \pi$, 3π, and so on—$\cos(2x)$ *again* reaches its maximum, but $\cos x$ reaches its minimum. So perfect destructive interference occurs at $x = (2n + 1)\pi$. (See Figure 3.)

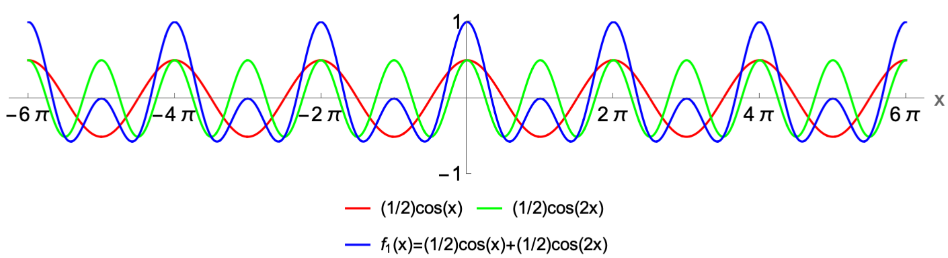

Figure 3: At $x = 2n\pi$, both $(1/2)\cos(x)$ (in red) and $(1/2)\cos(2x)$ (in green) reach their maxima, so the two interfere constructively. At the points exactly in between the two individual cosines are opposite, so they interfere destructively.

What if we add more waves?

Active Reading Exercise: Lots of Cosines

Our next function adds many more cosines, but still a finite number.

$$f_2(x) = \frac{1}{10}\cos(x) + \frac{1}{10}\cos(2x) + \frac{1}{10}\cos(3x) + \ldots + \frac{1}{10}\cos(10x)$$

1. List the periods of all ten cosine functions that make up $f_2(x)$. (This isn't as onerous as it sounds: once you see the pattern, the whole list will take less than thirty seconds.)
2. What are all the x-values at which this function reaches perfect constructive interference, i.e. $f_2(x) = 1$?
3. When x is not at or very near the values you listed in Part 2, $f_2(x)$ stays very close to zero. Based on the behavior of its component cosine functions, explain why.
4. Extrapolating from what you've seen so far, draw a rough sketch of what you think the following function would look like.
$$f_3(x) = \frac{1}{30}\cos(x) + \frac{1}{30}\cos(2x) + \frac{1}{30}\cos(3x) + \ldots + \frac{1}{30}\cos(30x)$$

We'll start with the sum of 10 cosines that began the exercise. Those individual cosine functions have periods 2π, $2\pi/2$, $2\pi/3$, and so on. So they all come back to their maxima at $x = 2\pi$. The entire function therefore still has a period of 2π, and reaches its maximum value $f_2(x) = 1$ at $x = 2n\pi$ for all integers n.

But at every point *except* $x \approx 2n\pi$, you have a large collection of waves all with different phases. Some are positive and some are negative, and they tend to cancel. The result is that $f_2(x)$ stays between -0.1 and 0.1 until it is very close to one of its peaks.

That trend becomes more pronounced as you add more cosines. Below is a graph of $f_3(x)$, the sum of thirty cosine functions.

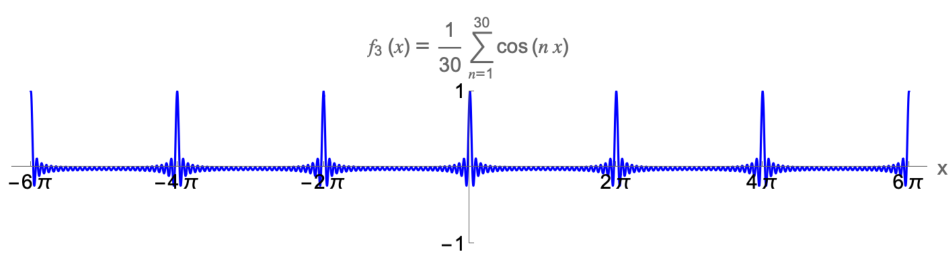

Figure 4: The sum of 30 cosine functions stays very close to zero except at points where they all interfere constructively. (Think about why the plot goes up to 1 but not down to -1; see Question 3.)

Remember that our purpose in the first half of this section is to consider how periodic, oscillating functions can lead to localized wave packets. Looking at Figure 4 you can see that we are making progress toward that goal. As we add even more cosines, the peaks will become sharper, and the regions between the peaks will be more suppressed. In the limit of an infinite series, the function will approach zero at all x-values away from its peaks.

But we will still have infinitely many peaks, occurring at every $x = 2n\pi$. A localized particle should have only one peak (or at least only have peaks in one finite region). Our next step in that direction is to use cosines whose frequencies are closer together.

Active Reading Exercise: Half-Integer Frequencies

The previous Active Reading Exercise introduced the function $f_3(x)$, the sum of thirty cosine functions. Figure 4 above shows the graph of $f_3(x)$.

The function f_4 below is *also* the sum of thirty cosine functions, but their frequencies are half the frequencies of the corresponding cosines in f_3:

$$f_4(x) = \frac{1}{30}\cos\left(\frac{x}{2}\right) + \frac{1}{30}\cos(x) + \frac{1}{30}\cos\left(\frac{3x}{2}\right) + \frac{1}{30}\cos(2x) + \ldots + \frac{1}{30}\cos(15x).$$

How does the graph of f_4 differ from the graph of f_3?

The answer is that f_4 looks identical to f_3 in every way, except that the peaks are twice as far apart: they appear at $x = 4n\pi$ (Figure 5).

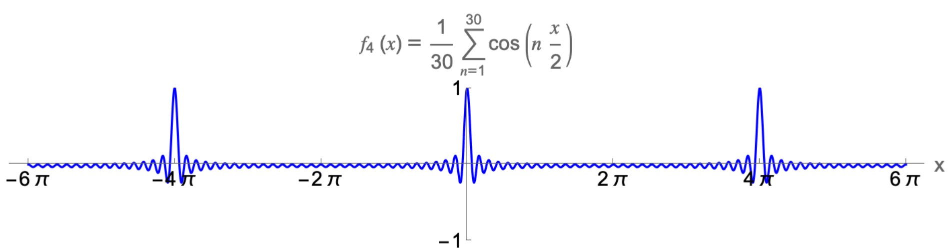

Figure 5: In a series of cosines, the closer their frequencies come to each other, the farther apart the peaks of constructive interference get.

Here are a few ways you can see why that happens.

- Given any function $f(x)$, the function $f(x/2)$ always looks the same but stretched out horizontally.
- The periods of the cosine components of $f_4(x)$ are 4π, $4\pi/2$, $4\pi/3$, and so on. They all reach peaks at every multiple of 4π. In between, as before, they mix positive and negative contributions in a way that tends to cancel out.

If you get that, you can see how we could push the peaks to 8π apart, by using quarter-integer frequencies. As the frequencies are pushed tighter together, the wave peaks spread as far apart as we like. But no matter how far we continue that progression, there will still be infinitely many peaks. As we mentioned in Section 6.2, an infinite sum of cosines cannot be normalized.

But in the limit, as the distance between frequencies approaches zero, the distance between peaks approaches infinity—which is to say, we approach a wave packet with only one peak. In that limit, our sum becomes an integral.

The leap from a series to an integral is not an easy one, but it's a vital step to become comfortable with. The connection is a Riemann sum. An integral is, by definition, the limit of a series, as the distance between terms in the series *approaches* (but can never reach) zero. So look again at $f_3(x)$ and $f_4(x)$ as defined above, and then consider an $f_5(x)$ function that starts with $\cos(x/1000)$. Then $x/10^6$. The distance between peaks grows without bound, so the periodic function approaches a localized wave packet.

To finish the story, we have to mention the amplitudes of our various cosine waves. If you build a sum of 100 cosines, you can change their 100 various amplitudes to anything you like. The results will be different functions, so you can tailor the amplitudes to fit the behavior you're looking for. (Until you turn your series into an integral, however, the resulting functions will still necessarily be periodic.)

With infinitely many functions, of course, you can't specify their amplitudes one at a time. Instead you write a function $A(k)$, the amplitude of each wave as a function of its angular frequency. Different $A(k)$ functions lead to different final wavefunctions. Appendix C gives the formula for choosing the $A(k)$ function that will lead to the wave packet you want; you will derive that formula when you study Fourier transforms.

From Real-Valued Cosines to Complex Exponentials

Quantum mechanical wavefunctions are complex. The energy eigenstates of a free particle (and the momentum eigenstates for *any* particle) are, in general, of the form e^{ikx}.

As we mentioned above, you can think of e^{ikx} in terms of its real part (a cosine) and its imaginary part (a sine), both oscillating with period $2\pi/k$. But it's cleaner to think about that function in terms of its modulus (always 1) and its phase (kx). As x increases, e^{ikx} travels in a circle of radius 1 around the origin of the complex plane.

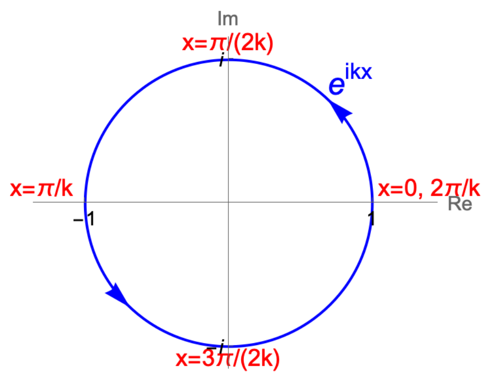

Figure 6: When $x = 0$, the function e^{ikx} is 1. When $x = 2\pi/k$, the function returns to 1.

Figure 6 shows all the points generated by this function on the complex plane. Figure 7 offers a more powerful visualization: the complex plane is now perpendicular to an x-axis, so each point shows both an x-value and the corresponding complex number.

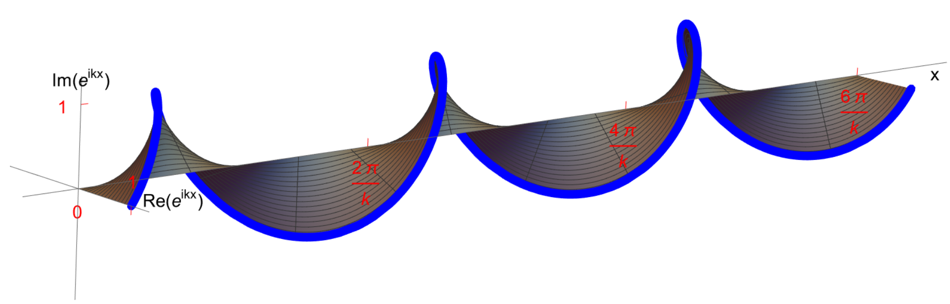

Figure 7: The function e^{ikx} with axes for the real and imaginary parts of the function. (The function is in blue; the gray surface is there to guide your eye.)

Now, just as we did with cosines, let's start adding these functions.

Active Reading Exercise: The Sum of Two Complex Exponentials

1. What is the period of the function e^{ix}? What is the period of e^{2ix}?

Now consider the following combination of those two functions:

$$f_5(x) = \frac{1}{2}e^{ix} + \frac{1}{2}e^{2ix}.$$

2. At $x = 0$ our function has its maximum modulus, $|f_5(0)| = 1$, because of perfect constructive interference. What are all the other x-values at which this function has modulus 1?
3. What is the first x-value at which this function reaches perfect *destructive* interference, i.e. $e^{ix} = -e^{2ix}$ (which implies $f_5(x) = 0$)?

Yes, those are basically the exact same questions we asked about $f_1(x)$, the sum of two cosines—and the answers are also the same. But make sure you can see your answers in terms of the spirals in Figure 8 below.

- At $x = 0$ both waves are real and positive, so they interfere constructively.
- At $x = \pi$ the second wave has rotated one full turn around the x-axis, while the first wave has only rotated a half-turn. So one is real and positive and the other real and negative, and they interfere destructively.
- At $x = 2\pi$ they are back in phase again and interfere constructively, and the whole system repeats from there.

From here the story is essentially identical to what we described with cosines. Add waves with ever-higher frequencies and you'll generally create more destructive interference in between the high peaks. Place your frequencies closer together and you'll push those high peaks farther apart. In the limit where the frequencies are infinitesimally close, your sum becomes an integral and you can produce a function $\psi(x)$ with only one peak. By convention, the amplitudes of those complex exponentials are called $\hat{\psi}(k)/(\sqrt{2\pi})$, so:[1]

$$\psi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{\psi}(k) e^{ikx} dk. \tag{1}$$

The function $\hat{\psi}(k)$ is called the "Fourier transform" of $\psi(x)$, as discussed in Section 6.2.

The Time Evolution of a Wave Packet

We have seen that by properly combining complex exponentials—functions that are themselves periodic, and certainly not normalizable—we can end up with a wave packet, a single bump that is localized and normalizable. Now we turn our attention to the question: if those exponential functions all move, how does that move the resulting wave packet?

The mathematics of real-valued traveling waves are discussed in Section 6.1, and summarized in Appendix F. In complex form, a traveling wave is represented as:

$$\Psi(x,t) = e^{i(kx-\omega t)}. \tag{2}$$

The parameters k and ω in Equation 2 might both be referred to as "frequencies." But the first is spatial and the second is temporal, so they describe very different properties.

[1] Technically the dk is part of the amplitude as well. That makes sense because adding a continuously infinite set of complex exponentials and getting a finite result requires that each one have an infinitesimal amplitude. That dk plays the same role that the 1/2, 1/10, or 1/30 played in our sums above.

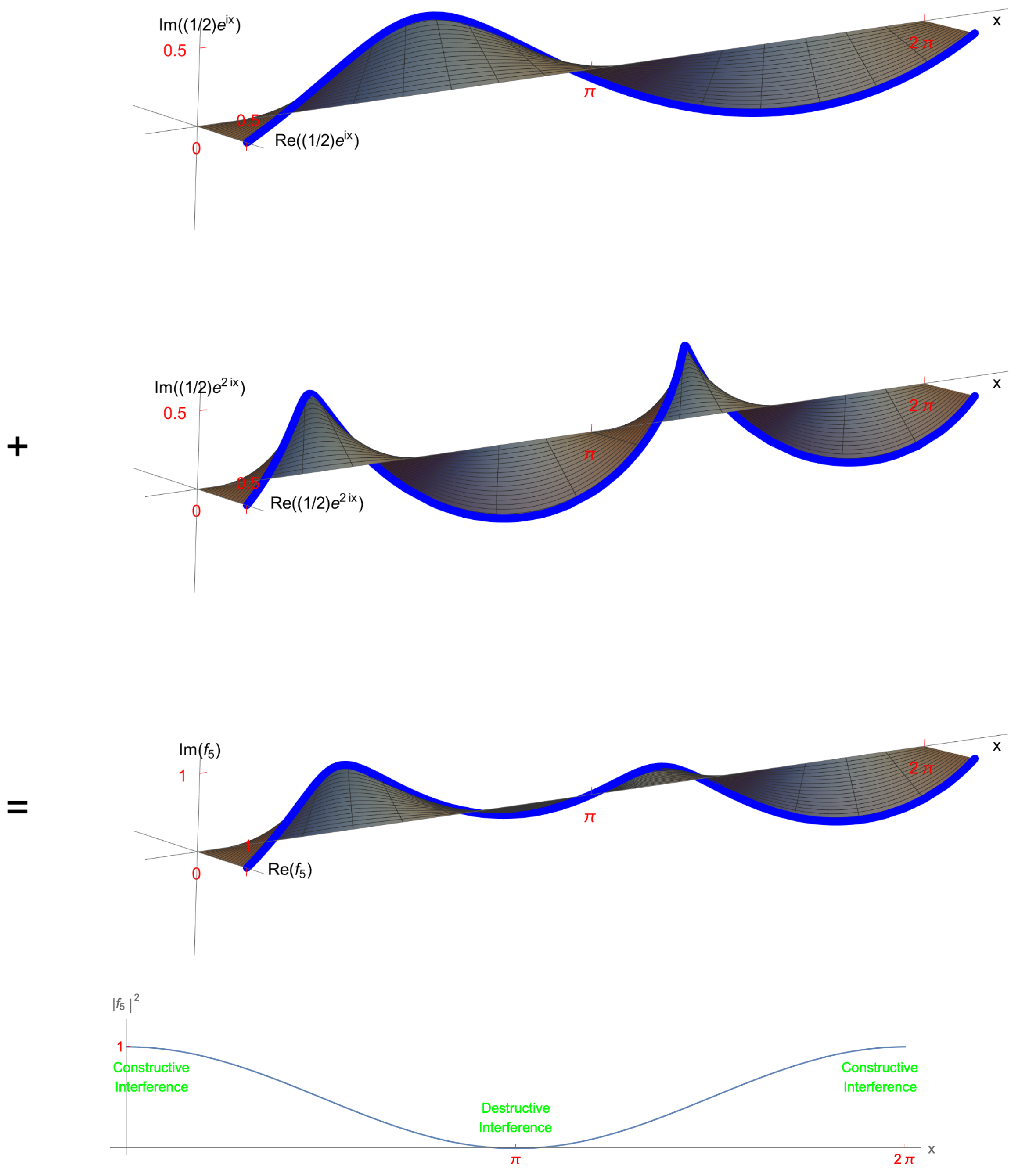

Figure 8: Sums of complex exponentials produce areas of constructive and destructive interference just like sums of sines or cosines do. The top three figures show plots of complex functions with axes for the real and imaginary parts. The bottom plot shows the modulus squared of the sum of two complex functions, illustrating the effect of constructive interference at $x = 0$ and $x = 2\pi$ and destructive interference at $x = \pi$.

- k is the "wave number": it is 2π times the *number of wavelengths per unit length.* A cosine wave with a high k-value has its peaks packed close together; a wave with a low k-value is widely spread out.

- ω is the "angular frequency." As the wave travels along the x-axis, any given point oscillates up and down, and ω is 2π times the *number of oscillations per unit time.* A wave with a high ω bobs up and down many times in a second; a wave with a low ω oscillates slowly.

Mathematically, you can set these parameters independently of each other. For instance, $k = 1$ m^{-1} and $\omega = 2\pi$ s^{-1} describes a certain shape oscillating once per second; $k = 1$ m^{-1} and $\omega = 10\pi$ s^{-1} describes the exact same shape oscillating five times per second.

But for any given *physical* system, the angular frequency is determined by the wave number. Bend a guitar string into a cosine wave with a particular k-value, and when you release the string it will vibrate with a corresponding ω-value. So that particular guitar string is characterized by a particular $\omega(k)$ function, its "dispersion relation."[2] Tighten the string and you will change its dispersion relation, thus changing the note it produces.

We are emphasizing the roles of ω and k, and the relationship between them, because those quantities also describe how a wave travels.

A wave packet is an integral over infinitely many energy eigenstates, each of which individually looks like Equation 2. The waves in that integral are governed by a particular dispersion relation. That is, each eigenstate is defined by a particular wave number k, and that in turn determines its angular frequency ω. The speed of each individual eigenfunction, which is called its "phase velocity," can be read directly from Equation 2 above:

$$v_p = \frac{\omega}{k} \qquad \textit{phase velocity of a traveling, sinusoidal wave.} \tag{3}$$

If that isn't clear to you, review the discussion of traveling waves in Section 6.1, and/or see Problem 9.

But the actual velocity of the particle, as you might measure it in a lab, is not the velocity of any individual energy eigenfunction. It is the velocity of the entire wave packet, called its "group velocity."

As a first example of this idea, consider a periodic function (not a localized wave packet) built from two waves.

$$f_6(x) = A_1 e^{i(k_1 x - \omega_1 t)} + A_2 e^{i(k_2 x - \omega_2 t)}$$

The resulting function may change its shape over time, but its antinodes (points of perfect constructive interference) and its nodes (perfect destructive interference) will travel with velocity:

$$v_g = \frac{\omega_2 - \omega_1}{k_2 - k_1} = \frac{\Delta\omega}{\Delta k} \qquad \textit{group velocity of the sum of two waves.} \tag{4}$$

You derived that result (using cosines instead of complex exponentials) in Problem 14 of Section 6.4. In Problem 10 of this section, you will reach the same conclusion a different way.

Quite apart from any derivation, Equation 4 offers a significant insight: when you sum two waves, their collective velocity depends on the *difference* between their ω values. That's because if the two component waves oscillate with the same angular frequency, then their points of constructive and destructive interference remain at the same places. If the angular frequencies are different, then at any given point the two waves go in and out of phase with each other, so each point oscillates between constructive and destructive interference. That's what moves the combined wave along: not the motion of its component waves, but the change in their interference pattern.

The group velocity can be very different from the phase velocities. You could have two waves with nearly identical phase velocities, but their sum might have a group velocity much larger, much smaller, or even in the opposite direction from those phase velocities. If you haven't encountered phase and group velocity much before, we recommend you pause here to play with the interactive animation "Group Velocity vs. Phase Velocity":

`science.smith.edu/physics_demos/groupVSPhase/groupVSPhase.html`

We'll assume you've spent some time with that animation and move on.

[2] The name comes from the idea that waves with different velocities will "disperse," i.e. move through space, differently.

You may quite reasonably wonder why we're dwelling on the sum of two waves, when our real interest is a sum (or really an integral) over infinitely many waves. But you can learn a lot by thinking of all the *pairs* of waves in that infinite sum. Each individual wave in the series has its own values of k, ω, and v_p. For each pair of waves the points of constructive and destructive interference move with velocity $\Delta\omega/\Delta k$, and that velocity may be different for each pair.

So what does that mean about the entire series? With the points of constructive and destructive interference for each pair moving at different speeds or even in different directions, the overall function will not necessarily look like a set of moving peaks. It may look like a constantly evolving shape, changing almost randomly from one moment to the next.

But—still keeping an eye on Equation 4—can you see the exception? Under what circumstance would a series of waves, all with different k and ω values, have a consistent group velocity, and therefore retain its basic shape over time? That will happen if every pair of waves has the same group velocity, the same $\Delta\omega/\Delta k$. It's easy to show (Problem 13) what kind of dispersion relation that implies.

If the dispersion relation $\omega(k)$ is a *linear function,* then the resulting series will move with a consistent group velocity $v_g = d\omega/dk$, meaning all the nodes and antinodes will move with that velocity.

In most cases $\omega(k)$ isn't linear, and therefore $d\omega/dk$ isn't constant, but for many physical situations $d\omega/dk$ is *approximately* constant. The resulting wave will move smoothly with a fairly well-defined velocity, while changing its shape gradually.

To see why $d\omega/dk$ is often approximately constant, look back at Equation 1. That equation shows how we build a wave packet as an integral over infinitely many complex exponential waves, each with its own wave number k, and each with its own amplitude proportional to the Fourier transform $\hat{\psi}(k)$. Any particular physical system has a dispersion relation that gives ω as a function of k, which means $d\omega/dk$ is also a function of k. Unless $\omega(k)$ is linear, that means there will be no single group velocity for the wave packet.

But in order for the integral in Equation 1 to converge, the amplitudes $\hat{\psi}(k)$ must approach zero in the limit $k \to \pm\infty$, so the wave packet only has significant contributions from a limited range of wave numbers k. If $d\omega/dk$ doesn't vary much within that range, then you can approximate $d\omega/dk$ as a constant, and meaningfully speak about "the group velocity" of your wave packet.

Put another way, a wave packet is in general made up of complex exponential functions with all values of k from $-\infty$ to ∞, each with its own individual value of $d\omega/dk$. But in practice any individual wave packet is built up almost entirely by waves within some finite range of k, and that gives rise to two possibilities.

1. If the range of k that contributes significantly to the wave packet is small enough and/or if $d\omega/dk$ varies slowly enough with k, then the wave packet will have a well defined group velocity, all the nodes and antinodes will move with that velocity, and the basic shape of the wave packet will stay approximately constant over time.
2. If the range of k that contributes significantly to the wave packet is large enough and/or $d\omega/dk$ varies quickly enough that $d\omega/dk$ changes appreciably within the relevant range, then the wave packet won't have a well defined group velocity. In this case the time evolution of the wave packet can look arbitrarily complicated.

The Time Evolution of a Quantum Mechanical Free Particle

To conclude, let's circle back to where we started and apply what we've learned to the case of a free particle.

You may recall that, based on Schrödinger's equation, we wrote the energy eigenstates of such a particle as traveling waves with $\omega = \hbar k^2/(2m)$. We now recognize that equation as the dispersion relation for our free particle's eigenfunctions. This dispersion relation is not linear, so $d\omega/dk$ is not constant, and in general a free particle wave packet does not necessarily move with a single group velocity.

But as we noted above, if the wave packet is made almost entirely from waves within a small range of k values, then we can treat $d\omega/dk$ as approximately constant within that range. In that case the wave packet will move along the

x-axis with a roughly constant shape at a single, well-defined velocity:

$$v_g = \frac{d\omega}{dk} = \frac{\hbar k}{m} \quad \textit{group velocity of a free particle wave packet with a narrow range of } k\textit{-values.} \tag{5}$$

We can give a physical interpretation to the fact that group velocity is only defined if the range of k is sufficiently narrow. Remember that e^{ikx} is not just the energy eigenfunction of a free particle; it's also the *momentum* eigenfunction of any particle, with eigenvalue $p = \hbar k$. So, saying that all the important waves for a given wave packet have similar k-values is the same as saying that the particle has a limited (but of course nonzero!) uncertainty in momentum.

Earlier we listed two possibilities for how a wave packet can behave, purely based on math. We can now restate what those two possibilities mean for the physical behavior of a free particle.

1. If a particle has a small enough momentum uncertainty (small range of k-values), then its wave packet will have a well defined group velocity $v_g = \hbar k/m = p/m$, and the wave packet will move along the x-axis at that speed with roughly constant shape.
2. If a particle has a large momentum uncertainty (large range of k-values), then its wave packet will change shape significantly as it moves. In practice, if you start with a single bump such as the one shown in Figure 2, it will spread out, becoming wider over time.

These conclusions come directly from the math, but they also make physical sense. If a free particle (one with no forces on it) has a well defined momentum p (narrow range of k-values making up its wave packet), then it will move at a constant velocity $v = p/m$, just as a classical particle would. But if a free particle does not have a well-defined momentum (wide range of k-values), then it will not simply move; its wavefunction will broaden over time. If you start in the state "I know where the particle is but I have no idea how fast it's moving," you won't know where it is for long!

Questions and Problems: Wave Packets

Conceptual Questions and ConcepTests

1. Define the phrases "phase velocity" and "group velocity" in your own words.
2. Figure 1 on p. 2 shows the wavefunction of a particle. Which of the following best describes the energy eigenfunctions that make up this wave packet at $x = 4$? (Choose one.)
   - A. The eigenfunctions are defined on a domain that does not include $x = 4$.
   - B. The eigenfunctions are all at or near zero at $x = 4$.
   - C. Individual eigenfunctions may have very high or low values at $x = 4$, but they all add up to a near-zero value there.
   - D. The eigenfunctions may add up to a very high or low value at $x = 4$, or they may add up to a near-zero value there.
3. Figure 4 on p. 4 shows the sum of 30 cosine functions.
   - (a) Why does the plot go up to $f = 1$ but not down to $f = -1$?
   - (b) If you were to plot a similar function, but with all the sines replaced by cosines, would the plot go up to 1? Down to -1? (Choose one and explain.)
     - A. Up to 1 but not down to -1
     - B. Down to -1 but not up to 1
     - C. Both

D. Neither

(c) Write a sum of 30 sinusoidal functions (other than the two we've already discussed) that goes down to -1 but not up to 1.

4. Figure 9 shows two Gaussian wavefunctions.

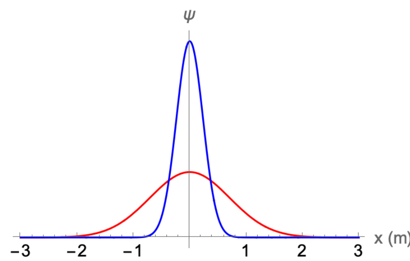

Figure 9: Two wavefunctions

(a) Which wavefunction has a greater position uncertainty?

(b) Which wavefunction has a greater momentum uncertainty? How do you know?

(c) Assume both wavefunctions have an expectation value of velocity $\langle v \rangle = 1$ m/s. Describe using words and a picture how the two will evolve over time. What will be the same and what will be different about their evolution?

Problems

5. Water waves in shallow water (meaning the depth is much less than the wavelength) move with roughly the dispersion relation $\omega = k\sqrt{gh}$, where h is the depth and g is the acceleration due to gravity.

(a) Does a shallow water wave move at the same speed, faster than, or slower than the individual sinusoidal waves that make it up? You must do a calculation to justify your answer.

(b) Deep below the surface, water waves have the approximate dispersion relation $\omega = \sqrt{kh}$. Repeat Part (a) for deep water waves.

6. We began this section by reminding you that a free particle has eigenstates $\psi(x) = e^{ikx}$ with eigenvalues $\hbar^2 k^2/(2m)$.

(a) Derive those formulas by plugging the definition of "free particle" (no external forces) into the time-independent Schrödinger equation.

(b) Is $\psi(x) = 2e^{ikx}$ also an eigenstate of a free particle? Show the work that leads to your answer.

(c) Show that the function Ae^{ikx} cannot be normalized on the domain $(-\infty, \infty)$.

(d) What does the result in Part (c) imply about a free particle?

(e) The time evolution rule for quantum mechanics states that each energy eigenstate with eigenvalue E evolves by multiplying $\psi(x)$ by $e^{-iEt/\hbar}$. Show that the eigenstates of a free particle are traveling waves with velocity $k\hbar/(2m)$.

7. Prove that the following function must repeat itself every 2π.

$$f(x) = e^{ix} + e^{2ix} + e^{3ix} + \ldots$$

That is, prove that for any x-value, $f(x) = f(x + 2\pi)$.

8. What is the period of the following function?

$$f(x) = e^{ix/5} + e^{2ix/5} + e^{3ix/5} + \ldots$$

9. If a function $f(x,t)$ is moving to the right with a constant speed of v, that means that if you wait Δt seconds, you will see the same function moved to the right by $v\Delta t$.

$$f(x + v\Delta t, t + \Delta t) = f(x,t) \qquad \textit{A function moving to the right with velocity } v$$

Use that fact to prove the claim, made in the Explanation for this section, that the function $\Psi(x,t) = e^{i(kx-\omega t)}$ moves with phase velocity $v_g = \omega/k$.

10. Consider two waves $\Psi_1 = e^{i(k_1 x - \omega_1 t)}$ and $\Psi_2 = e^{i(k_2 x - \omega_2 t)}$. Assume all k and ω-values are real.
    (a) What is the phase of Ψ_1 on the complex plane? (Section 5.5.)
    (b) The two waves interfere constructively when and where their phases are equal. Write an equation that describes such a point in terms of x, t, k, and ω. Then solve that equation for x.
    (c) Based on your answer to Part (b), with what group velocity do the points of constructive interference occur?

11. *[This problem depends on Problem 10.]* For the two-function sum in Problem 10, write a relationship between x, t, k, and ω that describes a point of perfect *destructive* interference, where the phases of Ψ_1 and Ψ_2 differ by π. With what velocity does that point move?

12. Equation 4 on p. 9 suggests a fascinating case: if two waves have $\omega_1 = \omega_2$ and $k_1 \neq k_2$, their collective group velocity is zero. As an example, consider the following function:

$$z(x,t) = e^{i(x-t)} + e^{i(2x-t)}.$$

    (a) Calculate the real-valued function $|z(x,t)|$, the modulus of the complex-valued function. Simplify your answer as much as possible.
    (b) Graph your modulus function at times $t = 0$, $t = 1/2$, and $t = 1$. What is happening to the function over time?

13. One of the central results in this section is that a series of waves with a linear dispersion relation adds up to a wave that moves with a constant group velocity. To show that, consider an arbitrary series of traveling waves.

$$f(x) = A_1 e^{i(k_1 x - \omega_1 t)} + A_2 e^{i(k_2 x - \omega_2 t)} + A_3 e^{i(k_3 x - \omega_3 t)} + \ldots$$

Assume that the dispersion relation is a linear function $\omega(k)$. Show that the group velocity $\Delta\omega/\Delta k$ is the same for every *pair* of waves, and that this group velocity is equal to $d\omega/dk$.